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Hamiltonian circle action with self-indexing moment map

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Let (M,ω) be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points, and let $\mu : M \to \mathbb{R}$ be a moment map. Then it is well-known that μ is a Morse function whose critical point set coincides with the fixed point set M^{S^1}. Let Λ_{2k} be the set of all fixed points of Morse index $2k$. In this paper, we will show that if μ is constant on Λ_{2k} for each $k \leq n$, then (M,ω) satisfies the hard Lefschetz property. In particular, if (M,ω) admits a self-indexing moment map, i.e. $\mu(z) = 2k$ for every $k \leq n$ and $z \in \Lambda_{2k}$, then (M,ω) satisfies the hard Lefschetz property.

1. Introduction

Throughout this paper, we denote by T a compact torus, T^k a k-dimensional compact torus, and S^1 the unit circle group. Let (M,ω) be a $2n$-dimensional closed symplectic manifold. We say that ω satisfies the *hard Lefschetz property* if

$$\begin{array}{rccc} \wedge[\omega]^{n-k}: & H^k(M) & \longrightarrow & H^{2n-k}(M) \\ & \alpha & \longmapsto & \alpha \wedge [\omega]^{n-k} \end{array}$$

is an isomorphism for every $k = 0, 1, \ldots, n$ where $[\omega]$ is the cohomology class in $H^2(M;\mathbb{R})$ represented by ω. According to [4], the following question has been around for many years:

Question 1.1. [4, Problem 4.2] Let (M,ω) be a closed symplectic manifold with a Hamiltonian circle action. Assume that all fixed points are isolated. Then does (M,ω) satisfy the hard Lefschetz property?

In this paper, we discuss about conditions under which a Hamiltonian circle action with isolated fixed points satisfies the hard Lefschetz property. A leading candidate for these conditions might be '*semifree*', i.e. the action is free outside the fixed point set. We note that a Hamiltonian circle action with isolated fixed points is semifree if and only if the weights on the normal bundles to the fixed points are all ± 1, hence 'semifree' is a condition on weights of tangential S^1-representations at fixed points. Actually, any Hamiltonian semifree circle action with isolated fixed points satisfies the hard Lefschetz property as R. Sjamaar pointed out in [4]. But there is no other result related to Question 1.1 as far as the authors know. A motivation of this paper is a recent result of the authors [2] which says that we can determine whether a given closed Hamiltonian T-manifold with isolated fixed points satisfies the hard Lefschetz property just by looking at its moment map image in some cases. More precisely, the authors studied the hard Lefschetz property of Hamiltonian GKM-manifolds. A GKM *T-manifold*, defined by Goresky, Kottwitz, and MacPherson in [3], is an equivariantly formal T-manifold such that the fixed point set is finite, and irreducible components of the tangential T-representation on each fixed point are pairwise linearly independent. In [2], the authors proved that a six-dimensional closed Hamiltonian GKM T^2-manifold with index-increasing GKM graph satisfies the hard Lefschetz property. Hence one might guess that a moment map image, specially the image of the *1-skeleton* determines the hard Lefschetz property. In this point of view, it is also conceivable to find such conditions on the moment map image of the fixed point set in the case of a Hamiltonian circle action. An elementary theory of Hamiltonian group actions says that a moment map for a Hamiltonian circle action with isolated fixed points is a Morse function. And, the critical point set of a moment map is equal to the fixed point set, see [1] for the details. In this paper, we prove the following:

Theorem 1.2. *Let (M,ω) be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points, and let $\mu : M \to \mathbb{R}$ be a corresponding moment map. Let Λ_{2k} be the set of all fixed points of Morse index $2k$ with respect to μ. If μ is constant on Λ_{2k} for each $k \leq n$, then (M,ω) satisfies the hard Lefschetz property.*

Note that the condition '*constant μ on each Λ_{2k}*' is weaker than the well-known condition 'self-indexing' on Morse functions. A Morse function $f : M \to \mathbb{R}$ is called *self-indexing* if $\lambda(z) = f(z)$ for every critical point $z \in M$, where $\lambda(z)$ is a Morse index at z, see [7, p.44, Definition 4.9]. Hence if (M,ω) admits a self-indexing moment map μ, then it satisfies the condition in Theorem 1.2 automatically so that we have a corollary as follows:

