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Inflexibility, Weil-Petersson distance, and volumes of fibered 3-manifolds

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A recent preprint of S. Kojima and G. McShane [27] observes a beautiful explicit connection between Teichmüller translation distance and hyperbolic volume. It relies on a key estimate which we supply here: using geometric inflexibility of hyperbolic 3-manifolds, we show that for S a closed surface, and $\psi \in \text{Mod}(S)$ pseudo-Anosov, the double iteration $Q(\psi^{-n}(X), \psi^n(X))$ has convex core volume differing from $2n\,\text{vol}(M_\psi)$ by a uniform additive constant, where M_ψ is the hyperbolic mapping torus for ψ. We combine this estimate with work of Schlenker, and a branched covering argument to obtain an explicit lower bound on *Weil-Petersson translation distance* of a pseudo-Anosov $\psi \in \text{Mod}(S)$ for general compact S of genus g with n boundary components: we have

$$\text{vol}(M_\psi) \le \frac{3}{2}\sqrt{\text{area}(S)}\,\|\psi\|_{\text{WP}}$$

where $\text{area}(S) = 2\pi(2g-2+n)$ is the usual Poincaré area of any complete finite area hyperbolic structure on $\text{int}(S)$. This gives the first explicit estimates on the lengths of Weil-Petersson systoles of moduli space, of the minimal distance between nodal surfaces in the completion of Teichmüller space, and explicit lower bounds to the Weil-Petersson diameter of the moduli space via [20]. In the process, we recover the estimates of [27] on Teichmüller translation distance via a Cauchy-Schwarz estimate (see [29]).

1. Introduction

Let S be a closed surface of genus $g > 1$. Let $\psi : S \to S$ be a pseudo-Anosov element of $\text{Mod}(S)$, $Q_n = Q(\psi^{-n}(X), \psi^n(X))$ quasi-Fuchsian simultaneous uniformizations, and M_ψ the hyperbolic mapping torus for ψ. Let $\text{core}(Q_n)$ denote the convex core of Q_n. We will prove the following theorem.

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Theorem 1.1. *The quantity*

$$|\operatorname{vol}(\operatorname{core}(Q_n)) - 2n \operatorname{vol}(M_\psi)|$$

is uniformly bounded.

The possibility of such a result was suggested in [9, §1]. It gives an alternative, more direct proof of the main result of that paper comparing hyperbolic volume of M_ψ and *Weil-Petersson translation distance* of ψ as a direct corollary of a similar comparison in the quasi-Fuchsian case [8]. A recent preprint of S. Kojima and G. McShane shows how this suggestion can be used to give sharper bounds between volume of M_ψ and *normalized entropy*, or the translation distance in the Teichmüller metric of ψ.

In addition to supplying a proof of Theorem 1.1, we focus here on volume implications for the Weil-Petersson metric on Teichmüller space. Indeed, by analyzing the *renormalized volume*, a variant of convex core volume introduced by E. Witten [41], Jean-Marc Schlenker improved the upper bound in [9], for closed S with genus at least 2.

Theorem 1.2. (Schlenker) *Let S be a closed surface of genus $g > 1$ and let X, Y lie in* $\operatorname{Teich}(S)$. *There is a constant $K_S > 0$ so that*

$$\operatorname{vol}(\operatorname{core}(Q(X,Y))) \leq 3\sqrt{\pi(g-1)}\, d_{\mathrm{WP}}(X,Y) + K_S.$$

(See [35, Cor. 1.4]).

The *Weil-Petersson translation length* of ψ as an automorphism of $\operatorname{Teich}(S)$ is defined by taking the infimum

$$\|\psi\|_{\mathrm{WP}} = \inf_{X \in \operatorname{Teich}(S)} d_{\mathrm{WP}}(X, \psi(X)).$$

Daskalopoulos and Wentworth [23] showed this infimum is realized by some X in $\operatorname{Teich}(S)$ when ψ is pseudo-Anosov.

When $S = S_{g,n}$ has genus g and n boundary components and $\chi(S) < 0$, the Teichmüller space $\operatorname{Teich}(S)$ parametrizes finite area marked hyperbolic structures on $\operatorname{int}(S)$ up to marking preserving isometry. We take $\operatorname{area}(S)$ to denote the Poincaré area of any $X \in \operatorname{Teich}(S)$, namely

$$\operatorname{area}(S) = 2\pi(2g - 2 + n).$$

Combining Theorem 1.1, Theorem 1.2, and a branched covering argument, we obtain the following Theorem.

Theorem 1.3. *Let S be a compact surface with genus g and n boundary components $\chi(S) < 0$ and let $\psi \in \mathrm{Mod}(S)$ be pseudo-Anosov. Then we have*

$$\mathrm{vol}(M_\psi) \leq \frac{3}{2}\sqrt{\mathrm{area}(S)}\,\|\psi\|_{\mathrm{WP}}.$$

The case when S is closed readily follows from Theorem 1.1 and Theorem 1.2. When S has boundary, a branched covering argument allows us to recover the estimates from the closed case; we defer the proof to section 4.

Given X, $Y \in \mathrm{Teich}(S)$, it is frequently natural to consider the *normalized Weil-Petersson distance*

$$d_{\mathrm{WP}^*}(X, Y) = \frac{d_{\mathrm{WP}}(X, Y)}{\sqrt{\mathrm{area}(S)}}.$$

Passage to finite covers of S yields an isometry of normalized Weil-Petersson metrics, as is the case with the Teichmüller metric.

Then by an application of the Cauchy-Schwarz inequality (see [29]), we have for each X, Y in $\mathrm{Teich}(S)$ the bound

$$d_{\mathrm{WP}^*}(X, Y) \leq d_T(X, Y)$$

from which we conclude

$$\|\psi\|_{\mathrm{WP}^*} \leq \|\psi\|_T,$$

where $\|\psi\|_{\mathrm{WP}^*}$ denotes the translation distance of ψ in the normalized Weil-Petersson metric. As it follows from Theorem 1.3 that

$$\mathrm{vol}(M_\psi) \leq \frac{3}{2}\,\mathrm{area}(S)\,\|\psi\|_{\mathrm{WP}^*} \leq \frac{3}{2}\,\mathrm{area}(S)\,\|\psi\|_T,$$

we recover the Theorem of [27] concerning volumes and Teichmüller translation distance for arbitrary compact S.

We note that the study of normalized entropy and dilatation has seen considerable interest of late, note in particular the papers of [2], and [24] which have greatly improved our understanding of fibered 3-manifolds of low dilatation. The work of [27] has been particularly important here, giving a new proof of a weaker version of the finiteness theorem of [24], specifically, the statement that all mapping tori arising from pseudo-Anosov monodromy of *bounded dilatation* arise from Dehn filling of those from a finite list. We remark that an analogous Theorem where a bound on the normalized Weil-Petersson translation distance replaces a bound on the dilatation is immediate from [9], by applying Gromov's theorem that hyperbolic volume decreases under Dehn-filling [39, Thm. 6.5.6].

