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Global normally hyperbolic invariant cylinders in Lagrangian systems

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In this paper, we study Tonelli Lagrangian $L \in C^r(T\mathbb{T}^2, \mathbb{R})$ with $r \geq 5$. For a generic perturbation of Lagrangian $L \to L + P$ where $P \in C^r(\mathbb{T}^2, \mathbb{R})$, we get simultaneous hyperbolicity of a family of minimal periodic orbits which share the same first homology class. Consequently, these periodic orbits make up one or more pieces of normally hyperbolic invariant cylinder in $T\mathbb{T}^2$.

1. Introduction

The configuration manifold considered here is a two-dimensional torus $M = \mathbb{T}^2$. We assume that L: $TM \to \mathbb{R}$ is an autonomous C^r-Lagrangian with $r \geq 5$, strictly convex on each tangent fiber, namely, the Hessian matrix of L in $\dot{x}$ is positive definite for each $x \in \mathbb{T}^2$. Therefore, it makes sense to study the minimal measures by applying the Mather theory for Tonelli Lagrangian [Mat].

Let $\mathfrak{M}(L)$ be the set of Borel probability measures on TM, which are invariant for the Lagrange flow ϕ_L^t produced by L. Each $\mu \in \mathfrak{M}(L)$ is associated with a rotation vector $\rho(\mu) \in H_1(\mathbb{T}^2, \mathbb{R})$ such that for every closed 1-form η on $\mathbb{T}^2$ one has

$$\langle [\eta], \rho(\mu) \rangle = \int \eta d\mu.$$

Let $\mathfrak{M}_\omega(L) = \{\mu \in \mathfrak{M}(L) : \rho(\mu) = \omega\}$, an invariant measure μ is called minimal with the rotation vector ω if

$$\int L d\mu = \inf_{\nu \in \mathfrak{M}_\omega(L)} \int L d\nu.$$

A rotation vector $\omega \in H_1(\mathbb{T}^2, \mathbb{R})$ is called resonant if there exists a non-zero integer vector $k \in \mathbb{Z}^2$ such that $\langle \omega, k \rangle = 0$. For two-dimensional space, it uniquely determines an irreducible element $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and a positive number $\lambda > 0$ such that $\omega = \lambda g$ if the rotation vector ω is resonant. Let α and β denote the α- and β-function of the Lagrangian L respectively. The Fenchel-Legendre transformation $\mathscr{L}_\beta$: $H_1(\mathbb{T}^2,\mathbb{R}) \to H^1(\mathbb{T}^2,\mathbb{R})$ is defined as follows

$$c \in \mathscr{L}_\beta(\omega) \quad \Longleftrightarrow \quad \alpha(c) + \beta(\omega) = \langle c, \omega \rangle.$$

For generic L, the set $\cup_{\lambda\in\mathbb{R}^+}\mathscr{L}_\beta(\lambda g)$ turns out to be a channel if $g \in H_1(\mathbb{T}^2,\mathbb{Z})$.

Since the configuration manifold is two-dimensional, each orbit $(\gamma, \dot\gamma)$ in the support of minimal measure μ is periodic if $\rho(\mu)$ is resonant. Indeed, if $\rho(\mu) = \lambda g$ where g is irreducible and $\lambda > 0$, all periodic orbits in the support share the same homology class. It is guaranteed by the Lipschitz property of Aubry set due to Mather.

To introduce λg-*minimal* curve, let us recall the definition of c-minimal curve for $c \in H^1(\mathbb{T}^2,\mathbb{R})$. For autonomous system, a curve γ: $\mathbb{R} \to M$ is called c-minimal if for any $t < t'$ and any absolutely continuous curve ζ: $[t,t'] \to M$ such that $\zeta(t) = \gamma(t)$, $\zeta(t') = \gamma(t')$ one has

$$\int_t^{t'} \Big(L(\gamma(t),\dot\gamma(t)) - \langle c, \dot\gamma(t)\rangle\Big)dt \le \int_t^{t'} \Big(L(\zeta(t),\dot\zeta(t)) - \langle c, \dot\zeta(t)\rangle\Big)dt.$$

The orbit $(\gamma, \dot\gamma)$ is called c-minimal if the curve γ is c-minimal. If $M = \mathbb{T}^2$, each c-*minimal* curve in Mather set must be periodic if the rotation vector is resonant.

To specify the topological information of minimal orbit, we call a periodic orbit λg-*minimal* if some $g \in H_1(M,\mathbb{Z})$ and $\lambda \in \mathbb{R}^+$ exist such that this orbit is c-minimal for each $c \in \mathscr{L}_\beta(\lambda g)$. The main result of this paper is the following

Theorem 1.1. *There exists a residual set $\mathfrak{P} \subset C^r(\mathbb{T}^2,\mathbb{R})$ with $r \ge 5$ such that for each $P \in \mathfrak{V}$, it holds simultaneously for all $(\lambda, g) \in \mathbb{R}^+ \times H_1(\mathbb{T}^2,\mathbb{Z})$ that all λg-minimal periodic orbits of $L + P$ are hyperbolic. Thus, for each $g \in H_1(\mathbb{T}^2,\mathbb{Z})$, all λg-minimal periodic orbits constitute one or more pieces of normally hyperbolic invariant cylinder in $T\mathbb{T}^2$.*

Remark 1. The existence of such normally hyperbolic invariant cylinders is crucial in the proof of Arnold diffusion in *a priori* stable systems with three degrees of freedom [Ch]. Near double resonant point, one is not able to get the cylinders by using KAM technique. It is also expected to play an important role in the study of Arnold diffusion with arbitrarily many degrees of freedom.

Remark 2. The Mañé conjecture [Man] is raised for Tonelli Lagrangian with fixed first cohomology class. When the manifold is a closed surface, the conjecture is solved in C^2-topology [CFR, Con]. Certain non-degeneracy of periodic orbits on fixed energy level is also obtained in [Ol]. However, the hyperbolicity we studied here is not only for a fixed cohomology class, but simultaneous for a path of first cohomology classes.

We use variational method to prove the theorem. For a first homology class g and an energy $E > \min\alpha$, we reduce L to a time-periodically dependent Lagrangian $\bar{L}$ of one degrees of freedom and get a functional $F(x, E, g) \to \mathbb{R}$. It measures the minimal action of $\bar{L}$ along those closed curves passing through x. The global minimal point of $F(\cdot, E, g)$ uniquely determines a minimal periodic orbit which support a minimal measure μ for certain rotation vector λg. It shall be proved in Theorem 4.1 that the non-degeneracy of the minimal point is equivalent to the hyperbolicity of the periodic orbit. Therefore, the proof becomes a problem to see whether it holds simultaneously for all E that the minimal of F in x is non-degenerate. We choose a four parameters family of trigonometric functions as the candidates of potential perturbation. Fix a class g and a fixed interval for energy $[E_0, E_1]$, there is an open-dense set in this four dimensional space such that for each perturbation from this set the non-degeneracy of minimal point of F holds for all $E \in [E_0, E_1]$. As there are countably many first homology class in $H_1(\mathbb{T}^2, \mathbb{Z})$, the residual property is obtained by taking a countable intersection of open-dense sets.

2. Variational set-up

Let us fix an irreducible class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and prove that, for any $0 < \lambda_0 < \lambda_1$, all λg-minimal orbits are simultaneously hyperbolic for all $\lambda \in [\lambda_0, \lambda_1]$. It is proved by variational method. By adding a constant and a closed 1-form to L, we assume $\alpha(0) = \min\alpha = 0$.