Corollary 1.3. *Let (M,ω) be a $2n$-dimensional closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points, and* *the corresponding moment map* $\mu : M \to \mathbb{R}$ *is self-indexing. Then* (M, ω) *satisfies the hard Lefschetz property.*

This paper is organized as follows. In Section 2, we will give a brief introduction to equivariant cohomology theory for Hamiltonian circle actions. In Section 3, we will prove Theorem 1.2. And in Section 4, we will give several examples of Hamiltonian circle actions satisfying the condition of Theorem 1.2.

2. Equivariant symplectic forms and Canonical classes

In this section, we briefly review an elementary equivariant cohomology theory for Hamiltonian circle actions which will be used in the rest of the paper. Throughout this section, we will assume that every coefficient of any cohomology theory is $\mathbb{R}$. Let M be an S^1-manifold. Then the equivariant cohomology $H^*_{S^1}(M)$ is defined by

$$H^*_{S^1}(M) := H^*(M \times_{S^1} ES^1)$$

where ES^1 is a contractible space on which S^1 acts freely. Note that $M \times_{S^1} ES^1$ has a natural M-bundle structure over the classifying space $BS^1 := ES^1/S^1$ so that $H^*_{S^1}(M)$ admits an $H^*(BS^1)$-module structure. For the fixed point set M^{S^1}, the inclusion map $i : M^{S^1} \hookrightarrow M$ induces an $H^*(BS^1)$-algebra homomorphism

$$i^* : H^*_{S^1}(M) \to H^*_{S^1}(M^{S^1}) \cong \bigoplus_{F \subset M^{S^1}} H^*(F) \otimes H^*(BS^1)$$

and we call i^* *the restriction map to the fixed point set.* Note that for any fixed component $F \subset M^{S^1}$, the inclusion map $i_F : F \hookrightarrow M^{S^1}$ induces a natural projection

$$i_F^* : H^*_{S^1}(M^{S^1}) \to H^*_{S^1}(F) \cong H^*(F) \otimes H^*(BS^1).$$

For every $\alpha \in H^*_{S^1}(M)$, we will denote by $\alpha|_F$ the image $i_F^*(i^*(\alpha))$. From now on, we assume that (M, ω) is a closed symplectic manifold with a Hamiltonian circle action with moment map $\mu : M \to \mathbb{R}$. Choose a Riemannian metric g on M which is compatible with ω and S^1-invariant. For each fixed component $F \subset M^{S^1}$, let ν_F be a normal bundle of F in M. Then the *negative normal bundle* ν_F^- of F can be defined as a sub-bundle of ν_F whose fiber over $p \in F$ is a subspace of T_pM tangent to an unstable submanifold of M at

F with respect to g and μ. We denote by $e_F^- \in H^*_{S^1}(F)$ the equivariant Euler class of ν_F^-. In the Hamiltonian case, $H^*_{S^1}(M)$ has remarkable properties as follows:

Theorem 2.1. *[5] Let (M, ω) be a closed symplectic manifold with a Hamiltonian circle action. Then the restriction map $i^* : H^*_{S^1}(M) \to H^*_{S^1}(M^{S^1})$ is injective.*

Theorem 2.2. *[1] Let (M, ω) be a closed symplectic manifold with a Hamiltonian circle action. For an inclusion $f : M \hookrightarrow M \times_{S^1} ES^1$ as a fiber, the induced map $f^* : H^*_{S^1}(M) \to H^*(M)$ is surjective. Equivalently, $H^*_{S^1}(M)$ is a free $H^*(BS^1)$-module, i.e. M is equivariantly formal. Moreover, the kernel of f^* is given by $u \cdot H^*_{S^1}(M)$, where u is the degree-two generator of $H^*(BS^1)$ and $\cdot$ is the scalar product as an $H^*(BS^1)$-module.*