We will focus our attention primarily on implications for Teichmüller space with the Weil-Petersson metric.

Weil-Petersson geometry. Work of the first author relating Weil-Petersson geometry to volumes of quasi-Fuchsian and fibered hyperbolic 3-manifolds [8, 9] factors through relationships with a combinatorial structure related to each, the *pants graph* $P(S)$ whose vertices are pants decompositions of the surfaces S and whose edges connect pants decompositions related by certain *elementary moves.*

More direct interrelations between Weil-Petersson geometry and the uniformization of hyperbolic 3-manifolds had been developed from a different point of view by Taktajan, Zograf and Teo, ([37, 38]) in their study of the *Liouville action* which agrees with renormalized volume. In different contexts, this renormalized volume plays the role of a potential for the Weil-Petersson symplectic form. In a related construction, C. McMullen showed that the difference between Fuchsian and quasi-Fuchsian projective structures yields a bounded 1-form on moduli space that is a primitive for the Weil-Petersson metric, showing the Moduli space is *Kähler hyerbolic* in the sense of Gromov [32].

Schlenker's Theorem (Theorem 1.2) was the first to relate the notion of renormalized volume to the connection between Weil-Petersson *distance* and convex core volume (see [28] for a detailed account of renormalized volume from a geometric perspective). All the above connections raise further questions about a more direct relationship between these quantities. Indeed, in [30] Manin and Marcolli raise the expectation of an exact formula relating the two, but joint work of the first author with Juan Souto [15] shows

Theorem 1.4. ([15]) *There is no continuous function* $f\colon \mathbb{R} \to \mathbb{R}$ *so that given* $\psi \in \mathrm{Mod}(S)$*,*

$$f(\mathrm{vol}(M_\psi)) = \|\psi\|_{\mathrm{WP}}.$$

Similar results hold for the quasi-Fuchsian case.

Nevertheless, for a given S, the first author and Yair Minsky show the following further similarity with the distribution of lengths of closed Weil-Petersson geodesics in the Riemann moduli space $\mathcal{M}(S)$ with the set of hyperbolic volumes of fibered manifolds [14].

Theorem 1.5. *[14]* (Length Spectrum) *The extended Weil-Petersson geodesic length spectrum of* $\mathcal{M}(S)$ *is a well ordered subset of* $\mathbb{R}$*, with order type* ω^ω*.*

Here, the extended length spectrum refers to the set of lengths of closed geodesics together with lengths of *extended mapping classes*, automorphisms of a Teichmüller-Coxeter complex introduced by Yamada, where Dehn-twist iterations can take infinite powers. Such limiting elements behave as *billiard paths* on the moduli space with the Weil-Petersson completion, intersecting the compactification with equal angle of incidence and reflection (see [44], [46]).

It is natural to speculate regarding the value of the bottom of this spectrum, or the length of the Weil-Petersson *systole* of the moduli space $\mathcal{M}(S)$: Theorem 1.3 gives the first explicit estimates on the length of this shortest closed geodesic. It was shown by Gabai, Meyerhoff, and Milley [25] that the smallest volume closed orientable hyperbolic 3-manifold is the Weeks manifold $\mathcal{W}$, obtained by $(5,2)$ and $(5,1)$ Dehn surgeries on the Whitehead link. An explicit formula for its volume is given by

$$\mathrm{vol}(\mathcal{W}) = \frac{3 \cdot 23^{3/2}\zeta_k(2)}{4\pi^4} = 0.9427\ldots$$

where ζ_k is the Dedekind zeta function of k, where k is a number field $\mathbb{Q}(\theta)$ generated by θ, and θ satisfies $\theta^3 - \theta + 1 = 0$ [21].

Applying Theorem 1.3, we conclude the following lower bound on the length of the Weil-Petersson systole of $\mathcal{M}(S)$ for S a closed surface.

Theorem 1.6. (Weil-Petersson Systole - Closed Case) *Let S be a closed surface with genus $g > 1$, and let γ be the shortest closed Weil-Petersson geodesic in the moduli space $\mathcal{M}(S)$. Then we have*

$$\frac{\mathrm{vol}(\mathcal{W})}{3\sqrt{\pi(g-1)}} \leq \ell_{\mathrm{WP}}(\gamma).$$

We remark that a recent result of Agol, Leininger and Margalit [2] provides an upper bound:

$$\ell_{\mathrm{WP}}(\gamma) \leq \frac{2\sqrt{\pi}\log(\frac{3+\sqrt{5}}{2})}{\sqrt{(g-1)}}.$$

Similarly, Cao and Meyerhoff [19] show that the smallest volume among orientable cusped hyperbolic 3-manifolds is realized by the *figure eight knot complement* and its sibling, obtained by $(5,1)$ Dehn surgery on the Whitehead link complement. Their common volume is $2\mathcal{V}_3$ where $\mathcal{V}_3$ is the volume of the regular ideal hyperbolic tetrahedron. An application of this bound yields a similar result for the Weil-Petersson systole of the moduli space of punctured surfaces.

Theorem 1.7. (Weil-Petersson Systole - Punctured Case) *Let S be a surface of genus g with $n > 0$ boundary components and $\chi(S) < 0$, and let γ be the shortest closed Weil-Petersson geodesic in the moduli space $\mathcal{M}(S)$. Then we have*

$$\frac{4\mathcal{V}_3}{3\sqrt{\text{area}(S)}} \leq \ell_{\text{WP}}(\gamma).$$

Known upper bounds require a more involved discussion, which we omit here.

The Weil-Petersson inradius of Teichmüller space. It is remarkable that even to estimate the distance between nodal surfaces at infinity in the Weil-Petersson metric has been an elusive problem. Theorem 1.3 provides the first explicit means by which to do this, through a limiting process involving Dehn-twist iterates about a longitude-meridian pair (α, β) on the punctured torus.