Given an irreducible element $g \in H_1(\mathbb{T}^2, \mathbb{Z})$, we have a channel

$$\mathbb{C}_g = \bigcup_{\lambda>0} \mathscr{L}_\beta(\lambda g) \quad \text{or} \quad \mathbb{C}_g = \bigcup_{\lambda\geq\lambda_0>0} \mathscr{L}_\beta(\lambda g).$$

In former case, $\alpha(c) > \min\alpha$ for each $c \in \mathscr{L}_\beta(\lambda g)$ and each $\lambda > 0$. In latter case, $\alpha(c) = \min\alpha$ for $c \in \mathscr{L}_\beta(\lambda_0 g)$. In general, $\mathbb{C}_g$ has a foliation of lines $I_{g,E} = \mathbb{C}_g \cap \alpha^{-1}(E)$. Indeed, as the configuration manifold is $\mathbb{T}^2$, each ergodic minimal measure is supported on a periodic orbit if the rotation vector is resonant. Let γ be a periodic curve so that $(\gamma, \dot{\gamma})$ is a periodic orbit located in the Mather set for the class $c \in I_{g,E}$, the group $H_1(\mathbb{T}^2, \gamma, \mathbb{Z})$ is an one-dimensional lattice, two generators are denoted by g_0 and $g_0' = -g_0$. For a homology class $g' \in H_1(\mathbb{T}^2, \gamma, \mathbb{Z})$ we consider the quantity

$$h(g', c) = \lim_{T\to\infty} \inf_{\substack{\zeta(0)=\zeta(T)\in\gamma \\ [\zeta]=g'}} \int_0^T (L - \eta_c)(\zeta(t), \dot{\zeta}(t))dt + T\alpha(c).$$

The identity $h(g_0, c) = h(g_0', c)$ holds for all $c \in I_{g,E}$ if and only if the set $I_{g,E}$ is a singleton. In this case, there exists an invariant two-dimensional torus foliated into a family of periodic orbits with the same homological class g, each of them is c-minimal. According to the result of [BC], it is generic that all minimal measures consist of at most 3-ergodic components. Therefore, the set $I_{g,E}$ is a segment of line in general case. At one endpoint of the interval $I_{g,E}$ one has $h(g_0, c) = 0$ and $h(g_0', c) > 0$ and at another endpoint one has $h(g_0', c) = 0$ and $h(g_0, c) > 0$. Since all cohomology classes on $I_{g,E}$ share the same Mather set, it makes sense to write $\tilde{\mathcal{M}}(c) = \tilde{\mathcal{M}}(E, g)$ with $E = \alpha(c)$ and $c \in \mathbb{C}_g$.

Theorem 2.1. *Given a class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and a closed interval $[E_a, E_d] \subset \mathbb{R}_+$ with $E_a > \min \alpha$, there exists an open-dense set $\mathfrak{O} \subset C^r(\mathbb{T}^2, \mathbb{R})$ with $r \geq 5$ such that for each $P \in \mathfrak{O}$, it holds simultaneously for all $E \in [E_a, E_d]$ that the Mather set $\tilde{\mathcal{M}}(E, g)$ for $L + P$ consists of hyperbolic periodic orbits. Indeed, except for finitely many $E_j \in [E_a, E_d]$ where the Mather set consists of two hyperbolic periodic orbits, for all other $E \in [E_a, E_d]$ it consists exactly one hyperbolic periodic orbit.*

This theorem will be proved by showing the non-degeneracy of the minimal point of certain action function. Toward this goal, let us split the interval into suitably many subintervals $[E_a, E_d] = \cup_{i=0}^{k} [E_i - \delta_{E_i}, E_i + \delta_{E_i}]$ with suitably small $\delta_{E_i} > 0$. Once the open-dense property holds for each small subinterval, then it hold for the whole interval.

Let us explain how the interval $[E_a, E_d]$ is split. In the channel, one can choose a path along which the α-function monotonely increases. Restricted on this path, we obtain a family of Lagrangians with one parameter. By using the method of [BC], we can see that it is typical that the minimal measure is supported at most on two periodic orbits for each class on this path. Thus, the Mather set $\tilde{\mathcal{M}}(E, g)$ consists of at most two periodic orbits for each $E \in [E_a, E_d]$.

Without loss of generality, we assume $g = (0, 1)$, all these minimal curves are then associated with the homological class. Restricted on the neighborhood $\mathbb{S}_{\gamma_{E_i}} \subset \mathbb{T}^2$ of a minimal curve $\gamma_{E_i} \in \mathcal{M}(E_i, g)$ for certain energy E_i, we introduce a configuration coordinate transformation $x = X(u)$ such that along the curve γ_{E_i} one has $u_1 =$ constant. In the new coordinates, the Lagrangian reads

$$L'(\dot{u}, u) = L(DX(u)\dot{u}, X(u))$$

which is obviously positive definite in $\dot{u}$. As $\gamma_E(t)$ is a solution of the Euler-Lagrange equation determined by L, the curve $X^{-1}(\gamma_E)(t)$ solves the equation determined by L' and is minimal for the action of L'. As there are at most two minimal curves for each energy, the neighborhood of these two curves can be chosen not to overlap each other. Thus one can extend the coordinate transformation to the whole torus.

Let H' be the Hamiltonian determined by L' through the Legendre transformation. the minimal curve determines a periodic solution for the Hamiltonian equation. By construction, $\partial_{v_2} H' > 0$ holds along the periodic solution which entirely stays in the energy level set $H'^{-1}(E)$. We choose suitably small $\delta_{E_i} > 0$ such that for $E \in [E_i - \delta_{E_i}, E_i + \delta_{E_i}]$ each minimal periodic curve in $\mathcal{M}(E, g)$ falls into the strip $\mathbb{S}_{\gamma_{E_i}}$ and $\partial_{v_2} H' > 0$ holds along each minimal periodic orbit.

For brevity of notation, we still use x, L and H to denote the configuration coordinates, the Lagrangian and the Hamiltonian respectively. The condition $\partial_{y_2} H > 0$ holds in a neighborhood of minimal periodic orbits for $E \in [E_i - \delta_{E_i}, E_i + \delta_{E_i}]$. So, the Lagrangian and the Hamiltonian can be reduced to a time-periodic system with one degree of freedom when it is restricted on energy level set. The new Hamiltonian $\bar{H}(x_1, y_1, \tau, E)$ solves the equation $H(x_1, y_1, x_2, \bar{H}) = E$ with $\tau = -x_2$, from which one obtains a new Lagrangian $\bar{L} = \dot{x}_1 y_1 - \bar{H}(x_1, y_1, \tau, E)$ where $y_1 = y_1(x_1, \dot{x}_1, \tau)$ solves the equation $\dot{x}_1 = \partial_{y_1} \bar{H}(x_1, y_1, \tau)$. In the following we omit the subscript "1", i.e. let $(x, y, \dot{x}) = (x_1, y_1, \dot{x}_1)$ if no danger of confusion.

We introduce a function of Lagrange action $F(\cdot, E)$: $\mathbb{T} \to \mathbb{R}$:

$$F(x, E) = \inf_{\gamma(0)=\gamma(2\pi)=x} \int_0^{2\pi} \bar{L}(d\gamma(\tau), \tau, E)d\tau.$$

A curve $\gamma_E(t, x)$ is called the minimizer of F at x if $\gamma_E(0, x) = \gamma_E(2\pi) = x$ and the Lagrange action along this curve reaches the quantity $F(x, E)$. There might be more than one minimizer if x is not a minimal point as it is by no means a system without conjugate point [CI], the function F may not be globally smooth. However, if we let

$$S_i = \{(x, E) \in \mathbb{T} \times [E_i - \delta_{E_i}, E_i + \delta_{E_i}] : x \in F^{-1}(\min F(\cdot, E))\},$$

Lemma 2.1. *There is a neighborhood U of S_i in $\mathbb{T} \times [E_i - \delta_{E_i}, E_i + \delta_{E_i}]$, on which F is C^{r-1}, the minimizer $\gamma_E(\cdot, x)$ C^{r-1}-smoothly depends on x if $(x, E) \in S_i$.*

Proof. In the new coordinate system, $\partial_{y_2} H > 0$ holds in a neighborhood the minimal curve. It guarantees the smoothness of $\bar{H}$ and then $\bar{L}$ on E when $(x_1, y_1, x_2 = -\tau)$ stays in the neighborhood, since $\bar{H}$ solves the equation $H(x_1, y_1, x_2, \bar{H}) = E$.

