Math. Res. Lett.
Volume 23, Number 2, 473–497, 2016

Deformations of Lagrangian subvarieties of holomorphic symplectic manifolds

Christian Lehn

We generalize Voisin's theorem on deformations of pairs of a symplectic manifold and a Lagrangian submanifold to the case of Lagrangian normal crossing subvarieties. Partial results are obtained for arbitrary Lagrangian subvarieties. We apply our results to the study of singular fibers of Lagrangian fibrations.

Introduction

In [Vo92], Voisin studied deformations of pairs $Y \subset X$ where X is an irreducible symplectic manifold and Y a complex Lagrangian submanifold. She showed that, roughly speaking, deformations of X where Y stays a complex submanifold are exactly those deformations, where Y stays Lagrangian. We generalize Voisin's results to Lagrangian subvarieties with normal crossings.

Let $\pi : \mathfrak{X} \longrightarrow M = \mathrm{Def}(X)$ be the universal deformation of X. By the Bogomolov-Tian-Todorov theorem, see [Bog78, Tia87, Tod89], we know that M is smooth. Let $\omega \in R^2\pi_*\mathbb{C}_{\mathfrak{X}} \otimes \mathcal{O}_M$ be a class restricting to a symplectic form on the fibers of π. For a subvariety $i : Y \hookrightarrow X$ denote by $\mathrm{Def}^{\mathrm{lt}}(i)$ the base of the universal locally trivial deformation of i and by $p : \mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M$ the forgetful map. Then we have

Theorem 4.3. *Let $i : Y \hookrightarrow X$ be a normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, let $\nu : \widetilde{Y} \longrightarrow Y$ be the normalization and denote $j = i \circ \nu$. Consider the subspaces*

$$M_Y := \mathrm{im}(\mathrm{Def}^{\mathrm{lt}}(i) \xrightarrow{p} M) \text{ and } M'_Y := \left\{ t \in M : j^*\omega_t = 0 \text{ in } H^2(\widetilde{Y}, \mathbb{C}) \right\}$$

2010 Mathematics Subject Classification: 53D05, 32G10, 13D10, 14C30.
Key words and phrases: irreducible symplectic manifolds, lagrangian subvarieties, deformations.

of M. Then $M'_Y = M_Y$ and this space is smooth of codimension

$$\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y = \operatorname{rk}\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right)$$

in M.

The space M'_Y can be thought of as parametrizing those deformations for which Y remains Lagrangian. We are especially interested in the space M_Y; it parametrizes deformations of X such that Y deforms along with it in a locally trivial manner, so in particular, keeping its singularities. We interpret this space as an invariant of the singularities of Y. Therefore, considering locally trivial deformations is not a restriction but has a geometric meaning. Note that if Y is smooth, then every deformation is locally trivial. This is why the above theorem is a generalization of [Vo92, 0.1 Théorème].

Many of the intermediate steps in the proof of Theorem 4.3 are essentially as in [Vo92], but for the smoothness of M_Y we have to argue differently. For this, we develop ideas of Ran [Ra92Lif], [Ra92Def] by exploiting the interplay between deformation theory and Hodge theory.

This is also where there normal crossing hypothesis comes from. We show in Proposition 3.6 that locally trivial deformations of the Lagrangian subvariety Y inside X are determined by the sheaf $\widetilde{\Omega}_Y$. Its relation to Hodge theory is specific to the normal crossing case. Easy examples show that this is no longer the case for other types of singularities, see Example 5.8. The necessary tools to apply Hodge theoretical arguments over an Artinian base were developed in [Le12].

As in [Vo92], we deduce the following

Corollary 4.4. *Let $K := \ker\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right)$, let q be the Beauville-Bogomolov quadratic form and consider the period domain*

$$Q := \{\alpha \in \mathbb{P}(H^2(X,\mathbb{C})) \mid q(\alpha) = 0, q(\alpha + \bar{\alpha}) > 0\}$$

of X. Then the period map $\wp : M \longrightarrow Q$ identifies M_Y with $\mathbb{P}(K) \cap Q$ locally at $[X] \in M$.