Specifically, letting S be the one-holed torus, we identify the upper-half-plane $\mathbb{H}^2$ with Teich(S). Then $\text{Mod}(S) = \text{SL}_2(\mathbb{Z})$ acts by isometries, and we consider the family

$$\psi_n = \tau_\alpha^n \circ \tau_\beta^{-n}$$

of composed n-fold Dehn-twists about simple closed curves α and β on S with $i(\alpha, \beta) = 1$ on S. Up to conjugation, we have

$$\psi_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

in $\text{SL}_2(\mathbb{Z})$. Then Theorem 1.3 gives

$$\text{vol}(M_{\psi_n}) \leq 3\sqrt{\frac{\pi}{2}}\|\psi_n\|_{\text{WP}}.$$

The manifolds M_{ψ_n} converge geometrically to the compelement of the Borromean rings in S^3, which has volume $2\mathcal{V}_8$ or twice the volume $\mathcal{V}_8 = 3.6638\ldots$ of the regular ideal hyperbolic octahedron, while the right hand side converges to twice the (finite) length of the imaginary axis in the upper half plane $\mathbb{H}^2$ in the Weil-Petersson metric on the Teichmüller space of the one-holed torus (see [10]).

Then we obtain:

Theorem 1.8. (Weil-Petersson Inradius) *Let $S = S_{1,1}$ be the one-holed torus. The Weil-Petersson length of the imaginary axis I satisfies*

$$\frac{1}{3}\sqrt{\frac{2}{\pi}}\mathcal{V}_8 \leq \ell_{\mathrm{WP}}(I) \leq 2\sqrt{30}\,\pi^{\frac{3}{4}}.$$

Numerically this gives

$$0.9744\ldots \leq \ell_{\mathrm{WP}}(I) \leq 25.8496\ldots$$

illustrating that there is room for improvement in these estimates. The upper bound arises from estimates on the length of the systole $\mathrm{sys}(X)$ of a hyperbolic surface X, together with Wolpert's upper bound of $\sqrt{2\pi\,\ell_X(\mathrm{sys}(X))}$ [45, Cor 4.10] on the distance from X to a nodal surface where a curve of shortest length $\ell_X(\mathrm{sys}(X))$ on X is pinched to a cusp (see [20]).

The axis I is isometric to each edge e of the *Farey graph* $\mathbb{F} = \Gamma(I)$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, joining pairs of rationals $(\frac{p}{q}, \frac{r}{s})$ with

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = \pm 1$$

or the extended distance

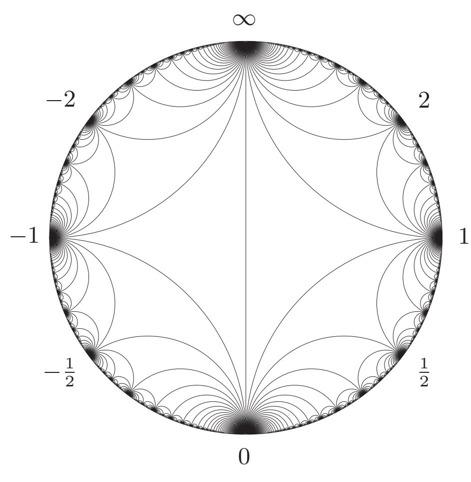

Figure 1: The Farey graph under stereographic projection to Δ.

$$d_{\overline{\mathrm{WP}}}\left(\frac{p}{q}, \frac{r}{s}\right) = d_{\overline{\mathrm{WP}}}(0, \infty)$$

between closest points in the *completion*

$$\overline{\text{Teich}(S)} = \mathbb{H}^2 \cup (\mathbb{Q} \cup \infty)$$

of the Teichmüller space of the punctured torus with the Weil-Petersson metric (see Figure 1). In section 5, we apply Theorem 1.8 to obtain the first explicit constants of comparison of distance in the Farey graph with Weil-Petersson distance between rational points in the completion, first appearing in [8].

Weil-Petersson diameter of Moduli space. The length $\ell_{\text{WP}}(I)$ is instrumental in the estimation of the Weil-Petersson diameter of moduli space [20]. Noting that the Weil-Petersson length of I in the Teichmüller space of the four-holed sphere is twice its length in the Teichmüller space of the one-holed torus, we may combine Theorem 1.8 with results of [20] relating this length to the diameter of moduli space to obtain the following explicit estimates:

Theorem 1.9. *Let $S = S_{g,n}$ have $\chi(S) < 0$, and let $\mathcal{M}_{g,n} = \mathcal{M}(S_{g,n})$, the moduli space of genus g Riemann surfaces with n punctures. Then we have the following:*

$$\text{diam}_{\text{WP}}(\mathcal{M}_{1,1}) \geq \frac{1}{6}\sqrt{\frac{2}{\pi}}\mathcal{V}_8,$$

$$\text{diam}_{\text{WP}}(\mathcal{M}_{0,4}) \geq \frac{1}{3}\sqrt{\frac{2}{\pi}}\mathcal{V}_8,$$

and otherwise for $3g - 3 + n \geq 2$,

$$\text{diam}_{\text{WP}}(\mathcal{M}_{g,n}) \geq \frac{1}{3\sqrt{\pi}}\mathcal{V}_8\sqrt{2g + n - 4}.$$

Proof. The imaginary axis in $\mathbb{H}$ projects 2-to-1 to a geodesic in $\mathcal{M}_{1,1}$ of half its original Weil-Petersson length, which is estimated in Theorem 1.8. The Weil-Petersson metric on $\mathcal{M}_{0,4}$ is isometric to twice that of $\mathcal{M}_{1,1}$. The general estimate follows from the totally geodesically embedded $\mathcal{M}_{0,2g+n}$ strata in the completion of $\mathcal{M}_{g,n}$, as observed in [20, Prop. 5.1]. □

We note that dividing by $\sqrt{\text{area}(S)}$ gives an explicit, positive lower bound to the *normalized Weil-Petersson diameter*

$$\text{diam}_{\text{WP}^*}(\mathcal{M}(S)) = \text{diam}_{\text{WP}}(\mathcal{M}(S))/\sqrt{\text{area}(S)}$$

of moduli spaces $\mathcal{M}(S)$ that is independent of S (cf. [20, Prop. 5.1]).

Corollary 1.10. *The normalized Weil-Petersson diameter of* $\mathcal{M}_{g,n}$ *satisfies*

$$\mathrm{diam}_{\mathrm{WP}^*}(\mathcal{M}_{g,n}) \geq \frac{1}{\pi\, 3\sqrt{2}} \mathcal{V}_8.$$

History. The original version of [27] relied without proof on a remark in [9] suggesting a proof of Theorem 1.1 should be possible using the idea of *geometric inflexibility* from [31] and [12]. The present paper supplies such a proof, as a means toward employing Schlenker's improvement [35, Cor. 1.4] to the upper bound in [8, Thm. 1.2] to obtain new explicit estimates on the Weil-Petersson geometry of Teichmüller and moduli space. After we presented our arguments in Curt McMullen and Martin Bridgeman's Informal Seminar at Harvard, a revision to [27] presented an independent proof of Theorem 1.1 (as well as version of Theorem 1.3 restricted to closed surfaces) and McMullen provided a succinct argument for a slightly weaker version of Theorem 1.1 directly from Thurston's *Double Limit Theorem* and the strong convergence of Q_n to the fiber Q_∞ ([18], see also [12, Thm. 1.2]). His argument appears in his Seminar Notes available on his webpage, together with other estimates for L^p metrics on Teichmüller space (and alongside notes from our lecture). We have retained the inflexibility approach here to illustrate its utility.