Now, let us fix an energy E first, choose $T_i = \frac{2\pi i}{m}$ and define the function of action $F_i(x, x', E)$

$$F_i(x, x', E) = \inf_{\substack{\gamma(T_i)=x \\ \gamma(T_{i+1})=x'}} \int_{T_i}^{T_{i+1}} \bar{L}(d\gamma(\tau), \tau) d\tau.$$

There will be two or more minimizers of $F_i(x, x', E)$ if the point x is in the "cut locus" of the point x'. However, the minimizer is unique if x is suitably close to x', denoted by $\gamma_i(\cdot, x, x', E)$. In this case, it uniquely determines a speed $v = v(x, x')$ such that $\dot{\gamma}_i(T_i, x, x', E) = v(x, x')$. Let $\vec{x} = (x_0, x_1, \ldots, x_m)$ denote a periodic configuration ($x_0 = x_m$), we introduce a function of action

$$\mathbf{F}(\vec{x}, E) = \sum_{i=0}^{m-1} F_i(x_i, x_{i+1}, E).$$

As $T_{i+1} - T_i$ is suitably small and the Lagrangian is positive definite in the speed, the boundary condition $\gamma(T_j) = x_j$ for $j = i, i+1$ uniquely determines the speed $v_j = \dot{\gamma}(T_j)$ for $j = i, i+1$. Indeed, the function F_i generates an area-preserving twist map from the time-T_i-section to the time-T_{i+1}-section Φ_i: $(x_i, y_i) \to (x_{i+1}, y_{i+1})$

$$y_{i+1} = \partial_{x_{i+1}} F_i(x_i, x_{i+1}), \qquad y_i = -\partial_{x_i} F_i(x_i, x_{i+1}).$$

where $y_i = \partial_{\dot{x}} L(x_i, v_i, T_i)$. As the Lagrangian is positive definite in $\dot{x}$, it implies that the initial condition (x_i, v_i) smoothly depends on the boundary condition (x_i, x_{i+1}) in this case. Because of the smooth dependence of solution of ordinary differential equation on initial condition, the function is smooth. Obviously, each minimal point of $\mathbf{F}(\cdot, E)$ uniquely determines a c-minimal measure with $c \in \alpha^{-1}(E) \cap \mathbb{C}_g$, supported on a periodic orbit $(\gamma_E, \dot\gamma_E)$ with $[\gamma_E] = g$. Let $x_i = \gamma_E(T_i)$, it satisfies the equation

$$\frac{\partial F_i}{\partial x'}(x_{i-1}, x_i, E) + \frac{\partial F_{i+1}}{\partial x}(x_i, x_{i+1}, E) = 0.$$

Its linearized variational equation turns out to be $\mathbf{J}\delta\vec{x} = 0$ where

$$\mathbf{J} = \begin{bmatrix} A_0 & B_0 & 0 & \cdots & B_{m-1} \\ B_0 & A_1 & B_1 & \cdots & 0 \\ 0 & B_1 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & B_{m-2} \\ B_{m-1} & 0 & 0 & B_{m-2} & A_{m-1} \end{bmatrix},$$

$$A_i = \frac{\partial^2 F_{i-1}}{\partial x'^2}(x_{i-1}, x_i) + \frac{\partial^2 F_i}{\partial x^2}(x_i, x_{i+1}), \quad B_i = \frac{\partial^2 F_i}{\partial x \partial x'}(x_i, x_{i+1})$$

and $x_{-1} = x_{m-1}$.

Let $\vec{x}$ be a minimal configuration of the function $\mathbf{F}(\cdot, E)$, then the Jacobi matrix of $\mathbf{F}$ at the configuration $\vec{x}$ is non-negative. As the generating function $F_i(x, x', E)$ determines an area-preserving and twist map Φ_i, we have $B_i < 0$. Consequently, by using a theorem in [vM], we find that the smallest eigenvalue is simple. Let λ_i denote the i-th eigenvalue of the matrix, at the minimal configuration one has

$$0 \le \lambda_0 < \lambda_1 \le \lambda_2 < \cdots \le \lambda_{m-1}.$$

Let $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,m-1})$ be the eigenvector for λ_i. By choosing $\xi_{0,0} = 1$ we have $\xi_{0,i} > 0$ for $1 \le i < m$ (see Lemma 3.4 in [An]). At the minimal configuration, we find the following matrix which is positive definite:

$$\mathbf{J}_{m-1} = \begin{bmatrix} A_1 & B_1 & \cdots & 0 \\ B_1 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & B_{m-2} \\ 0 & 0 & B_{m-2} & A_{m-1} \end{bmatrix}.$$

If not, there will be a vector $\hat{v} = (v_1, \ldots, v_{m-1}) \in \mathbb{R}^{m-1} \setminus \{0\}$ such that $\hat{v}^t \mathbf{J}_{m-1} \hat{v} = 0$. It follows that $v^t \mathbf{J} v = 0$ if we set $v = (0, \hat{v}) \in \mathbb{R}^m$. As the matrix $\mathbf{J}$ is non-negative, it implies that $v = \mu\xi_0$, but it contradicts the fact that all entries of ξ_0 have the same sign, either positive or negative.

In a suitably small neighborhood $\vec{U}$ of the minimal configuration, let us consider the equations

$$\frac{\partial \mathbf{F}}{\partial x_i}(x_0, x_1, \dots, x_{m-1}, E) = 0, \qquad \forall\, i = 1, 2, \dots, m-1. \tag{2.1}$$

Since the matrix $\{\frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j}\}_{i,j=1,\dots,m-1} = \mathbf{J}_{m-1}$ is positive definite at the minimal point, by the implicit function theorem, the equation has a unique solution $x_i = X_i(x_0, E)$ for $x_0 \in U_0 = \pi\vec{U}$ which is C^{r-1}-smooth because the Hamiltonian equation is C^{r-1}. Let γ: $[0, 2\pi] \to \mathbb{R}$ be a minimizer of $F(x_0)$ with $\gamma(0) = \gamma(2\pi) = x_0$, we obtain a configuration $x_i = \gamma(2i\pi/m)$. Clearly, $\partial_{x_i}\mathbf{F} = 0$ holds at this configuration for each $i \geq 1$. It implies the uniqueness of the minimizer of $F(x_0, E)$ for $x_0 \in U_0$. The minimal point of F uniquely determines a minimal configuration of $\mathbf{F}$, so the function F is C^{r-1}-smooth in certain neighborhood of its minimal point. When $x_0 = x$ uniquely determines x_i, the initial speed $\dot{\gamma}_E(0, x)$ smoothly depends on x, namely, $\gamma_E(t, x)$ smoothly depends on x.