A normal crossing Lagrangian subvariety in a symplectic manifold is quite special: it cannot have more than two local branches, see Lemma 3.2. I am grateful to Claire Voisin for pointing this out. However, these are still the most important degenerations of Lagrangian submanifolds. For example, the majority of singular fibers of Lagrangian fibrations have normal crossings by the results of Hwang-Oguiso [HO09]; so our results apply. A considerable part of Theorem 4.3 holds true for arbitrary Lagrangian subvarieties. More precisely, we have for any Lagrangian subvariety $Y \subset X$ that

$$(M_Y)_{\mathrm{red}} \subset M'_Y \text{ and } \operatorname{codim}_M M'_Y = \operatorname{rk}\left(H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right),$$

so that we can at least bound the codimension of M_Y, the space we are interested in, from below, see Theorem 2.6. This enables us to deduce results about Lagrangian fibrations.

Theorem 5.7. *Let X be an irreducible symplectic manifold and let $f : X \longrightarrow B$ be a Lagrangian fibration. Then X can be deformed, keeping the fibration, to an irreducible symplectic manifold X' with a Lagrangian fibration $f' : X' \longrightarrow B'$ such that outside a codimension 2 subset $Z \subset B'$, all singular fibers of f' over the complement of Z are of Kodaira type I, II, III or IV.*

This result is based on the Kodaira-type classification of singular fibers by Hwang-Oguiso, see Section 5 for details. Similar results on Lagrangian fibrations were independently obtained by Justin Sawon [Saw15] by completely different methods.

Furthermore, the projectivity of arbitrary Lagrangian subvarieties of an irreducible symplectic manifold is shown.

Theorem 1.1. *Let $i : Y \hookrightarrow X$ be a complex Lagrangian subvariety in an irreducible symplectic manifold. Then Y is a projective algebraic variety.*

This is used to apply results from [Le12], but is also interesting in its own right. Again, the statement was known to Voisin in the smooth case.

Let us spend some words about the structure of this article. In Section 1 we show that a Lagrangian subvariety in an irreducible symplectic manifold is always projective. Section 2 is basically an adaptation of Voisin's results from [Vo92] to our setting. The main new results of this article are contained in Sections 3 and 4. In Section 3 we prove smoothness of $\mathrm{Def}^{\mathrm{lt}}(i)$ in case Y has normal crossings using the T^1-lifting principle. It also enables us to deduce that the canonical map $p : \mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M$ has constant rank in a neighbourhood of the distinguished point, which implies the smoothness of the image M_Y. Section 4 finally puts together all previous theory to prove Theorem 4.3 along the lines of Voisin's original argument with some additional input from Hodge theory and deformations of normal crossing varieties. We give applications to Lagrangian fibrations in Section 5. First, we relate deformations of a singular fiber to deformations of the fibration and then we try to deform away from very singular fibers. Our results can be applied to most types of the general singular fibers of a Lagrangian fibration in the sense of Hwang-Oguiso [HO09].

Notations and conventions

We work over the field $k = \mathbb{C}$ of complex numbers. The term *algebraic variety* will stand for a separated reduced k-scheme of finite type. In particular, a variety may have several irreducible components. Similarly, a *complex variety* will be a separated reduced complex space. If there is no danger of confusion, we will drop the adjectives *algebraic* respectively *complex*. A variety Y of equidimension n is called a *normal crossing variety* if for every closed point $y \in Y$ there is an $r \in \mathbb{N}_0$ such that $\widehat{\mathcal{O}}_{Y,y} \cong k[[y_1, \ldots, y_{n+1}]]/(y_1 \cdots y_r)$. It is called a *simple normal crossing variety* if in addition every irreducible component is nonsingular.