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2. Preliminaries

We review background for our results.

Weil-Petersson geometry. The above results give new explicit estimates on the geometry of Teichmüller space with the Weil-Petersson metric. The

Weil-Petersson metric arises from the hyperbolic L^2-norm on the space of quadratic differentials $Q(X)$ on a Riemann surface X, given by

$$\|\varphi\|^2_{\mathrm{WP}} = \int_X \frac{|\varphi|^2}{\rho_X}$$

with ρ_X the hyperbolic metric on X. Though known to be geodesically convex [43] it is not complete [22, 42]. It has negative curvature [4], but its curvatures are bounded away neither from 0 nor negative infinity. In [23], Daskalopoulos and Wentworth showed that a pseudo-Anosov automorphism $\psi \in \mathrm{Mod}(S)$ has an invariant geodesic axis along which ψ translates. A primitive pseudo-Anosov element $\psi \in \mathrm{Mod}(S)$, therefore, determines a closed Weil-Petersson geodesic in the quotient moduli space of Riemann surfaces $\mathcal{M}(S)$ whose length is the Weil-Petersson translation distance $\|\psi\|_{\mathrm{WP}}$ of ψ.

The complex of curves. Let S be a compact surface of genus g with n boundary components. The complex of curves $\mathcal{C}(S)$ is a $3g-4$ dimensional complex, each vertex of which is associated to a simple closed curve on the surface S up to isotopy, and so that k-simplices span collections of $k+1$ vertices whose associated curves are disjoint. Masur and Minsky proved $\mathcal{C}^1(S)$ is a δ-hyperbolic metric space with the distance $d_{\mathcal{C}}(.,.)$ given by the edge metric.

Given S there is a an L_S so that for each $X \in \mathrm{Teich}(S)$ there is a $\gamma \in \mathcal{C}(S)$ so that $\ell_X(\gamma) < L_S$. By making such a choice of γ for each X we obtain a coarsely well-defined projection

$$\pi_{\mathcal{C}} \colon \mathrm{Teich}(S) \to \mathcal{C}^0(S).$$

We refer to the distance between a point X in Teichmüller space and a curve γ in $\mathcal{C}^0(S)$ with the notation:

$$d_{\mathcal{C}}(X,\gamma) = d_{\mathcal{C}}(\pi_{\mathcal{C}}(X),\gamma).$$

Quasi-Fuchsian manifolds. Each pair $(X,Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(S)$ determines a quasi-Fuchsian *simultaneous uniformization* $Q(X,Y)$ with X and Y in its conformal boundary. This is the quotient

$$Q(X,Y) = \mathbb{H}^3/\rho_{X,Y}(\pi_1(S))$$

of a quasi-Fuchsian representation of the fundamental group

$$\rho_{X,Y} \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C}).$$

The quasi-Fuchsian representations sit as the interior of the space $\mathrm{AH}(S)$ of all marked hyperbolic 3-manifolds homotopy equivalent to S up to marking-preserving isometry, with the topology of convergence on generators of the fundamental group. For more information, see [1, 11, 13, 16, 17].

A complete hyperbolic 3-manifold M, marked by a homotopy equivalence

$$f\colon S \to M$$

determines a point in $\mathrm{AH}(S)$ up to equivalence - we denote such a marked hyperbolic 3-manifold by the pair (f, M). Equipping M with a baseframe (M, ω) determines a specific representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and a Kleinian surface group

$$\Gamma = \rho(\pi_1(S)).$$

The geometric topology on such based hyperbolic 3-manifolds records geometric information: a sequence (M_n, ω_n) converges to (M, ω_∞) if for each $\epsilon, R > 0$ there is an $N > 0$, and for all $n > N$ we have embeddings

$$\varphi_n\colon (B_R(\omega), \omega) \to (M_n, \omega_n)$$

from the R-ball around ω to M_n, whose derivatives send ω_∞ to ω_n and whose bi-Lipschitz constants are at most $1 + \epsilon$ at all points of $B_R(\omega_\infty)$.

The convergent sequence $(f_n, M_n) \to (f_\infty, M_\infty)$ in $\mathrm{AH}(S)$ converges *strongly* if there are baseframes ω_n in M_n and $\omega_\infty \in M_\infty$ so that the resulting ρ_n converge to the resulting ρ_∞ on generators, and the manifolds (M_n, ω_n) converge geometrically to $(M_\infty, \omega_\infty)$.

Convex core width. Given $M \in \mathrm{AH}(S)$, let $d_M(U, V)$ be the minimal distance between subsets U and V in M. We prove the following in [12].

Theorem 2.1. *Given ϵ, $L > 0$, there exist K_1 and K_2 so that if $M \in \mathrm{AH}(S)$ and α and β in $\mathcal{C}^0(S)$ have representatives α' and β' with $\ell_M(\alpha')$ and $\ell_M(\beta')$ bounded above by L and below by ϵ, then*

$$d_M(\alpha, \beta) \geq K_1 d_{\mathcal{C}}(\alpha, \beta) - K_2$$

It is due to Bers that

$$2\ell_X(\gamma) \geq \ell_{Q(X,Y)}(\gamma).$$

Thus Theorem 2.1 serves to bound from below the width of the convex core of $Q(X, Y)$ (the distance between its boundary components) in terms of the curve complex distance. Such convex core width estimates will be important to our application of the inflexibility theory outlined in the next section.

3. Geometric Inflexibility

To prove Theorem 1.1, our key tool will be the *inflexibility theorem* of [12].

Theorem 3.1. (Geometric Inflexibility) *Let M_0 and M_1 be complete hyperbolic structures on a 3-manifold M so that M_1 is a K-quasi-conformal deformation of M_0, $\pi_1(M)$ is finitely generated, and M_0 has no rank-one cusps.*

There is a volume preserving $K^{3/2}$-bi-Lipschitz diffeomorphism

$$\Phi\colon M_0 \to M_1$$

whose pointwise bi-Lipschitz constant satisfies

$$\log \mathrm{bilip}(\Phi, p) \leq C_1 e^{-C_2 d(p, M_0 \setminus \mathrm{core}(M_0))}$$

for each $p \in M^{\geq \epsilon}$, where C_1 and C_2 depend only on K, ϵ, and $\mathrm{area}(\partial\,\mathrm{core}(M_0))$.