Indeed, the minimizer $\gamma_E(\tau, x)$ depends C^{r-1}-smoothly on (x, E) if it is restricted in S_i. Along the curve $\gamma_E(\tau, x)|_{\tau\in[0,2\pi]}$, no point is in the "cut locus" of $(0, x)$, then, there is a small neighborhood of (x, E) so that for each (x', E') in this neighborhood, the minimizer $\gamma_{E'}(\cdot, x')$ is also uniquely determined. As $\bar{L}$ depends C^{r-1}-smoothly on E, F depends C^{r-1}-smoothly on (x, E) when it is restricted S_i. □

3. Non-degeneracy of minimizers

In a neighborhood of the minimal point, let us study what change the function of action undergoes when the Lagrangian is under a perturbation of potential $L \to L + \epsilon P$, where P: $\mathbb{T}^2 \to \mathbb{R}$ is a potential, $\epsilon > 0$ is a small number. Let $\bar{L}'$ denote the reduced Lagrangian of $L + \epsilon P$ and let $G = -(\partial_{y_2} H)^{-1}$, one has

$$\bar{L}' = \bar{L} + \epsilon G P + O(\epsilon^2).$$

Since the equation $\dot{x} = \partial_y \bar{H}(x, y, \tau)$ well defines a function $y = y(\dot{x}, x, \tau)$, substituting it in the formula $-\partial_{y_2} H^{-1}(x, y, \bar{H}(x, y, \tau), -\tau)$, we obtain the function $G(\dot{x}, x, \tau)$. We denote the minimizer of $F(x, E)$ by $\gamma(t, x, E)$ as the point x uniquely determines the minimizer when it is restricted in a neighborhood U_0. Let $\gamma'(t, x, E)$ and $F'(x, E)$ be the quantities defined for $\bar{L}'$ as the quantities $\gamma(t, x, E)$ and $F(x, E)$ defined for $\bar{L}$. By the definition of minimizer, we have

$$\begin{aligned}F'(x,E)-F(x,E)&=\int_0^{2\pi}\bar L'(d\gamma'(\tau))d\tau-\int_0^{2\pi}\bar L(d\gamma(\tau))d\tau\\&\geq\int_0^{2\pi}\bar L'(d\gamma'(\tau))d\tau-\int_0^{2\pi}\bar L(d\gamma'(\tau))d\tau\\&=\epsilon\int_0^{2\pi}G(d\gamma'(\tau),\tau)P(\gamma'(\tau))d\tau+O(\epsilon^2),\end{aligned}$$

$$\begin{aligned}F'(x,E)-F(x,E)&=\int_0^{2\pi}\bar L'(d\gamma'(\tau))d\tau-\int_0^{2\pi}\bar L(d\gamma(\tau))d\tau\\&\leq\int_0^{2\pi}\bar L'(d\gamma(\tau))d\tau-\int_0^{2\pi}\bar L(d\gamma(\tau))d\tau\\&=\epsilon\int_0^{2\pi}G(d\gamma(\tau),\tau)P(\gamma(\tau))d\tau+o(\epsilon).\end{aligned}$$

Since the distance between these two curves γ and γ' approaches zero as $\epsilon\to 0$, we finally obtain

$$F'(x,E)=F(x,E)+\mathscr{K}_E\epsilon P(x)+\mathscr{R}_E\epsilon P(x), \tag{3.1}$$

where $\mathscr{R}_E\epsilon P=o(\epsilon)$ and

$$\mathscr{K}_E\epsilon P(x)=\epsilon\int_0^{2\pi}G(d\gamma(\tau,x,E),\tau)P(\gamma(\tau,x,E))d\tau.$$

The family of smooth curves $\{\gamma(\cdot,x,E)\}$ defines an operator $P\to\mathscr{K}_EP$, which maps functions defined on $\mathbb{T}^2$ into the function space defined on $\mathbb{T}$. Obviously, both $\mathscr{K}_E\epsilon P$ and $\mathscr{R}_E\epsilon P$ are smooth in $x\in U_0$ and in E.

Unless the point x is a minimizer of $F(\cdot,E)$, the curve $\gamma(\cdot,x,E)$ may have corner at $\tau=0\mod 2\pi$. However, the size of the corner is bounded as follows:

Lemma 3.1. *There exist constants $\varepsilon,\theta>0$ such that if $F(x,E)-\min F(\cdot,E)<\varepsilon$ and if $\gamma:[0,2\pi]\to\mathbb{R}$ is a minimizer of $F(x,E)$, then*

$$|\dot\gamma(0)-\dot\gamma(2\pi)|<\theta\sqrt{F(x,E)-\min F(\cdot,E)}.$$

Proof. Let us consider the derivative of $F(\cdot,E)$. As the Lagrangian is positive definite, some positive constants $m_L>0$ exist such that

$$\frac{\partial^2\bar L}{\partial\dot x^2}\geq m_L,\qquad\forall\ (x,\dot x)\in T\mathbb{T}^2.$$

Since $\gamma(0,x,E)=\gamma(2\pi,x,E)=x$, one has $\partial_x\gamma(0)=\partial_x\gamma(2\pi)=1$ and

$$\begin{aligned}\left|\frac{\partial F}{\partial x}\right| &= \left|\int_0^{2\pi}\Big(\frac{\partial \bar L}{\partial \dot x}(d\gamma(\tau),\tau)\frac{\partial\dot\gamma}{\partial x}+\frac{\partial \bar L}{\partial x}(d\gamma(\tau),\tau)\frac{\partial\gamma}{\partial x}\Big)d\tau\right|\\ &= \left|\frac{\partial \bar L}{\partial \dot x}(\dot\gamma(0),\gamma(0),0)-\frac{\partial \bar L}{\partial \dot x}(\dot\gamma(2\pi),\gamma(2\pi),2\pi)\right|\\ &\ge m_L|\dot\gamma(0)-\dot\gamma(2\pi)|,\end{aligned}$$

where the second equality follows from that γ solves the Euler-Lagrange equation. If $\frac{\partial F}{\partial x}>0$ and if the lemma does not hold, by choosing $x'-x=-\sqrt{\Delta}$ ($\Delta=F(x,E)-\min F(\cdot,E)$) we obtain from the Taylor series up to second order that

$$\begin{aligned}F(x',E)-\min F(\cdot,E) &= F(x',E)-F(x,E)+F(x,E)-\min F(\cdot,E)\\ &\le -\partial_x F(x,E)\sqrt{\Delta}+\frac{M}{2}\Delta+\Delta<0\end{aligned}$$

if $\theta>\frac{1}{m_L}(1+\frac{M}{2})$, where $M=\max\partial_x^2F$. But it is absurd. The case $\frac{\partial F}{\partial x}<0$ can be proved by choosing $x'-x=\sqrt{\Delta}$. This completes the proof. □

Let $U_0=(x^*-\delta_{x^*},x^*+\delta_{x^*})$ where $F(\cdot,E_0)$ is smooth and x^* is a minimal point. As it was shown above, $\gamma(\frac{2i\pi}{m},x,E_0)$ smoothly depends on $x\in U_0$. So, we obtain a smooth foliation of curves in a neighborhood of the curve $\gamma(\cdot,x^*,E_0)$. The corner at $\gamma(0,x,E_0)$, $|\dot\gamma(0)-\dot\gamma(2\pi)|$ approaches to zero as $F(x,E_0)\downarrow\min F(\cdot,E_0)$. For each $x\in U_0$, we construct a curve γ_x that smoothly connects the point $\gamma(2\pi-\delta,x,E_0)$ to the point $\gamma(\delta,x,E_0)$ with $\gamma_x(0)=x$, where $\delta>0$ is suitably small. Replacing the segment $\gamma(\cdot,x,E_0)|_{[0,\delta]\cup[2\pi-\delta,2\pi]}$ by this curve, we get a smooth curve γ_x such that $\gamma_x(t)=\gamma(t,x,E_0)|_{[\delta,2\pi-\delta]}$ and $\gamma_x(0)=x$. Indeed, since the curve $\gamma(t,x,E_0)$ is C^4-smooth in x and $\gamma(t,x^*,E_0)$ is also C^4-smooth in t, for small number ε some $\mu_{x^*}>0$ exists such that the quantities

$$\left|\frac{d^k\gamma}{dt^k}(t,x,E_0)-\frac{d^k\gamma}{dt^k}(t,x^*,E_0))\right|<\varepsilon,\qquad\forall\ |x-x^*|<\mu_{x^*}\ \ k=0,1,2,3,4.$$

Let the curve ζ_x: $[-\delta,\delta]\to\mathbb{R}$ be an interpolation polynomial of degree ten so that

$$\frac{d^k\zeta_x}{dt^k}(t)=\frac{d^k\gamma}{dt^k}(t,x,E_0)-\frac{d^k\gamma}{dt^k}(t,x^*,E_0)\qquad\forall\ t=\pm\delta,$$

and $\zeta_x(0)=\gamma(0,x,E_0)-\gamma(0,x^*,E_0)$, then the coefficients of the polynomial are smooth in x. Let $\gamma_x(t)=\gamma(t,x^*,E_0)+\zeta_x(t)$ for $|t|\le\delta$ and $\gamma_x(t)=$

$\gamma(t,x,E_0)$ for $t\in[\delta,2\pi-\delta]$, we see that the foliation of the curves γ_x is smooth in x and as a function of t, $\gamma_x-\gamma(\cdot,x,E_0)$ is small in C^4-topology.