1. Projectivity of Lagrangian subvarieties

Let X be an irreducible symplectic manifold, that is, a compact, simply connected Kähler manifold such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega$ for a symplectic form ω. A Lagrangian subvariety $i : Y \hookrightarrow X$ is a subvariety of dimension $\frac{\dim X}{2}$ such that $i^*\omega \in H^0(\Omega_Y^2)$ vanishes on $Y^{\text{reg}} \subset Y$.

If $Y \subset X$ is a smooth Lagrangian subvariety, then by an argument of Voisin, Y is projective even if X is only Kähler, see [Cam06, Prop 2.1]. If $Y \subset X$ is a singular Lagrangian subvariety, it is natural to ask whether Y is still projective. The following affirmative answer to this question is used in the proof of our main theorem, but also interesting in its own right. The proof is a careful adaptation of Voisin's argument to the singular setting.

Theorem 1.1. *Let $i : Y \hookrightarrow X$ be a complex Lagrangian subvariety in an irreducible symplectic manifold. There is a line bundle L on Y such that $\mathrm{c}_1(L) = i^*\lambda$ for some Kähler class λ on X. In particular, Y is a projective algebraic variety.*

Proof. Isomorphism classes of line bundles on Y are classified by the group $H^1(Y, \mathcal{O}_Y^\times)$, see [GR77, Kap V, § 3.2]. This cohomology group appears in the commutative diagram

$$\begin{CD}
\cdots @>>> H^1(Y,\mathcal{O}_Y^\times) @>>> H^2(Y,\mathbb{Z}) @>>> H^2(Y,\mathcal{O}_Y) @>>> \cdots \\
@. @. @VVV @AAA \\
@. @. H^2(Y,\mathbb{C}) @>>> \mathbb{H}^2(Y,\Omega_Y^\bullet) \\
@. @. @. @AAA \\
@. @. @. \mathbb{H}^2(Y,\Omega_Y^{\geq 1})
\end{CD}$$

where the first line is the long exact sequence associated with the exponential sequence, see [GR77, Kap V, § 2.4], and the right vertical column comes from the short exact sequence

$$0 \longrightarrow \Omega_Y^{\geq 1} \longrightarrow \Omega_Y^\bullet \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

To obtain a holomorphic line bundle L on Y it is sufficient to find a class $\alpha \in H^2(Y,\mathbb{Z})$ whose image in $\mathbb{H}^2(Y,\Omega_Y^\bullet)$ comes from $\mathbb{H}^2(Y,\Omega_Y^{\geq 1})$. Such L will have $\mathrm{c}_1(L) = \alpha$.

As X is Kähler, so is Y. Hence, the $H^k(Y,\mathbb{Q})$ carry a mixed Hodge structure. Let us consider a resolution of singularities $\pi : \widetilde{Y} \longrightarrow Y$. If $W_m \subset H^2(Y,\mathbb{C})$ denotes the weight filtration, then π^* factors as

$$\pi^* : H^2(Y,\mathbb{C}) \twoheadrightarrow H^2(Y,\mathbb{C})/W_1 \hookrightarrow H^2(\widetilde{Y},\mathbb{C}).$$

As Y is Lagrangian, we have $i^*\omega = 0$ in $H^2(Y,\mathbb{C})$ where $\omega \in H^0(X,\Omega_X^2) = H^{2,0}(X)$ is the symplectic form on X. Indeed, it maps to 0 in $H^2(\widetilde{Y},\mathbb{C})$ and as X is smooth and morphisms of Hodge structures are strict, $i^*\omega$ is in W_1 if and only if it is zero. Consequently, also $H^{0,2}(X)$ maps to 0 in $H^2(Y,\mathbb{C})$, because it is the complex conjugate of $H^{2,0}(X)$.