The existence of a volume preserving, $K^{3/2}$ bi-Lipschitz diffeomorphism was established by Reimann [34], using work of Ahlfors [5] and Thurston [39] (see McMullen [31] for a self-contained account). That the bi-Lipschitz constant decays exponentially fast with depth in the convex core at points in the thick part follows from comparing L^2 and pointwise bounds on *harmonic strain fields* arising from extending a *Beltrami isotopy* realizing the deformation. Exponential decay of the L^2 norm in the core can be converted to pointwise bounds via mean value estimates, building on work in the cone-manifold deformation theory of hyperbolic manifolds due to Hodgson and Kerckhoff [26] and the second author [16].

Inflexibility was used in [12] to give a new, self-contained proof of Thurston's *Double Limit Theorem*, and the hyperbolization theorem for closed 3-manifolds that fiber over the circle with pseudo-Anosov monodromy.

Theorem 3.2. (Thurston) *Let S be a closed hyperbolic surface. The double iteration $Q(\psi^{-n}(X), \psi^n(X))$ converges algebraically and geometrically to the limit Q_∞, the infinite cyclic cover of the mapping torus M_ψ corresponding to the fiber S.*

The manifolds

$$Q_n = Q(\psi^{-n}(X), \psi^n(X))$$

admit volume preserving, uniformly bi-Lipschitz *Reimann maps*

$$\phi_n \colon Q_n \to Q_{n+1}$$

as in Theorem 3.1. The key to obtaining Theorem 3.2 from Theorem 3.1 is an analysis of the growth rate of the convex core diameter in terms of the curve complex.

We will employ the following key consequence of inflexibility [12, Proposition 9.7].

Proposition 3.3. *Given $\epsilon, R, L, C > 0$ there exist $B, C_1, C_2 > 0$ such that the following holds. Assume that $\mathcal{K}$ is a subset of Q_N such that $\operatorname{diam}(\mathcal{K}) < R$, $\operatorname{inj}_p(\mathcal{K}) > \epsilon$ for each $p \in \mathcal{K}$ and $\gamma \in \mathcal{C}^0(S)$ is represented by a closed curve in $\mathcal{K}$ of length at most L satisfying*

$$\min\{d_{\mathcal{C}}(\psi^{N+n}(Y), \gamma), d_{\mathcal{C}}(\psi^{-N-n}(X), \gamma)\} \geq K_\psi n + B$$

for all $n \geq 0$. Then we have

$$\log \operatorname{bilip}(\phi_{N+n}, p) \leq C_1 e^{-C_2 n}$$

for p in $\phi_{N+n-1} \circ \cdots \circ \phi_N(\mathcal{K})$ and

$$\frac{C_1}{1 - e^{-C_2}} < C.$$

The simple closed curve γ serves to control the depth of the compact set $\mathcal{K}$ in the convex core of Q_{N+n} as $n \to \infty$ via inflexibility and Theorem 2.1: if $\mathcal{K}$ starts out sufficiently deep, then the geometry freezes around it quickly enough that Theorem 2.1 guarantees its depth grows linearly, resulting in the exponential convergence of the bi-Lipschitz constant.

Double Iteration. The pseudo-Anosov double iteration $\{Q_n\}$ converges strongly to the doubly degenerate manifold Q_∞, invariant by the isometry

$$\Psi \colon Q_\infty \to Q_\infty$$

the isometric covering translation for Q_∞ over the mapping torus M_ψ for ψ (see [12, 18, 31, 40]). Likewise, McMullen showed the iteration $Q(X, \psi^n(X))$

also converges strongly to a limit Q_{X,ψ^∞} in the *Bers slice*

$$B_X = \{Q(X,Y) \;:\; Y \in \mathrm{Teich}(S)\}.$$

Each element $\tau \in \mathrm{Mod}(S)$ acts on $\mathrm{AH}(S)$ by *remarking*, or precomposition of the representation by the corresponding automorphism of the fundamental group. This action is denoted by

$$\tau(f, M) = (f \circ \tau^{-1}, M)$$

Then by Thurston's *Double Limit Theorem* [12, 33, 40], the remarking of Q_{X,ψ^∞} by ψ^{-n} produces a sequence

$$\psi^{-n}(Q_{X,\psi^\infty}) = Q_{\psi^{-n}(X),\psi^\infty}$$

converging strongly in $\mathrm{AH}(S)$ to Q_∞ (see [31]).

Bonahon's Tameness Theorem [6] provides a homeomorphism

$$F\colon S \times \mathbb{R} \to Q_\infty$$

equipping the limit Q_∞ with a product structure; indeed, as Q_∞ is the cyclic cover $Q_\infty \to M_\psi$ for the fibered manifold M_ψ, we may take this product structure so that the isometric covering transformation

$$\Psi\colon Q_\infty \to Q_\infty$$

in the homotopy class of ψ preserves this product structure and acts by integer translation $\Psi(S,t) = (S, t+1)$ in the second factor. We denote by $Q_\infty[a,b]$ the subset $F(S \times [a,b])$.

Proposition 3.4. *Let $\gamma \in \mathcal{C}^0(S)$ satisfy $\ell_X(\gamma) < L_S$. Then there exists $a > 0$ and $N_1 > 0$ so that for each $n > N_1$ the compact subset $Q_\infty[-a,a]$ contains γ^* and admits a marking preserving 2-bi-Lipschitz embedding*

$$\varphi_n\colon Q_\infty[-a,a] \to Q_{\psi^{-n}(X),\psi^\infty}.$$

Proof. The Proposition follows from the observation that the geodesic representatives of $\psi^n(\gamma)$ lie arbitrarily deep in the convex core of Q_{X,ψ^∞}, and the fact that the isometric remarkings $\psi^{-n}(Q_{X,\psi^\infty}) = Q_{\psi^{-n}(X),\psi^\infty}$ converge to the fiber Q_∞ (see [31, Thm. 3.11]). Choosing an interval $[-a,a]$ so that

$Q_\infty[-a,a]$ contains γ^*, the marking preserving bi-Lipschitz embeddings

$$\varphi_n \colon Q_\infty[-a,a] \to Q_{\psi^{-n}(X),\psi^\infty}$$

are eventually 2-bi-Lipschitz, giving the desired N_1. □

We note that we may argue symmetrically for $Q_{\psi^{-\infty},\psi^n(X)}$, the strong limit of $Q(\psi^{-m}(X),\psi^n(X))$ as $m \to \infty$.

4. The Proof

In this section we give the proof of Theorem 1.1.