The curve $\gamma_x(\tau)$ uniquely determines the speed $v_x(\tau)=\dot\gamma_x(\tau)$. The functions G, γ_x and v_x are at least C^4-smooth, can be approximated in C^4-topology by C^r-functions denoted by G^s, $\gamma^s_x(\tau)$ and $v^s_x(\tau)$ respectively. Since passing through the point (τ,x) there is a unique $x'\in\mathbb{T}$ such that $\gamma^s_{x'}(\tau)=x$, we obtain a function C^r-function $x'=x'(x,\tau)$ such that $\gamma^s_{x'(\tau,x)}(\tau)=x$.

With these preliminary works, we are ready to introduce perturbations. Given a C^r-function $\bar P\colon\mathbb{T}\to\mathbb{R}$ we get a C^r-function $P=\mathscr{T}_{E_0}\bar P\colon\mathbb{T}^2\to\mathbb{R}$ defined by

$$P(x,\tau)=\mathscr{T}_{E_0}\bar P(x')=\frac{\bar P(x'(\tau,x))}{G^s(v^s(x,\tau),x,\tau)}. \tag{3.2}$$

By the definition, we have

$$\begin{aligned}&\mathscr{K}_E\mathscr{T}_{E_0}\bar P(x)\\&=\int_0^{2\pi}\frac{G(d\gamma(\tau,x,E),\tau)}{G^s(v^s(\gamma(\tau,x,E),\tau),\gamma(\tau,x,E),\tau)}\bar P(x+\Delta\gamma^s(\tau,x,E))d\tau,\end{aligned} \tag{3.3}$$

where $\Delta\gamma^s(\tau,x,E)=x'-x$. We introduce a set of perturbations with four parameters:

$$\bar{\mathfrak{P}}=\left\{\sum_{\ell=1}^{2}(A_\ell\cos\ell x+B_\ell\sin\ell x):\ (A_1,B_1,A_2,B_2)\in\mathbb{I}^4\right\},$$

where $\mathbb{I}^4=[1,2]\times[1,2]\times[1,2]\times[1,2]$. By applying the formula (3.3) to the function $\cos\ell x$ and $\sin\ell x$ we find that

$$\begin{aligned}\mathscr{K}_E\mathscr{T}_{E_0}\cos\ell x&=u_\ell(x,E)\cos\ell x-v_\ell(x,E)\sin\ell x,\\ \mathscr{K}_E\mathscr{T}_{E_0}\sin\ell x&=u_\ell(x,E)\sin\ell x+v_\ell(x,E)\cos\ell x,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}u_\ell(x,E)&=\int_0^{2\pi}\frac{G(d\gamma(\tau,x,E),\tau)}{G^s(v^s(\gamma(\tau,x,E),\tau),\gamma(\tau,x,E),\tau)}\cos\ell\Delta\gamma^s(\tau,x,E)d\tau,\\ v_\ell(x,E)&=\int_0^{2\pi}\frac{G(d\gamma(\tau,x,E),\tau)}{G^s(v^s(\gamma(\tau,x,E),\tau),\gamma(\tau,x,E),\tau)}\sin\ell\Delta\gamma^s(\tau,x,E)d\tau.\end{aligned}$$

Let us study the dependence of the terms $u_\ell(x,E)$ and $v_\ell(x,E)$ on the point x. We claim that there exist small constants $\theta_1>0$, $\delta_{E_0}>0$ and $\mu_{x^*}>0$

such that for each $E \in (E_0 - \delta_{E_0}, E_0 + \delta_{E_0})$, each $x \in (x^* - \mu_{x^*}, x^* + \mu_{x^*})$, $j = 0, 1, 2, 3$ and $\ell = 1, 2$, we have

$$\begin{aligned} &|u_\ell(x,E)| \geq 1 - \theta_1\delta, \qquad |v_\ell(x,E)| \leq \theta_1\delta, \\ &\max_{j=1,2,3} \left\{ \left| \frac{\partial^j u_\ell}{\partial x^j}(x,E) \right|, \left| \frac{\partial^j v_\ell}{\partial x^j}(x,E) \right| \right\} \leq \theta_1\delta. \end{aligned} \tag{3.5}$$

Indeed, for the curve γ_x one can also define $\Delta\gamma(\tau, x, E)$ in the way that $\Delta\gamma^s(\tau, x, E)$ is defined, i.e. $\Delta\gamma(\tau, x, E) = x' - x$, where $x' \in \mathbb{T}$ is chosen so that $\gamma_{x'}(\tau) = x$. By the definition, for $\tau \in \mathbb{T} \backslash (-\delta, \delta)$ and for $x \in (x^* - \mu_{x^*}, x^* + \mu_{x^*})$ one has

$$\Delta\gamma(\tau, x, E_0) = 0 \quad \text{and} \quad \frac{G(d\gamma(\tau, x, E_0), \tau)}{G(v(\gamma(\tau, x, E_0), \tau), \gamma(\tau, x, E_0), \tau)} = 1$$

and $\partial_x^j \Delta\gamma(\tau, x, E_0)$ is finite for $\tau \in (-\delta, \delta)$ and for $j = 0, 1, 2, 3, 4$. Integrating the term over the interval with length equal to 2δ, we find that some small $\theta_1 > 0$ exists such that the formulae in (3.5) hold for $E = E_0$ with θ_1 being replaced by $\theta_1/4$ if G^s and v^s in the formula (3.3) are replaced by G and v respectively. Note that $\Delta\gamma^s(\tau, x, E)$ is in fact a C^r-approximation of $\Delta\gamma(\tau, x, E)$, both v and G are approximated by v^s and G^s in C^3-topology, by choosing $\delta_1 > 0$ suitably small, all formulae in (3.5) hold for $E = E_0$ with θ_1 being replaced by $\theta_1/2$.

For other energy E, let us recall the solution $x_i = X_i(x_0, E)$ of Equation (2.1) is smooth in E. As the map Φ_i: $(x_i, y_i) \to (x_{i+1}, y_{i+1})$ is area-preserving and twist, it uniquely determines the initial speed $v_0 = v_0(x_0, E)$, i.e. the initial speed smoothly depends in the initial position as well as in the parameter E. Since solution of ODE smoothly depends on its initial conditions, the minimal curve $\gamma(\cdot, x, E)$ of $F(x, E)$ smoothly depends on the parameters x and E. Thus, the formulae in (3.5) hold if the numbers $\delta_{E_0} > 0$ and $\mu_{x^*} > 0$ are suitably small.