Let us look at the composition $r : H^2(X,\mathbb{C}) \longrightarrow H^2(Y,\mathbb{C}) \longrightarrow \mathbb{H}^2(Y,\Omega_Y^\bullet)$ and let $H \subset \mathbb{H}^2(Y,\Omega_Y^\bullet)$ denote the image of $H^2(X,\mathbb{R})$. By the Hodge-theoretic considerations above, the image of the Kähler cone $r(\mathcal{K}_X)$ is open in $H = r(H^{1,1}(X,\mathbb{R}))$ and clearly, $r(H^2(X,\mathbb{Q}))$ is dense in H so that there is some $0 \neq \alpha' \in r(\mathcal{K}_X) \cap r(H^2(X,\mathbb{Q}))$. Then a multiple $\alpha = m \cdot \alpha'$ is contained in $r(H^2(X,\mathbb{Z})) \cap r(\mathcal{K}_X)$ and we obtain a line bundle L on Y with the desired property by using the exponential sequence as explained above.

We conclude that Y is projective by [GPR94, Chapter V, Corollary 4.5], see also [Gra62, 3, Satz 1 and Satz 2]. □

2. Deformations of symplectic manifolds and Lagrangian subvarieties

As $H^0(X, T_X) = 0$ for an irreducible symplectic manifold X, the Kuranishi family $\pi : \mathfrak{X} \longrightarrow M = \mathrm{Def}(X)$ is universal at the point $0 \in M$ corresponding to X. Close to $0 \in M$ the fibers of π are again irreducible symplectic manifolds, see [Bea83, § 8]. M is known to be smooth by the Bogomolov-Tian-Todorov theorem [Bog78, Tia87, Tod89], see also [GHJ, Thm 14.10]. Therefore, $\dim M = \dim T_{M,0} = h^1(T_X) = h^{1,1}(X)$.

2.1. Deformations of closed immersions

We refer to [Ser06] for an introduction to deformation theory. By Art_k we denote the category of local Artinian k-algebras with residue field k. Let $i : Y \hookrightarrow X$ be a closed immersion of algebraic k-schemes and suppose that X is smooth and proper. Let $R \in \mathrm{Art}_k$. A *deformation of* i over $S = \mathrm{Spec}\, R$ is a diagram

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & \mathcal{X} \\ & \searrow & \downarrow \\ & & S, \end{array} \tag{2.1}$$

where $\mathcal{X} \longrightarrow S$ and $\mathcal{Y} \longrightarrow S$ are flat and the fiber of (2.1) over $k = R/\mathfrak{m}_R$ is isomorphic to $i : Y \hookrightarrow X$. Such a deformation is *locally trivial* if for every $x \in X$, $y \in Y$ with $i(y) = x$ there is an open subset $U \subset X$ and such that $x \in U$ and the restriction $\mathcal{Y}|_U \hookrightarrow \mathcal{X}|_U$ is a trivial deformation of $Y \cap U \hookrightarrow U$.

The tangent space to the deformation space of locally trivial deformations of a proper k-variety Y is given by $H^1(T_Y)$, see [Ser06, Proposition 1.2.9]. Let $i : Y \hookrightarrow X$ be a closed immersion with smooth X and let $T_X\langle Y\rangle$ be the kernel of the natural map $T_X \longrightarrow i_* N_{Y/X}$, see [Ser06, 3.4.4]. Then the tangent space to the deformation space for locally trivial deformations of i is given by $H^1(T_X\langle Y\rangle)$, see [Ser06, Proposition 3.4.17]. More generally, given a diagram like (2.1), we define $T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle$ by the exact sequence of sheaves on $\mathcal{X}$

$$0 \longrightarrow T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle \longrightarrow T_{\mathcal{X}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}} \longrightarrow 0, \tag{2.2}$$

where

$$N'_{\mathcal{Y}/\mathcal{X}} := \ker(N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S}) \tag{2.3}$$

is the *equisingular normal sheaf*.