Proof. The proof is a straightforward application of Proposition 3.3. Making an initial choice of N, we will find, for each k, a subset $\mathcal{K}_k$ of Q_{N+k} accounting for all but a uniformly bounded amount of the volume of the core of Q_{N+k}. Fixing k, the volume preserving Reimann maps ϕ_{N+k+n} of Proposition 3.3 applied to $\mathcal{K}_k$ produce subsets of Q_{N+k+n} that converge as $n \to \infty$ to a subset of Q_∞ within bounded volume of $2k$ copies of the fundamental domain for the action of ψ. This yields the desired comparison.

Step I. *Choose constants.* As the input for Proposition 3.3, let

$$R > 4\,\mathrm{diam}(Q_\infty[-a,a]),$$

take $L > 4L_S$, and fix $\epsilon < \epsilon_\psi/4$, where

$$\epsilon_\psi = \mathrm{inj}(M_\psi) = \mathrm{inj}(Q_\infty)$$

(here $\mathrm{inj}(M) = \inf_{p\in M}\mathrm{inj}_p(M)$). Finally, taking $C = 2$, we take B, C_1 and C_2 satisfying the conclusion of Proposition 3.3. Recall that $\gamma \in \mathcal{C}^0(S)$ satisfies $\ell_X(\gamma) < L_S$. Then applying [12, Thm. 8.1] there is an $N_0 > 0$ so that

$$\min\{d_{\mathcal{C}}(\psi^{-N_0-n}(X),\gamma), d_{\mathcal{C}}(\psi^{N_0+n}(X),\gamma)\} \geq K_\psi n + B$$

for all $n \geq 0$.

Step II. *Geometric convergence.* Applying Proposition 3.4, we may take $N_1 > N_0$ so that for each $N > N_1$ there are marking preserving 2-bi-Lipschitz embeddings

$$\varphi_N^-\colon Q_\infty[-a,a] \to \operatorname{core}(Q_{\psi^{-N}(X),\psi^\infty})$$
$$\varphi_N^+\colon Q_\infty[-a,a] \to \operatorname{core}(Q_{\psi^{-\infty},\psi^N(X)}).$$

Applying strong convergence of

$$Q(Y,\psi^n(X)) \to Q_{Y,\psi^\infty} \quad \text{and} \quad Q(\psi^{-n}(X),Y) \to Q_{\psi^{-\infty},Y},$$

we take $N_2 > N_1$ so that for each $\delta > 0$, $D > 0$, and $N > N_2$, we have k_0 so that for $k > k_0$ there are diffeomorphisms

$$\eta_{N,k}^-\colon Q_{\psi^{-N}(X),\psi^\infty} \to Q(\psi^{-N}(X),\psi^{N+2k}(X))$$

and

$$\eta_{N,k}^+\colon Q_{\psi^{-\infty},\psi^N(X)} \to Q(\psi^{-N-2k}(X),\psi^N(X))$$

so that $\eta_{N,k}^-$ has bi-Lipschitz constant satisfying $\log \operatorname{bilip}(\eta_{N,k}^-, p) < \delta$ for all points in the D-neighborhood of $\varphi_N^-(Q_\infty[-a,a])$, and likewise for $\eta_{N,k}^+$.

It follows that if we fix N satisfying $N > N_2$ for the remainder of the argument, the images $\varphi_N^-(Q_\infty[-a,a])$ and $\varphi_N^+(Q_\infty[-a,a])$ determine product regions in $\operatorname{core}(Q_{\psi^{-N}(X),\psi^\infty})$ and $\operatorname{core}(Q_{\psi^{-\infty},\psi^N(X)})$ whose complements contain one product region of volume bounded by $\mathcal{V} > 0$.

Noting that the action by $\psi^{\pm k}$ on $\operatorname{AH}(S)$ gives

$$\psi^{-k}(Q(\psi^{-N}(X),\psi^{N+2k}(X))) = Q_{N+k} = \psi^k(Q(\psi^{-N-2k}(X),\psi^N(X))),$$

we let, for each $k > 0$, the subsets $\mathcal{K}_k^-$ and $\mathcal{K}_k^+$ in Q_{N+k} be given by

$$\psi^{-k}(\eta_{N,k}^- \circ \varphi_N^-(Q_\infty[-a,a])) \quad \text{and} \quad \psi^k(\eta_{N,k}^+ \circ \varphi_N^+(Q_\infty[-a,a]))$$

by following the embeddings of $Q_\infty[-a,a]$ from geometric convergence with the isometric remarkings ψ^{-k} and ψ^k. Then geometric convergence implies that for each $k > k_0$ the component of $Q_{N+k} \setminus \mathcal{K}_k^-$ facing $\psi^{-N-k}(X)$ has intersection with the convex core bounded by $2\mathcal{V}$ for k large, and likewise for $\mathcal{K}_k^+$.

Step III. *Apply Inflexiblility (Proposition 3.3).* We take B as in Proposition 3.3 given the above choices for ϵ, L, R and C.

For our choice of N, we know $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ each have diameter at most R, injectivity radius at least ϵ, and as $L = 4L_S$, $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ contain representatives γ_k^- of $\psi^{-k}(\gamma)$ and γ_k^+ of $\psi^k(\gamma)$ of length less than L. As B is chosen as in the output of Proposition 3.3 and N is chosen as above we have

$$\min\{d_{\mathcal{C}}(\psi^{-N-k-n}(X), \gamma_k^-), d_{\mathcal{C}}(\psi^{N+k+n}(X), \gamma_k^-)\} \geq K_\psi n + B$$

is satisfied for all $n \geq 0$ and likewise for γ_k^+.

Let $\phi_N \colon Q_N \to Q_{N+1}$ denote the (marking preserving) Reimann map furnished by Proposition 3.3. Then the composition of Reimann maps

$$\Phi_n = \phi_{N+n} \circ \cdots \circ \phi_N \colon Q_N \to Q_{N+n+1}$$

is globally volume preserving.

Furthermore, since $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ each satisfy the hypotheses of Proposition 3.3 the compositions are uniformly bi-Lipschitz as $n \to \infty$. It follows from Arzela-Ascoli that we may extract a limit limit Φ_∞ on $\mathcal{K}_k^+$ and that Φ_∞ sends γ_k^+ to a curve of length at most $8L$ and likewise for $\mathcal{K}_k^-$ and γ_k^-. Since Φ_∞ is 2-bi-Lipschitz on $\mathcal{K}_k^-$ and $\mathcal{K}_k^+$, it follows that $\Phi_\infty(\mathcal{K}_k^-)$ has diameter $8R$, and contains a representative of $\psi^{-k}(\gamma)$ of length $8L_S$ and likewise for $\Phi_\infty(\mathcal{K}_k^+)$ and $\psi^k(\gamma)$. There is thus a $d > 0$ depending only on R and L_S and ϵ_ψ so that we have

$$\Phi_\infty(\mathcal{K}_k^-) \subset Q_\infty[-k-d, -k+d] \quad \text{and} \quad \Phi_\infty(\mathcal{K}_k^+) \subset Q_\infty[k-d, k+d].$$

Furthermore, if we take k large enough, we may apply Theorem 2.1 to conclude that

$$d_{Q_{N+k+n}}(\gamma_k^-, \gamma_k^+) > 16R$$

which ensures that $\Phi_{n+k}(\mathcal{K}_k^-)$ and $\Phi_{n+k}(\mathcal{K}_k^+)$ are disjoint for all $n \geq 0$. The complement $Q_{N+k} \setminus \mathcal{K}_k^- \cup \mathcal{K}_k^+$ contains one subset O_{N+k} with compact closure 'between' the product regions $\mathcal{K}_k^-$ and $\mathcal{K}_k^+$.