Theorem 3.1. *There exists an open-dense set $\mathfrak{O} \subset C^r(M, \mathbb{R})$ with $r \geq 5$ such that for each $P \in \mathfrak{O}$ and each $E \in [E_0 - \delta_{E_0}, E_0 + \delta_{E_0}]$, all minimizers of $F(\cdot, E)$, determined by $L + P$, are non-degenerate.*

Proof. As one assumes $r \geq 5$, the function of action $F(\cdot, E)$ is at least C^4-smooth in x. To show the non-degeneracy of the global minimum of $F(\cdot, E)$ located at the point x, we only need to verify that

$$F(x + \Delta x, E) - F(x, E) \geq M|\Delta x|^4 \tag{3.6}$$

holds for small $|\Delta x|$, where $M = 12^{-1}\max \partial_x^4 F$. If it was degenerate, then one would have $F(x+\Delta x, E) - F(x,E) \le \frac{1}{2}M|\Delta x|^4$. Assume I is an interval, we define $\mathrm{Osc}_I F = \max_{x,x'} |F(x) - F(x')|$. So, to show the non-degeneracy, it is sufficient to verify that

$$\mathrm{Osc}_I F(\cdot, E) \ge M|I|^4$$

if the minimal point $x \in I$, where the length of the interval, $|I|$ is small.

The openness of $\mathfrak{O}$ is obvious. To show the density, we are concerned only about the configurations where F takes the value close to the minimum and consider small perturbations from the following set where the parameters (A_1, B_1, A_2, B_2) range over the cube $\mathbb{I}^4 = [1,2]\times[1,2]\times[1,2]\times[1,2]$

$$\mathfrak{V}_E = \left\{ (\mathscr{K}_E + \mathscr{R}_E)\mathscr{T}_{E_0} \sum_{\ell=1}^{2} \epsilon(A_\ell \cos \ell x + B_\ell \sin \ell x) : (A_1, B_1, A_2, B_2) \in \mathbb{I}^4 \right\}$$

in which each element is a function of (x, E), see the formulae (3.4). Recall that both operators $\mathscr{K}_E$ and $\mathscr{T}_{E_0}$ are linear and $\|\mathscr{R}_E(\epsilon P)\| = o(\epsilon)$, see the formula (3.1) and the formula (3.2).

We choose sufficiently large integer K such that $\epsilon = \sqrt[4]{\pi/K}$ can be arbitrarily small. Let $x_k = \frac{2k\pi}{K}$, $I_k = [x_k - d, x_k + d]$ and $d = \pi/K$, then $\bigcup_{k=0}^{K-1} I_k = \mathbb{T}$. Restricted on each interval I_k, each C^4-function $V \in \mathfrak{V}_E$ is approximated by the Taylor series (module constant)

$$V_k(x) = \epsilon\Big(a_k(x-x_k) + b_k(x-x_k)^2 + c_k(x-x_k)^3 + O(|x-x_k|^4)\Big).$$

Given two points (a_k, b_k, c_k) and (a'_k, b'_k, c'_k), we have two functions V_k and V'_k in the form of Taylor series. Let $\Delta V = V'_k - V_k$, $\Delta a = a'_k - a_k$, $\Delta b = b'_k - b_k$ and $\Delta c = c'_k - c_k$, we have $\Delta V(x_k) = 0$ and

$$\begin{aligned}
&\Delta V(x_k + d) + \Delta V(x_k - d) = 2\epsilon \Delta b d^2 + O(\epsilon d^4),\\
&\Delta V(x_k + d) - \Delta V(x_k - d) = 2\epsilon(\Delta a + \Delta c d^2)d + O(\epsilon d^4),\\
&\Delta V\Big(x_k \pm \frac{1}{2}d\Big) = \epsilon\Big(\pm\frac{1}{2}\Delta a + \frac{1}{4}\Delta b d \pm \frac{1}{8}\Delta c d^2\Big)d + O(\epsilon d^4).
\end{aligned}$$

It follows that

$$\mathrm{Osc}_{I_k}(V'_k - V_k) \ge \frac{\epsilon}{4}\max\Big\{|\Delta a|, |\Delta b|d, |\Delta c|d^2\Big\}d. \tag{3.7}$$

We construct a grid for the parameters (a_k, b_k, c_k) by splitting the domain for them equally into a family of cuboids and setting the size length by

$$\Delta a_k = 12Md^{\frac{11}{4}}, \quad \Delta b_k = 12Md^{\frac{7}{4}}, \quad \Delta c_k = 12Md^{\frac{3}{4}}.$$

These cuboids are denoted by $\mathbf{c}_{kj}$ with $j \in \mathbb{J}_k = \{1, 2, \dots\}$, the cardinality of the set of the subscripts is up to the order

$$\#(\mathbb{J}_k) = N[d^{-\frac{21}{4}}],$$

where the integer $0 < N \in \mathbb{N}$ is independent of d. If $\mathrm{Osc}_{I_k} F(\cdot, E) \leq Md^4$, we obtain from the formula (3.7) that

$$\mathrm{Osc}_{I_k}(F(x, E) + V(x)) \geq 2Md^4$$

if $V(x) = \epsilon(a(x - x_k) + b(x - x_k)^2 + c(x - x_k)^3 + O(|x - x_k|^4))$ with

$$\max\left\{|a|d^{-\frac{11}{4}}, |b|d^{-\frac{7}{4}}, |c|d^{-\frac{3}{4}}\right\} \geq 12M.$$

The coefficients (a_k, b_k, c_k) depend on the parameters (A_1, B_1, A_2, B_2), the energy E and the position x_k. The gird for (a_k, b_k, c_k) induces a grid for the parameters (A_1, B_1, A_2, B_2), determined by the equation

$$\begin{bmatrix} a_k \\ b_k \\ c_k \end{bmatrix} = (\mathbf{C_1}\mathbf{U} + \mathbf{C_2}) \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} \Big(1 + T_{\epsilon,E,x_k}(A_1, B_1, A_2, B_2)\Big) \tag{3.8}$$

where the map $T_{\epsilon,E,x_k} \colon \mathbb{R}^4 \to \mathbb{R}^3$ is as small as of order $O(\epsilon)$,

$$\mathbf{C_1} = \begin{bmatrix} -\sin x_k & \cos x_k & -2\sin 2x_k & 2\cos 2x_k \\ -\cos x_k & -\sin x_k & -4\cos 2x_k & -4\sin 2x_k \\ \sin x_k & -\cos x_k & 8\sin 2x_k & -8\cos 2x_k \end{bmatrix},$$

$$\mathbf{U} = \mathrm{diag}\left\{ \begin{bmatrix} u_1(x_k) & v_1(x_k) \\ -v_1(x_k) & u_1(x_k) \end{bmatrix}, \begin{bmatrix} u_2(x_k) & v_2(x_k) \\ -v_2(x_k) & u_2(x_k) \end{bmatrix} \right\},$$

each entry of $\mathbf{C_2}$ is a linear function of $\partial_x^j u_\ell \cos \ell x_k$, $\partial_x^j v_\ell \cos \ell x_k$, $\partial_x^j u_\ell \sin \ell x_k$ and $\partial_x^j v_\ell \sin \ell x_k$ with $j = 1, 2, 3$, $\ell = 1, 2$. Both matrices $\mathbf{U}$ and $\mathbf{C}_2$ depend on the energy E, $\mathbf{U}$ is close to the identity matrix. Let $\mathbf{M_1}$ be the matrix composed by the first three columns of $\mathbf{C_1}\mathbf{U} + \mathbf{C_2}$, $\mathbf{M_2}$ be the matrix composed by the first, the second and the fourth column of $\mathbf{C_1}\mathbf{U} + \mathbf{C_2}$. As we are concerned about those positions where F takes value close to the minimum and about the energy E close to E_0, in virtue of (3.5) we obtain

$$\det(\mathbf{M}_1)(x_k) = 6\sin 2x_k(1 - O(\theta_1\delta)),$$
$$\det(\mathbf{M}_2)(x_k) = -6\cos 2x_k(1 - O(\theta_1\delta)).$$

Because $\inf_{x_k}\{|\det\mathbf{M}_1(x_k)| + |\det\mathbf{M}_2(x_k)|\} = 3\sqrt{2}(1 - O(\theta_1\delta))$, the grid for (a_k, b_k, c_k) induces a grid for (A_1, B_1, A_2, B_2) which contains as many as $N_1[d^{-\frac{21}{4}}]$ 4-dimensional strips ($N_1 > 0$ is independent of d). The induced partition for (A_1, B_1, A_2, B_2) depends on the energy.