Let $i : Y \hookrightarrow X$ be the inclusion of a closed subvariety in an irreducible symplectic manifold. Then, as a consequence of [FK87], there is a universal locally trivial deformation of i over a (germ of a) complex space $\mathrm{Def}^{\mathrm{lt}}(i)$. The inclusion $Y \hookrightarrow X$ gives a point $0 \in \mathrm{Def}^{\mathrm{lt}}(i)$. By construction there is a forgetful morphism $p : \mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M$ of complex spaces with $p(0) = 0$.

Definition 2.2. We denote by $M_Y \subset M$ the image of p, that is, the smallest closed complex subspace such that p factors through $M_Y \hookrightarrow M$.

2.3. The locus where a subvariety is Lagrangian

Let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian subvariety in an irreducible symplectic manifold X of dimension $2n$.

We take a flat section $0 \neq \omega \in R^0\pi_*\Omega^2_{\mathfrak{X}/M} \hookrightarrow \mathscr{H}^2 := R^2\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \otimes \mathcal{O}_M$ and write $\omega_t := \omega|_{X_t}$ for the symplectic form on the fiber $X_t = \pi^{-1}(t)$. By parallel transport, we interpret ω_t as an element of $H^2(X, \mathbb{C})$. Let $[Y] \in H^{2n}(X, \mathbb{Z})$ denote the Poincaré dual of the fundamental cycle of Y. It has a unique flat lift to $\mathscr{H}^{2n} := R^{2n}\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \otimes \mathcal{O}_M$ and we denote by $[Y]_t$ the restriction of this lift to the fiber $\mathscr{H}^{2n}_t = H^{2n}(X_t, \mathbb{C})$ of $\mathscr{H}^{2n}$ at t. Let us denote by $\nu : \widetilde{Y} \longrightarrow Y$ a resolution of singularities and put $j = i \circ \nu$.

Definition 2.4. With these notations following Voisin [Vo92] we define

$$M'_Y := \left\{ t \in M \mid j^*\omega_t = 0 \text{ in } H^2(\widetilde{Y}, \mathbb{C}) \right\}. \tag{2.4}$$

Clearly, this definition is independent of the resolution $\nu : \widetilde{Y} \longrightarrow Y$.

If we think of deformations of X as deformations of its complex structure on the underlying differentiable manifold, then we may interpret the space M'_Y as parametrizing those deformations X_t for which the differentiable subvariety underlying Y remains Lagrangian with respect to the (holomorphic) symplectic structure ω_t on X_t.

Lemma 2.5. The tangent space of M'_Y at 0 is given by

$$T_{M'_Y,0} = \ker\left(H^1(T_X) \xrightarrow{\omega'} H^1(\Omega_X) \xrightarrow{j^*} H^1(\Omega_{\widetilde{Y}}) \right) \tag{2.5}$$

where ω' is the isomorphism induced by the symplectic form on X.

Proof. Locally at $0 \in M$ the space M'_Y is cut out by the equation $j_t^*\omega_t = 0$. Therefore, the tangent space at 0 is given by the equation

$$0 = (\nabla j_t^* \omega_t)\,|_{t=0} = j^*\,(\nabla \omega_t)\,|_{t=0},$$

where ∇ is the Gauß-Manin connection. At 0 it can be identified with the map $H^0(\Omega_X^2) \longrightarrow \mathrm{Hom}(H^1(T_X), H^1(\Omega_X))$ given by cup product and contraction, which concludes the proof. □

Theorem 2.6. *Let $i : Y \hookrightarrow X$ be a Lagrangian subvariety in a compact irreducible symplectic manifold X, let $\nu : \widetilde{Y} \longrightarrow Y$ be a resolution of singularities and denote $j = i \circ \nu$. Then $(M_Y)_{\mathrm{red}} \subset M'_Y$ and M'_Y is smooth of codimension*

$$\mathrm{codim}_M\, M'_Y = \mathrm{rk}\left(H^2(X, \mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y}, \mathbb{C})\right) \tag{2.6}$$

in M.