By Theorem 2.2, the equivariant cohomology $H^*_{S^1}(M)$ is isomorphic to $H^*(M) \otimes H^*(BS^1)$ as an $H^*(BS^1)$-module. Note that $H^*(BS^1)$ is isomorphic to $\mathbb{R}[u]$ where $-u$ is the first Chern class of the principal S^1-bundle $ES^1 \to BS^1$. McDuff and Tolman found a remarkable family of equivariant cohomology classes as follows:

Theorem 2.3. *[6, Lemma 1.12] Let (M, ω) be a closed symplectic manifold equipped with a Hamiltonian circle action with moment map $\mu : M \to \mathbb{R}$. For each fixed component $F \subset M^{S^1}$, let k_F be the index of F with respect to μ. Then given any cohomology class $Y \in H^i(F)$, there exists a unique class $\widetilde{Y} \in H^{i+k_F}_{S^1}(M)$ such that*

1) *$\widetilde{Y}|_{F'} = 0$ for every fixed component $F' \subset M^{S^1}$ with $\mu(F') < \mu(F)$,*
2) *$\widetilde{Y}|_F = Y \cup e_F^-$, and*
3) *the $H^*(BS^1)$-degree of $\widetilde{Y}|_{F'} \in H^*_{S^1}(F')$ is less than the index $k_{F'}$ of F' for all fixed components $F' \neq F$. Here, the $H^*(BS^1)$-degree of $\widetilde{Y}|_{F'} \in H^*_{S^1}(F')$ is the smallest integer j such that*

$$\widetilde{Y}|_{F'} \in \bigoplus_{i=0}^{j} H^i(BS^1) \otimes H^*(F').$$

Moreover, these classes generate $H^*_{S^1}(M)$ as an $H^*(BS^1)$-module.

We call such a class $\widetilde{Y}$ *the canonical class* with respect to Y. In the case when all the fixed points are isolated, Theorem 2.3 implies the following corollary:

Corollary 2.4. *Let (M,ω) be a closed symplectic manifold equipped with a Hamiltonian circle action with moment map $\mu : M \to \mathbb{R}$. Assume that all fixed points are isolated. For each fixed point $F \in M^{S^1}$, there exists a unique class $\alpha_F \in H^{k_F}_{S^1}(M)$ such that*

1) *$\alpha_F|_{F'} = 0$ for every $F' \in M^{S^1}$ with $\mu(F') < \mu(F)$,*
2) *$\alpha_F|_F = e^-_F = \prod w^-_i u$, where $\{w^-_i\}_{1\leq i \leq \frac{k_F}{2}}$ are the negative weights of the S^1-representation on $T_F M$, and*
3) *$\alpha_F|_{F'} = 0$ for every $F' \neq F \in M^{S^1}$ with $k_{F'} \leq k_F$.*

Moreover, $\{\alpha_F\}_{F\in M^{S^1}}$ is a basis of $H^*_{S^1}(M)$ as an $H^*(BS^1)$-module.

Proof. The first and second statements are straightforward by Theorem 2.3. For the third one, let F be an isolated fixed point. Since $H^*_{S^1}(F) \cong H^*(BS^1) \otimes H^0(F)$, the $H^*(BS^1)$-degree of $\beta|_F \in H^*_{S^1}(F)$ is the same as the degree (as a cohomology class) of β for every $\beta \in H^*_{S^1}(M)$. Therefore, the $H^*(BS^1)$-degree of $\alpha_F|_{F'}$ is $\deg \alpha_F = k_F$ and $k_F < k_{F'}$ by Theorem 2.3-(3), i.e. $\alpha_F|_{F'} = 0$ if $k_{F'} \leq k_F$. □

There is another important class, called the *equivariant symplectic class* $\widetilde{\omega}_\mu \in H^2_{S^1}(M)$ *with respect to* μ, which satisfies the following:

Proposition 2.5. *[1] For a given closed Hamiltonian S^1-manifold with only isolated fixed points with moment map μ, there exists an equivariant symplectic class $\widetilde{\omega}_\mu$ such that*