Letting

$$\mathcal{K}_k = \mathcal{K}_k^- \cup O_{N+k} \cup \mathcal{K}_k^+,$$

the images $\Phi_{k+n}(\mathcal{K}_k)$ satisfy

$$\text{vol}(\mathcal{K}_k) = \text{vol}(\Phi_{k+n}(\mathcal{K}_k))$$

since Φ_{k+n} is the composition of volume preserving maps.

But strong convergence of Q_{N+k+n} to Q_∞ as $n \to \infty$ guarantees that for large n there are nearly isometric marking-preserving embeddings

$$G_n \colon Q_\infty[-k-d, k+d] \to Q_{N+k+n}$$

that are surjective onto $\Phi_{k+n}(\mathcal{K}_k)$ for n sufficiently large.

We conclude that

$$(2k-2d)\,\mathrm{vol}(M_\psi) \le \mathrm{vol}(\Phi_\infty(\mathcal{K}_k)) \le (2k+2d)\,\mathrm{vol}(M_\psi)$$

and that

$$\mathrm{vol}(\mathrm{core}(Q_{N+k})) - 4\mathcal{V} \le \mathrm{vol}(\mathcal{K}_k) \le \mathrm{vol}(\mathrm{core}(Q_{N+k})$$

for all k sufficiently large. Thus we conclude

$$|\,\mathrm{vol}(\mathrm{core}(Q_{N+k})) - 2(N+k)\,\mathrm{vol}(M_\psi)| < 2(d+N)\,\mathrm{vol}(M_\psi) + 4\mathcal{V}$$

completing the proof. □

To complete the proof of Theorem 1.3, we conclude the section by addressing the case when S has boundary.

Proof of Theorem 1.3. We now complete the proof of Theorem 1.3. It remains to treat the case when S has boundary. We thank Ian Agol for suggesting such an argument applies in the setting of the Teichmüller metric; we employ a similar line of reasoning for the Weil-Petersson metric, recovering the Teichmüller case as a consequence.

We note the following: by Ahlfors Lemma [3], for a surface $S = S_{g,n}$ with genus $g > 1$ and $n > 0$ boundary components, the natural forgetful map

$$\mathrm{Teich}(S_{g,n}) \to \mathrm{Teich}(S_{g,0})$$

obtained by filling in the n punctures on a surface $X \in \mathrm{Teich}(S_{g,n})$ is a contraction of Poincaré metrics and thus of Weil-Petersson metrics (see e.g. [36]). Assuming an even number of punctures, we may branch at the punctures to obtain degree-k covers $\tilde{S}_k$.

Lifting to finite covers of S induces natural maps between Teichmüller spaces that are local isometries with respect to the *normalized Weil-Petersson distance* $d_{\mathrm{WP}^*}(.,.)$, obtained by taking

$$d_{\mathrm{WP}^*}(.,.) = \frac{d_{\mathrm{WP}}(.,.)}{\sqrt{\mathrm{area}(S)}}.$$

Given ψ pseudo-Anosov, let $\|\psi\|_{\mathrm{WP}^*}$ denote its translation length in the normalized Weil-Petersson metric.

Letting $\psi \in \mathrm{Mod}(S)$, then, we let $\tilde{\psi}_k$ denote the lift to $\mathrm{Mod}(\tilde{S}_k)$, and $\hat{\psi}_k \in \mathrm{Mod}(\hat{S}_k)$ obtained by filling in the punctures of $\tilde{S}_k$ to obtain $\hat{S}_k$.

Then we have

$$\|\psi\|_{\mathrm{WP}^*} = \|\tilde{\psi}_k\|_{\mathrm{WP}^*} \geq C_k \cdot \|\hat{\psi}_k\|_{\mathrm{WP}^*}$$

where $C_k = \sqrt{\operatorname{area}(\hat{S}_k)/\operatorname{area}(\tilde{S}_k)} \to 1$ as $k \to \infty$. Applying Theorem 1.3 in the closed case we obtain,

$$\|\psi\|_{\mathrm{WP}^*} \geq C_k \frac{2}{3} \frac{\operatorname{vol}(M_{\hat{\psi}_k})}{\operatorname{area}(\hat{S}_k)}.$$

As $M_{\hat{\psi}_k}$ admits an order-k isometry corresponding to the k-fold branched covering, it covers a fibered orbifold with n order-k orbifold loci, which converges geometrically to the fibered 3-manifold M_ψ as $k \to \infty$. Likewise, $\hat{S}_k$ covers an orbifold with n cone points with cone-angle $2\pi/k$, whose area is $\operatorname{area}(\hat{S}_k)/k$, which converges to $\operatorname{area}(S)$ as $k \to \infty$.

Thus, dividing the top and the bottom by k, the right hand side of the inequality tends to

$$\frac{2}{3} \frac{\operatorname{vol}(M_\psi)}{\operatorname{area}(S)}$$

as $k \to \infty$, and the estimate holds.

Since any $S = S_{g,n}$ with $n > 0$ is finitely covered by $S_{g',n'}$ with $g' > 1$ and n' even, the proof is complete. □

5. Applications

We note the following applications to the Weil-Petersson geometry of $\operatorname{Teich}(S)$ and its quotient $\mathcal{M}(S)$ by the isometric action of the mapping class group $\operatorname{Mod}(S)$.

When α and β are a longitude and meridian pair on the punctured torus, the estimate of Theorem 1.8 gives a lower bound

$$\frac{\mathcal{V}_8}{3\sqrt{\pi/2}} \leq \ell_{\mathrm{WP}}(e)$$

to any edge e in the Farey graph $\mathbb{F}$. We remark that this estimate has implications for effective combinatorial models for $\operatorname{Teich}(S)$.