Given an energy $E \in [E_0 - \delta_{E_0}, E_0 + \delta_{E_0}]$, if there exist Taylor coefficients (a_k, b_k, c_k) which determines a perturbation V such that

$$\mathrm{Osc}_{I_k}(F(\cdot, E) + V) \leq Md^4$$

then for (a'_k, b'_k, c'_k) which determines a perturbation $\Delta V'$ one obtains from the formula (3.7) that

$$\mathrm{Osc}_{I_k}(F(\cdot, E) + V') \geq 2Md^4 \tag{3.9}$$

provided

$$\max\left\{\frac{|a_k - a'_k|}{12Md^{\frac{11}{4}}}, \frac{|b_k - b'_k|}{12Md^{\frac{7}{4}}}, \frac{|c_k - c'_k|}{12Md^{\frac{3}{4}}}\right\} \geq 1.$$

Under the map defined by the formula (3.8), the inverse image of a cuboid c_k with the size $24Md^{\frac{11}{4}} \times 24Md^{\frac{7}{4}} \times 24Md^{\frac{3}{4}}$ is a strip in the parameter space of (A_1, B_1, A_2, B_2) denoted by $\mathsf{S}_k(E)$, with the Lebesgue measure as small as $N_1^{-1}d^{\frac{21}{4}}$. If the cuboid c_k is centered at (a_k, b_k, c_k), then for $(a'_k, b'_k, c'_k) \notin \mathsf{c}_k$ the inequality (3.9) holds.

Splitting the interval $[E_0 - \delta_{E_0}, E_0 + \delta_{E_0}]$ equally into small sub-intervals $I_{E,j}$ with the size $|I_{E,j}| = M_1^{-1}d^4$, we obtain as many as $[M_1 d^{-4}]$ small intervals. As the function F is Lipschitz in E, suitably large positive number M_1 can be chosen so that

$$\max_{x\in I_k}|F(x, E) - F(x, E')| < \frac{1}{2}Md^4, \qquad \forall\ E, E' \in I_{E,j}.$$

Therefore, for $V \in \mathfrak{V}_E$ with $(\Delta A_1, \Delta B_1, \Delta A_2, \Delta B_2) \notin \mathsf{S}_k(E)$, one has

$$\mathrm{Osc}_{I_k}(F(\cdot, E) + \Delta V') \geq Md^4. \tag{3.10}$$

Pick up one energy E_j in each small interval $I_{E,j}$, there are $[M_1 d^{-4}]$ strips $\mathsf{S}_k(E_j)$. Finally, by considering all small intervals I_k with $k = 1, \ldots,$

K ($K^{-1} = d$), we find

$$\text{meas}\left(\bigcup_{k,j} \mathbf{S}_k(E_j)\right) \le M_1 N_1^{-1} \sqrt[4]{d}.$$

Let $\mathbf{S}^c = \mathbb{I}^4 \setminus \cup_{j,k} \mathbf{S}_k(E_j)$, we obtain the Lebesgue measure estimate

$$\text{meas}(\mathbf{S}^c) \ge 1 - M_1 N_1^{-1} \sqrt[4]{d} \to 1, \qquad \text{as} \ \ d \to 0.$$

Obviously, for any $(A_1, B_1, A_2, B_2) \in \mathbf{S}^c$, any $E \in [E_0 - \delta_{E_0}, E_0 + \delta_{E_0}]$ and any $k = 1, 2, \ldots, K$ the formula (3.10) holds. This proves the open-dense property that all minimal points of $F(\cdot, E)$ are simultaneous non-degenerate when the energy ranges over the interval $[E_0 - \delta_{E_0}, E_0 + \delta_{E_0}]$. □

The method used here can be generalized for higher dimensional case [CZ].

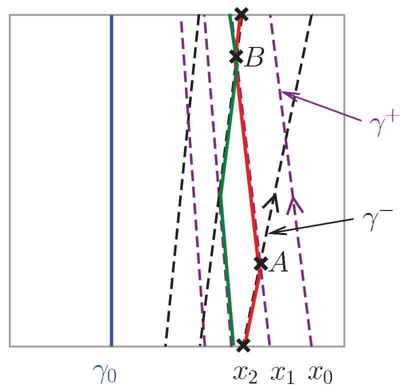
4. Hyperbolicity

Let x^* be a minimal point of the function $F(\cdot, E)$ and let the curve $\gamma(\cdot, x^*, E)$: $\mathbb{T} \to \mathbb{R}$ be the minimizer of $F(x^*, E)$ which is smooth and determines a periodic orbit $(\tau, \gamma(\tau), \frac{d}{d\tau}\gamma(\tau))$ of the Lagrange flow ϕ_L^τ. Back to the autonomous system, it determines a periodic orbit $(\gamma_1(t), \dot{\gamma}_1(t), \gamma_2(t), \dot{\gamma}_2(t))$ of the Lagrange flow ϕ_L^t, where $\gamma_2(t) = -\tau$, $\gamma_1(t) = \gamma(\gamma_2(t))$.

Theorem 4.1. *If x^* is a non-degenerate minimal point of the function $F(\cdot, E)$, then the periodic orbit $\gamma(\cdot, x^*, E)$ is hyperbolic.*

Proof. If a periodic orbit is hyperbolic, it has its stable and unstable manifold in the phase space. Consequently, any orbit staying on the stable (unstable) manifold approaches to the periodic orbit exponentially fast as the time approaches to positive (negative) infinity.

In a small neighborhood of the minimal periodic curve γ, each point x on the section $\{\tau = 0\}$ determines at least one forward (backward) semi-static curve $\gamma_x^+ \colon \mathbb{R}_+ \to \mathbb{T}$ ($\gamma_x^- \colon \mathbb{R}_- \to \mathbb{T}$) such that $\gamma_x^\pm(0) = x$. These curves determine forward (backward) semi-static orbits $d\gamma_x^\pm$ of which the ω-set (α-set) is the periodic orbit $d\gamma$. In the configuration space $(x, \tau) \in \mathbb{T}^2$, these two curves intersect with the section $\{\tau = 0\}$ infinitely many times at the points $\gamma_x^+(2k\pi)$ and $\gamma_x^-(-2k\pi)$. These points are denoted by x_i, they are well ordered $\cdots \prec x_{i+1} \prec x_i \cdots \prec x_0$. It is possible that $\gamma_x^+(2k\pi) = \gamma_x^-(-2k'\pi)$. In this case, we count the point twice. For each point x_i, there is a curve joining $(x_i, 0)$ to $(x_i, 2\pi)$ which is composed by some segments of γ_x^+ as well as of γ_x^-. For instance, in the following figure, by starting from the point $(x_2, 0)$ and following a segment of γ_x^- to the point A, then following a segment of γ_x^+ to the point B and finally following a segment of γ_x^- to the point $(x_2, 2\pi)$, we obtain a circle, see the figure. Clearly, the Lagrange action along this circle is not smaller than the quantity $F(x_2, E)$.



Let us consider the whole sequence $\{x_i\}$, we obtain infinitely many circles in that way. Therefore, the sum of the quantities $F(x_i, E)|_{i=0}^{\infty}$ is obviously not bigger than the total action along all of these circles

$$\sum_{i=0}^{\infty} F(x_i, E) \le \lim_{k\to\infty} \left\{ \int_0^{2k\pi} L(d\gamma^+(\tau), \tau) d\tau + \int_{-2k\pi}^{0} L(d\gamma^-(\tau), \tau) d\tau \right\}. \tag{4.1}$$

The right hand side is nothing else but the barrier function valued at x_0.