1) *$f^*\widetilde{\omega}_\mu = \omega$,*
2) *$\widetilde{\omega}_\mu|_F = -\mu(F)u$ for every $F \in M^{S^1}$.*

3. Proof of Theorem 1.2

Let (M,ω) be a closed symplectic manifold equipped with a Hamiltonian circle action with only isolated fixed points. Let $\mu : M \to \mathbb{R}$ be a moment map whose minimum is zero, and let

$$\Lambda_{2k} = \{F_1^{2k}, \ldots, F_{b_{2k}}^{2k}\}$$

be the set of all fixed points of index $2k$. Here, $b_{2k} := b_{2k}(M)$ is the $2k$-th Betti number of M. Then F_1^0 is the unique fixed point with $\mu(F_1^0) = 0$ by our assumption on μ. Throughout this section, we assume that μ is constant on each Λ_{2k} so that

$$\mu(F_1^{2k}) = \cdots = \mu(F_{b_{2k}}^{2k}) = c_{2k}$$

for every $k \leq n$ and some $c_{2k} \in \mathbb{R}$. Let $\{\alpha_F \mid F \in M^{S^1}\}$ be the set of all canonical classes described in Corollary 2.4. To simplify our proof, we will denote by $\beta_F := \frac{1}{\prod_i w_i^-}\alpha_F$, where $\{w_i^-\}_{1\leq i \leq \frac{k_F}{2}}$ are the negative weights of S^1-representation on $T_F M$ so that $\beta_F|_F = u^{k_F/2}$ by Corollary 2.4. To prove the main theorem, we need the following series of lemmas:

Lemma 3.1. *For the equivariant symplectic class $\widetilde{\omega}_\mu \in H^2_{S^1}(M)$, we have*

$$\widetilde{\omega}_\mu = -c_2 \cdot (\beta_{F_1^2} + \cdots + \beta_{F_{b_2}^2}).$$

Proof. Since M is equivariantly formal by Theorem 2.2, the equivariant cohomology $H^*_{S^1}(M)$ is isomorphic to the free $H^*(BS^1)$-module $H^*(M) \otimes H^*(BS^1)$. Hence we have

$$H^2_{S^1}(M) \cong \Big(H^0(M) \otimes H^2(BS^1)\Big) \oplus \Big(H^2(M) \otimes H^0(BS^1)\Big).$$

By the last statement of Corollary 2.4, the set

$$\left\{\ u\beta_{F_1^0} = u,\ \beta_{F_1^2}, \ldots, \beta_{F_{b_2}^2}\ \right\}$$

forms an $\mathbb{R}$-basis of $H^2_{S^1}(M)$. So we may let

$$\widetilde{\omega}_\mu = a_0 \cdot u\beta_{F_1^0} + \left(a_1 \cdot \beta_{F_1^2} + \cdots + a_{b_2} \cdot \beta_{F_{b_2}^2}\right)$$

for some real constants $a_0, a_1, \ldots, a_{b_2}$. Since each $\beta_{F_i^2}$ vanishes on F_1^0 for $i = 1, \ldots, b_2$ by Corollary 2.4, we have

$$\widetilde{\omega}_\mu|_{F_1^0} = a_0 \cdot u = -\mu(F_1^0)u = 0$$

by Proposition 2.5 and our assumption on μ. Hence we have $a_0 = 0$. Again by Corollary 2.4, each $\beta_{F_i^2}$ vanishes on F_j^2 for every $j \neq i$ so that we have

$$\widetilde{\omega}_\mu|_{F_i^2} = a_i \cdot u = -\mu(F_i^2)u = -c_2 u.$$

Therefore, we have $a_1 = \cdots = a_{b_2} = -c_2$. This finishes the proof. □

Lemma 3.2. *Let $\mu_{2k} := \mu - c_{2k}$ be a new moment map for each $k \leq n$. Then the equivariant symplectic class $\widetilde{\omega}_{\mu_{2k}}$ vanishes on Λ_{2k}.*