In particular, the main result of [8] guarantees the existence of K_1, K_2 depending only on S so that

$$\frac{d_P(P_1, P_2)}{K_1} - K_2 \leq d_{\mathrm{WP}}(N(P_1), N(P_2)) \leq K_1 d_P(P_1, P_2) + K_2.$$

Here, the distance d_P is taken in the *pants graph* $P(S)$ whose vertices are associated to pants decompositions of S and whose edges are associated to prescribed elementary moves (see [8], or [7] for an expository account) and $N(P_i)$ denotes the unique maximally noded Riemann surface in the boundary of Teichmüller space for which the curves in P_i have been pinched to cusps. To date, effective estimates on K_1 and K_2 have been elusive.

Theorem 1.3 gives the following estimate in the case of the punctured torus S, on which each pants decomposition is represented by a single non-peripheral simple closed curve.

Theorem 5.1. *Let S be a one-holed torus and let α and β denote essential simple closed curves on S. If $d_P(\alpha, \beta) = 1$ then*

$$\frac{\mathcal{V}_8}{3\sqrt{\pi/2}} \leq d_{\overline{\mathrm{WP}}}(N(\alpha), N(\beta)) \leq 2\sqrt{30}\,\pi^{\frac{3}{4}}$$

and if $d_P(\alpha, \beta) > 1$ then we have

$$\frac{\mathcal{V}_3}{3\sqrt{\pi/2}} d_P(\alpha, \beta) \leq d_{\overline{\mathrm{WP}}}(N(\alpha), N(\beta)) \leq 2\sqrt{30}\,\pi^{\frac{3}{4}} d_P(\alpha, \beta).$$

Remark: The first double inequality recapitulates of that of Theorem 1.8. Numerically, the second double inequality can be expressed as

$$(.2375\ldots) d_P(\alpha, \beta) \leq d_{\overline{\mathrm{WP}}}(N(\alpha), N(\beta)) \leq (25.8496\ldots) d_P(\alpha, \beta).$$

Proof. The space Teich(S) is naturally the unit disk Δ, and edges of the usual Farey graph are geodesics in the Weil-Petersson (as well as Teichmüller) metric. Once $d_P(\alpha, \beta)$ is at least 2, the completed Weil-Petersson geodesic g in $\overline{\mathrm{Teich}(S)}$ joining $N(\alpha)$ to $N(\beta)$ joins the endpoints of a *Farey sequence*, or a sequence $e_1, \ldots, e_n$ in $\mathbb{F}$ that joins α to β. Each pair of successive edges e_i and e_{i+1} determines a *pivot*, where they meet, and emanating from each pivot is a *bisector* b_i that meets the opposite edge of the ideal triangle determined by e_i and e_{i+1} perpendicularly (see Figure 2). For a Farey sequence

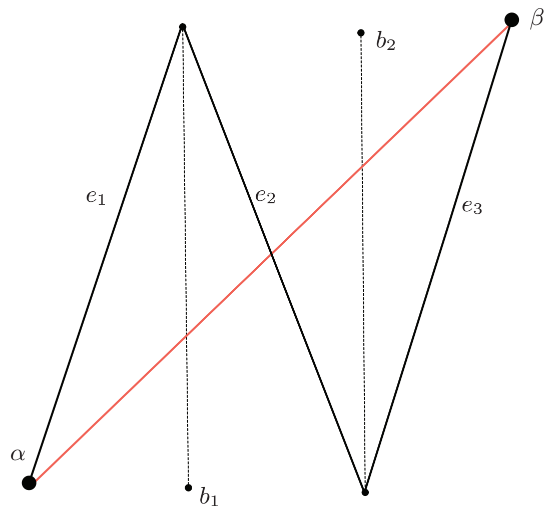

Figure 2: Bisectors of the Farey pivots have separation at least $\frac{1}{2}\|\psi_{\text{fig8}}\|_{\text{WP}}$.

that determines *exactly one* Farey triangle per pivot, these bisectors are perpendicular to the axis determined by a conjugate of the monodromy of the figure-8 knot complement, the mapping class

$$\psi_{\text{fig8}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and the intersections occur every half-period along the axis.

Thus, in this minimal case, successive bisectors have separation at least half the translation distance of ψ_{fig8} or at least

$$\frac{\mathcal{V}_3}{3\sqrt{\pi/2}}$$

by Theorem 1.7. When there are more triangles per pivot, the successive bisectors are further apart. Thus, the bisectors determined by the Farey sequence have at least the separation of the minimal case, as do the initial and terminal vertices α and β from the first and last bisector. The lower bound follows.

The upper bound follows from the triangle inequality, and the fact that each Farey edge has length bounded by $2\sqrt{30}\,\pi^{\frac{3}{4}}$ by Theorem 1.8. □

In the language of the introduction, if p/q has continued fraction expansion

$$\frac{p}{q} = [a_1, a_2, \ldots, a_n]$$

then p/q has distance n from 0 in the Farey graph $\mathbb{F}$; we say p/q has *Farey depth* n, or $\text{depth}_{\mathbb{F}}(p/q) = n$.

Then we have

$$\frac{\mathcal{V}_3 \; \text{depth}_{\mathbb{F}}(p/q)}{3\sqrt{\pi/2}} \leq d_{\overline{\text{WP}}}(0, p/q) \leq 2\sqrt{30}\,\pi^{\frac{3}{4}} \; \text{depth}_{\mathbb{F}}(p/q).$$

We conclude by noting that genus independent upper bounds are obtained in [20] on the extended Weil-Petersson distance between maximally noded surfaces in terms of the *cubical pants graph* $CP(S)$, a modification of the usual pants graph obtained by adding diagonals of standard Euclidean n-cubes corresponding to commuting families of elementary moves, as in [20, §4].

As a direct consequence of [20, Lemma 4.1] explicit constants can be given here in terms of the bounds on the length $\ell_{\text{WP}}(I)$, giving

$$d_{\overline{\text{WP}}}(N(P_1), N(P_2)) \leq \sqrt{2\ell_{\text{WP}}(I)}\, d_{CP}(P_1, P_2) \leq 2\sqrt[4]{30}\pi^{\frac{3}{8}} d_{CP}(P_1, P_2)$$

where $N(P)$ represents the point in the Weil-Petersson completion corresponding to the maximal noded surface pinched along P, and $d_{CP}(.,.)$ denotes the distance in the cubical pants graph.

The upper bounds on Weil-Petersson distance between noded surfaces in the above cases arise from a Lipschitz map of the pants graph (and cubical pants graph) into the completion of the Weil-Petersson metric on Teichmüller space. The lower bound in Theorem 5.1, however, makes use of separation properties for this Lipschitz embedding in the 2-dimensional ambient space, while the higher dimensional cases of the lower bound rely on compactness arguments in moduli space. It is interesting to imagine how one might attempt more explicit lower bounds in the general case without making use of the separation properties present in the cases of the one-holed torus and four-holed sphere.

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