As the periodic orbit supports the minimal measure, both $\gamma_x^+(2k\pi)$ and $\gamma_x^-(-2k\pi)$ approach the point x^* as $k \to \infty$, where x^* is the intersection point of the periodic curve with the section $\{\tau = 0\}$. If the periodic orbit is not hyperbolic, the sequence of $\{x_i\}$ approach x slower than exponentially, i.e., given any small $\lambda > 0$ there exists $\delta > 0$ such that

$$|\gamma_x^+(2(k+1)\pi) - x^*)| \ge (1-\lambda)|\gamma_x^+(2k\pi) - x^*)|,$$
$$|\gamma_x^-(-2(k+1)\pi) - x^*)| \ge (1-\lambda)|\gamma_x^-(-2k\pi) - x^*)|,$$

if $|\gamma_x^\pm(0) - x^*| \le \delta$. It follows that $|x_{i+1} - x^*| \ge (1-\lambda)|x_i - x^*|$. As the periodic curve is assumed non-degenerate minimizer, some $\lambda_0 > 0$ exists such that

$$(4.2)\qquad \sum_{i=0}^{\infty}(F(x_i,E)-F(x^*,E))\geq \lambda_0\sum_{i=0}^{\infty}(x_i-x^*)^2\geq \lambda_0\frac{(x_0-x^*)^2}{1-(1-\lambda)^2}.$$

By subtracting $\min F$ from the Lagrangian L we obtain that

$$\text{right-hand-side of (4.1)} = u^-(x,0)-u^+(x,0)$$

where $u^{\pm}$ represents the backward (forward) weak-KAM solution. Since u^- is semi-concave and u^+ is semi-convex, $u^- - u^+$ is semi-concave. Since $(x^*,0)$ is a minimal point where $u^-(x^*,0)-u^+(x^*,0)=0$, there exists some number $C_L>0$ such that (cf. [Fa])

$$u^-(x_0,0)-u^+(x_0,0)\leq C_L(x_0-x^*)^2.$$

Comparing this with the inequality (4.2), we obtain from (4.1) a contradiction

$$\lambda_0\frac{(x_0-x^*)^2}{1-(1-\lambda)^2}\leq C_L(x_0-x^*)^2$$

if $\lambda>0$ is suitably small. This proves the hyperbolicity of the periodic orbit. □

We are now ready to prove the main result.

Proof of Theorem 2.1. According to Theorem 3.1 and 4.1, for each $E_i\in[E_a,E_d]$, a neighborhood $[E_i-\delta_{E_i},E_i+\delta_{E_i}]$ of E_i and an open-dense set $\mathfrak{O}(E_i)\subset C^r(\mathbb{T}^2,\mathbb{R})$ exist such that for each $P\in\mathfrak{O}(E_i)$ and each $E\in[E_i-\delta_{E_i},E_i+\delta_{E_i}]$ each minimal orbit of ϕ^t_{L+P} with homological class g is hyperbolic. As each δ_{E_i} is positive, there exists finitely many E_i such that $[E_a,E_d]\subset\cup_i[E_i-\delta_{E_i},E_i+\delta_{E_i}]$. We take $P\in\cap\mathfrak{O}(E_i)$, the hyperbolicity for $L+P$ holds simultaneously for all $E\in[E_a,E_d]$.

Once a minimal point is non-degenerate for certain E, by the theorem of implicit function it has natural continuation to a neighborhood of E. Namely, there exists a curve of minimal points passing through this point, it either reaches to the boundary of $[E_a,E_d]$, or extends to some point E' where the critical point is degenerate. Since each global minimal point is non-degenerate, the critical point becomes local minimum when it enters into certain neighborhood of E'. As each non-degenerate minimal point is isolated to other minimal points for the same energy E, there are finitely many such curves, denoted by Γ_i.

For a curve Γ_i: $I_i = (E_i, E_i') \to \mathbb{T}$, the definition domain I_i contains finitely many closed sub-intervals $I_{i,j}$ such that $F(\Gamma_i(E), E) = \min_x F(\cdot, E)$ for all $E \in I_{i,j}$. By definition, $I_{i,j} \cap I_{i,j'} = \varnothing$ for $j \neq j'$. Let $\Gamma_{i,j} = \Gamma_i|_{I_{i,j}}$, we have finitely many curves $\{\Gamma_{i,j}\}$ such that $F(\cdot, E)$ reaches global minimum at the point x if and only if $x = \Gamma_{i,j}(E)$ for certain subscript (i,j).

For each $E \in \partial I_{i,j}$, by the definition of $I_{i,j}$, some other subscript (i', j') exists such that $F(\cdot, E)$ reaches the global minimum at the points $\Gamma_{i,j}(E)$ and $\Gamma_{i',j'}(E)$. It is obviously an open-dense property that

$$\frac{dF(\Gamma_{i,j}(E), E)}{dE} \neq \frac{dF(\Gamma_{i',j'}(E), E)}{dE}.$$

Thus, it is also open-dense that $[E_a, E_d] = \cup I_{i,j}$ and $[E_a, E_d] \backslash \cup \, \mathrm{int} I_{i,j}$ contains finitely points. This completes the whole proof. □

Proof of Theorem 1.1. According to Theorem 2.1, for each class $g \in H_1(\mathbb{T}^2, \mathbb{Z})$ and each $[E_a, E_d] \subset (0, \infty)$, an open-dense set $\mathfrak{V}(g, E_a, E_d) \subset C^r(\mathbb{T}^2, \mathbb{R})$ exists such that for each $V \in \mathfrak{V}(g, E_a, E_d)$ the minimal periodic orbits of ϕ^t_{L+V} are hyperbolic if they stay in the Mather set $\tilde{\mathcal{M}}(E, g)$ with $E \in [E_a, E_d]$.

Let $E_a^i \downarrow 0$, $E_d^i \uparrow \infty$ be sequences of numbers. Because there are countably many homological classes in the group $H_1(\mathbb{T}^2, \mathbb{R})$, the set

$$\mathfrak{P} = \bigcap_{\substack{g \in H_1(\mathbb{T}^2, \mathbb{R}) \\ i \in \mathbb{N}}} \mathfrak{V}(g, E_a^i, E_d^i)$$

is obviously residual in $C^r(\mathbb{T}^2, \mathbb{R})$. Therefore, to complete the proof we only need to consider periodic orbits lying in the Mather set $\tilde{\mathcal{M}}(c)$ with $c \in \mathbb{F}_0 = \{c \in H^1(\mathbb{T}^2, \mathbb{R}) : \alpha(c) = \min \alpha\}$.

The flat $\mathbb{F}_0$ is either two dimensional disk or a line. If it is a disk, it is a typical phenomenon that the minimal measure is uniquely supported on a fixed point for each $c \in \mathrm{int}\mathbb{F}_0$ which is guaranteed by a result of Massart [Mas]. It is also typical that this fixed point is hyperbolic. Let us consider $c \in \partial \mathbb{F}_0$ such that $\tilde{\mathcal{M}}(c)$ contains periodic orbits with the homological class g. As it was studied in [Ch], certain $\lambda_0 > 0$ exists such that the channel $\mathbb{C}_g = \cup_{\lambda \geq \lambda_0 > 0} \mathscr{L}_\beta(\lambda g)$ is connected to the flat $\alpha^{-1}(\min \alpha)$. As the speed of the periodic orbit is not zero, one can see that the proof of Theorem 2.1 applies to $[\min \alpha, E_d]$. If the flat $\mathbb{F}_0$ is a line, it is obviously typical that, for each $c \in \mathbb{F}_0$, the c-minimal measure is supported on two periodic orbits with different rotation vector, each of them is hyperbolic. □

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