Proof. By Proposition 2.5, we have $\widetilde{\omega}_{\mu_{2k}}|_F = -\big(\mu(F) - c_{2k}\big)u$ for every fixed point $F \in M^{S^1}$. Hence we have

$$\widetilde{\omega}_{\mu_{2k}}|_{F_i^{2k}} = -\big(\mu(F_i^{2k}) - c_{2k}\big)u = 0$$

for every $i = 1, \ldots, b_{2k}$. □

Lemma 3.3. *The $n+1$ numbers c_{2k}'s are all distinct.*

Proof. Assume that $c_{2i} = c_{2j}$ for some $i \neq j$. Then the class

$$\eta = \widetilde{\omega}_{\mu_0} \cdot \widetilde{\omega}_{\mu_2} \cdots \widehat{\widetilde{\omega}_{\mu_{2i}}} \cdots \widetilde{\omega}_{\mu_{2n}} \in H^{2n}_{S^1}(M)$$

is an equivariant extension of $\omega^n \in H^{2n}(M)$. By Lemma 3.2, η vanishes on Λ_{2k} for every $k = 0, 1, \ldots, n$ so that η vanishes on every fixed point, i.e. η is a zero class in $H^*_{S^1}(M)$ by Theorem 2.1. Since ω^n is nonzero in $H^{2n}(M)$, it contradicts that η is an extension of ω^n. □

Lemma 3.4. *If an element γ in $H^{2k}_{S^1}(M)$ vanishes on Λ_{2i} for every $i \leq k$, then γ is zero in $H^{2k}_{S^1}(M)$. Similarly, if γ in $H^{2k}_{S^1}(M)$ vanishes on Λ_{2i} for every $i \geq n-k$, then γ is zero in $H^{2k}_{S^1}(M)$.*

Proof. By Corollary 2.4, the set

$$\left\{ u^{k-i} \cdot \beta_{F_j^{2i}} \;\middle|\; 0 \leq i \leq k,\ 1 \leq j \leq b_{2i} \right\}$$

is an $\mathbb{R}$-basis of $H^{2k}_{S^1}(M)$ so that γ is uniquely expressed as follows:

$$\gamma = \sum_{0 \leq i \leq k,\ 1 \leq j \leq b_{2i}} p_{i,j} \cdot u^{k-i} \cdot \beta_{F_j^{2i}} \tag{3.1}$$

for some real coefficients $p_{i,j}$'s. By our assumption, γ vanishes on every fixed point of index less than or equal to $2k$. First, the restriction of γ on the fixed point F_1^0 with index zero is zero by our assumption so that

$$\gamma|_{F_1^0} = p_{0,1} \cdot u^k = 0.$$

Hence we have $p_{0,1} = 0$. Next, consider the restriction of γ on index two fixed points F_j^2 for any j. Then we have

$$\gamma|_{F_j^2} = p_{1,j} \cdot u^k = 0.$$

Hence we have $p_{1,j} = 0$ for every $j \leq b_2(M)$. In this way, every $p_{i,j}$ becomes zero. So, we obtain a proof for the first statement.

The proof of the second statement is done in exactly the same way with respect to the moment map $-\mu$. □

Now, we are ready to prove our main theorem.

Proof of Theorem 1.2. For some $2k < n$, assume that

$$\begin{array}{cccc} \wedge\omega^{n-2k}: & H^{2k}(M) & \longrightarrow & H^{2n-2k}(M) \\ & \alpha & \longmapsto & \alpha \wedge \omega^{n-2k} \end{array}$$

has a nonzero kernel $\gamma \in H^{2k}(M)$, i.e. $\gamma \wedge \omega^{n-2k}$ is zero in $H^{2n-2k}(M)$. By Theorem 2.2, M is equivariantly formal so that there is an equivariant extension $\widetilde{\gamma} \in H^{2k}_{S^1}(M)$ of γ such that $f^*(\widetilde{\gamma}) = \gamma$ where $f : M \hookrightarrow M \times_{S^1} ES^1$ is an inclusion as a fiber. Since the kernel of f^* is the ideal $u \cdot H^*_{S^1}(M)$ by Theorem 2.2, we may choose $\widetilde{\gamma}$ such that $\widetilde{\gamma}$ vanishes on Λ_{2i} for every $i < k$ by arguments similar to the proof of Lemma 3.4. We denote by μ_{2k} a moment map for the given circle action such that μ_{2k} vanishes on Λ_{2k}. Then the class

$$\delta = \widetilde{\gamma} \cdot \left(\widetilde{\omega}_{\mu_{2k}} \cdot \widetilde{\omega}_{\mu_{2k+2}} \cdots \widetilde{\omega}_{\mu_{2n-2k-2}}\right) \in H^{2n-2k}_{S^1}(M)$$

is an equivariant extension of $\gamma \wedge \omega^{n-2k}$ satisfying $\delta|_F = 0$ for every fixed point F of index less than $2n - 2k$ by the following two reasons:

1) $\widetilde{\gamma}$ vanishes on Λ_{2i} for every $i < k$ by definition,
2) $\widetilde{\omega}_{\mu_{2k}} \cdot \widetilde{\omega}_{\mu_{2k+2}} \cdots \widetilde{\omega}_{\mu_{2n-2k-2}}$ vanishes on Λ_{2i} for every $k \leq i \leq n-k-1$ by Lemma 3.2.

If we express δ by (3.1) as follows:

$$\delta = \sum_{0\le i\le n-k,\ 1\le j\le b_{2i}} p_{i,j}\cdot u^{n-k-i}\cdot \beta_{F_j^{2i}},$$

then the vanishing of δ on Λ_{2i} for $i<n-k$ implies $p_{i,j}=0$ for every $i<n-k$ by the same proof of Lemma 3.4. So we have

$$\delta = \sum_{1\le j\le b_{2n-2k}} p_{n-k,j}\cdot \beta_{F_j^{2n-2k}}.$$

But since $\gamma\wedge\omega^{n-2k}=0$ by our assumption, we have $\delta\in\ker f^*$ and therefore $\delta\in u\cdot H^*_{S^1}(M)$ by Theorem 2.2, i.e. $p_{n-k,j}=0$ for every j. Consequently, we have $\delta=0$ in $H^{2n-2k}_{S^1}(M)$, i.e. $\delta|_F=0$ for every fixed point F by Theorem 2.1.

Note that if $\widetilde{\gamma}|_F\neq 0$ for some fixed point F of index greater than or equal to $2n-2k$, then

$$\begin{aligned}\delta|_F &= \widetilde{\gamma}|_F\cdot\big(\widetilde{\omega}_{\mu_{2k}}|_F\cdot\widetilde{\omega}_{\mu_{2k+2}}|_F\cdots\widetilde{\omega}_{\mu_{2n-2k-2}}|_F\big)\\ &= \widetilde{\gamma}|_F\cdot\big(c_{2k}-\mu(F)\big)\cdots\big(c_{2n-2k-2}-\mu(F)\big)\neq 0\end{aligned}$$

by Proposition 2.5 and Lemma 3.3. Hence $\widetilde{\gamma}|_F=0$ for every fixed point F of index greater than or equal to $2n-2k$. By Lemma 3.4 again, we have $\widetilde{\gamma}=0$ in $H^{2k}_{S^1}(M)$ so that $\gamma=0$ in $H^{2k}(M)$. This finishes the proof. □

4. Examples

In the section, we give the examples that we mentioned in Section 1.

Example 4.1. First, we introduce a well-known example of a six-dimensional coadjoint orbit of SU(3). The Lie algebra $\mathfrak{su}(3)$ of SU(3) consists of all traceless 3×3 skew-Hermitian matrices. Let T and $\mathfrak{t}$ be the standard maximal torus of SU(3) and its Lie algebra, i.e. T and $\mathfrak{t}$ are subsets of diagonal matrices in SU(3) and $\mathfrak{su}(3)$, respectively. We identify $\mathfrak{su}(3)$ (resp. $\mathfrak{t}$) with its dual $\mathfrak{su}(3)^*$ (resp. $\mathfrak{t}^*$) through the Killing form on $\mathfrak{su}(3)$. Let $D_0\in\mathfrak{t}^*$ be the diagonal element with entries $\sqrt{-1}$, 0, $-\sqrt{-1}$. The *coadjoint orbit* $M\subset\mathfrak{su}(3)^*$ of SU(3) through the matrix D_0 is defined as the orbit of D_0 by the coadjoint action of SU(3) on $\mathfrak{su}(3)^*$, i.e. conjugation. The coadjoint orbit endowed with the Kostant-Kirillov symplectic form is a symplectic manifold and (trivially) invariant under the coadjoint T-action on $\mathfrak{su}(3)^*$, see [1, p.61].

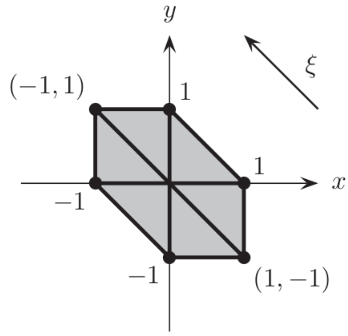

Figure 4.1: A moment map image of a six-dimensional coadjoint orbit of $SU(3)$.

The map

$$\mu : M \to \mathfrak{t}^*, \quad (a_{ij})_{1\le i,j\le 3} \longmapsto (\delta_{ij} \cdot a_{ij})_{1\le i,j\le 3}$$

is a moment map for the T-action on M, where δ_{ij} is the Kronecker delta function. Under the identification

$$\mathfrak{t}^* \to \mathbb{R}^2, \quad \begin{pmatrix} x\sqrt{-1} & 0 & 0 \\ 0 & y\sqrt{-1} & 0 \\ 0 & 0 & -(x+y)\sqrt{-1} \end{pmatrix} \longmapsto (x, y),$$

the moment map image is depicted in Figure 4.1: the black dots are images of the six fixed points. The vector $\xi = (-1, 1)$ in the Lie algebra $\mathfrak{t}$ defines a circle subgroup in T, and $\mu_\xi := \langle \mu, \xi \rangle$ is a moment map for the circle action, where $\langle\ ,\ \rangle$ is the evaluation pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. Then, we have

$$\begin{array}{ll} \mu_\xi = -2 \ \text{ on } \Lambda_0, & \mu_\xi = -1 \ \text{ on } \Lambda_2, \\ \mu_\xi = 1 \ \text{ on } \Lambda_4, & \mu_\xi = 2 \ \text{ on } \Lambda_6. \end{array}$$

So, the moment map μ_ξ satisfies the condition of Theorem 1.2.

The coadjoint orbit endowed with the Kostant-Kirillov symplectic form is Kähler, see [9, p.311]. So, readers might want to see a non-Kähler example. But, there is no known example of non-Kähler Hamiltonian circle action with nonempty isolated fixed point set yet. Instead, we give an example such that we do not know whether it is Kähler.

Example 4.2. In [8] (see also [9]), Tolman constructed a six-dimensional Hamiltonian T^2-manifold with isolated fixed points. She proved that her example does not admit any T^2-invariant Kähler structure. Its moment map image is Figure 4.2, see [9, Fig. 1.(3)]. The vector $\xi = (-1, 1)$ in the Lie algebra $\mathfrak{t}$ of T defines a circle subgroup in T^2, and μ_ξ is a moment map for the circle action. Then,

$$\mu_\xi = -4 \text{ on } \Lambda_0, \qquad \mu_\xi = -1 \text{ on } \Lambda_2,$$
$$\mu_\xi = 1 \text{ on } \Lambda_4, \qquad \mu_\xi = 4 \text{ on } \Lambda_6.$$

The moment map μ_ξ satisfies the condition of the main theorem. But it is still open whether Tolman's example might admit a non-invaraint Kähler structure, see [9, Section 4]. Hence Theorem 1.2 is nontrivial for the circle action.

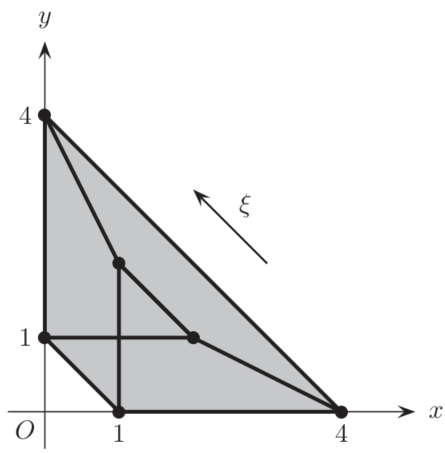

Figure 4.2: Tolman's example.

Next examples are monotone semifree Hamiltonian circle actions.

Example 4.3. Let (M, ω) be a $2n$-dimensional closed monotone symplectic manifold, i.e. a symplectic manifold satisfying $\omega = c_1(M)$, where $c_1(M)$ is the first Chern class of M with respect to some ω-tamed almost complex structure J. Assume that (M, ω) admits a semifree Hamiltonian circle action with only isolated fixed points. Then there exists a unique moment map $\mu : M \to \mathbb{R}$ such that $\widetilde{\omega}_\mu = c_1^{S^1}(M) \in H^2_{S^1}(M; \mathbb{R})$, where $c_1^{S^1}(M)$ is the equivariant first Chern class of M. Note that the Morse index $\lambda(F)$ is twice the number of negative weights of the tangential S^1-representation at F for each fixed point $F \in M^{S^1}$. Since we assumed that the action is semifree, we have

$$-\mu(F)u = \widetilde{\omega}_\mu|_F = c_1^{S^1}(M)|_F = (p_F - n_F)u = (n - 2n_F)u$$

by Proposition 2.5, where p_F (resp. n_F) is the number of positive (resp. negative) weights of the tangential representation at F. Hence we have $(\mu + n)(F) = 2n_F = \lambda(F)$ so that $\widetilde{\mu} := \mu + n$ is the self-indexing moment map.

Example 4.4. We give a concrete example of a monotone semifree Hamiltonian circle action with eight fixed points. Let ω be the Fubini-Study form divided by $\pi/2$ on $\mathbb{C}P^1$. Then, the first Chern class $c_1(\mathbb{C}P^1)$ of $\mathbb{C}P^1$ is equal to ω, and $\mathbb{C}P^1$ with ω is a symplectic manifold which is invariant under the usual action of S^1. And, a moment map $\mu : \mathbb{C}P^1 \to \mathbb{R}^1 = L(S^1)$ for the action takes $\mathbb{C}P^1$ onto $[0, 2]$, where $L(S^1)$ is the Lie algebra of S^1.

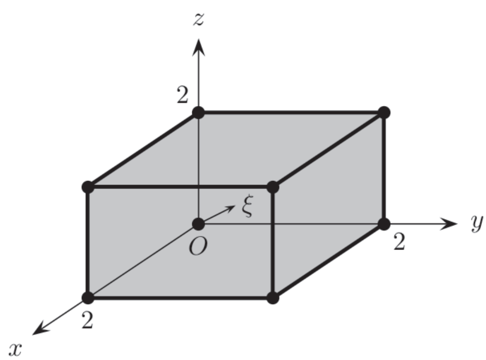

Figure 4.3: A moment map image of $(\mathbb{C}P^1)^3$.

Considering the diagonal action of $(S^1)^3$ on $(\mathbb{C}P^1)^3$, the map

$$\overline{\mu} : (\mathbb{C}P^1)^3 \longrightarrow \mathbb{R}^3, \qquad (z_1, z_2, z_3) \longmapsto \Big(\mu(z_1), \mu(z_2), \mu(z_3)\Big)$$

is a moment map for the action of $T = (S^1)^3$, where $\mathfrak{t}^* = \mathbb{R}^3$. The vector $\xi = (1, 1, 1)$ in the Lie algebra $\mathfrak{t}$ defines a circle subgroup in T, and μ_ξ is a moment map for the circle action. Then, μ_ξ is the self-indexing moment map.

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