Proof. First assume that Y is irreducible. Voisin shows in [Vo92, Propositions 1.2 and 1.7] that M'_Y is a smooth submanifold of M and that it coincides with the Hodge locus $M_{[Y]}$ associated with the class $[Y]$ of Y in $H^{2n}(X, \mathbb{C})$, see [Vo2, Ch 5.3]. Smoothness of Y is not needed for the first proposition, as its proof only uses the class of Y. The second one uses [Vo92, Lemme 1.5] which has to be replaced by Lemma 2.7 below.

Now let $Y = \cup_i Y_i$ be a decomposition into irreducible components. Then set-theoretically we have the following inclusions:

$$M_Y \subset \bigcap_i M_{Y_i} \subset \bigcap_i M_{[Y_i]} = \bigcap_i M'_{Y_i} = M'_Y. \tag{2.7}$$

Let us explain this briefly. Indeed, the first inclusion is a consequence of [Le12, Lemma 1.4], which states that a locally trivial deformation of a variety induces a flat (and, in fact, locally trivial) deformation of any of its irreducible components. In loc. cit. this is proven over an Artinian base scheme which immediately implies the analogous statement over an open subset in the base for general base spaces. The inclusion $M_{Y_i} \subset M_{[Y_i]}$ is obvious and the equalities in (2.7) follow from the irreducible case and the definition of M'_Y. The statement about the codimension is deduced from the description (2.5) of the tangent space of M'_Y. □

The following straight-forward generalization of [Vo92, Lemme 1.5] will complete the proof of Theorem 2.6. We include a full proof for convenience.

Lemma 2.7. Let X be a compact Kähler manifold of dimension m, let $Y \subset X$ be a closed subvariety of dimension n, and let $\widetilde{Y} \longrightarrow Y$ be a resolution of singularities. We denote by j the composition $\widetilde{Y} \longrightarrow Y \longrightarrow X$ and by $\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2(m-n)+2}(X,\mathbb{C})$ the map given by cup product with $[Y]$. If Y is irreducible, then

$$\ker\left(H^2(X,\mathbb{C}) \xrightarrow{\mu} H^{2(m-n)+2}(X,\mathbb{C})\right) = \ker\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right).$$

Proof. We show equality of the respective kernels with real coefficients. Let us first observe that μ is the composition $H^2(X,\mathbb{R}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{R}) \xrightarrow{j_*} H^{2(m-n)+2}(X,\mathbb{R})$. From this we immediately have $\ker j^* \subset \ker \mu$. For the other inclusion we have to show that j_* is injective on $\operatorname{im} j^*$. We consider two cases seperately, depending on the dimension n of Y.

Assume $n = 1$. As $\widetilde{Y}$ is connected, $H^2(\widetilde{Y},\mathbb{R}) \cong \mathbb{R}$ and the map $j_* : H^2(\widetilde{Y},\mathbb{R}) \longrightarrow H^{2m}(X,\mathbb{R})$ maps the class of a point to the class of a point and hence is an isomorphism so that the claim follows.

If $n \geq 2$, choose a Kähler class $\kappa \in H^2(X,\mathbb{R})$. We may assume that $\widetilde{Y} \longrightarrow Y$ is obtained by a sequence of blow-ups in smooth centers and that the exceptional locus is pure of codimension one. Hence there is a Kähler class of the form $\widetilde{\kappa} = j^*\kappa - \sum_i \delta_i E_i \in H^2(\widetilde{Y},\mathbb{R})$ where the E_i are exceptional divisors and $\delta_i \in \mathbb{Q}$ are positive. We define a bilinear form

$$q(\alpha,\beta) := \int_{\widetilde{Y}} \widetilde{\kappa}^{n-2}.\alpha.\beta \qquad \alpha,\beta \in H^2(\widetilde{Y},\mathbb{C})$$

on $H^2(\widetilde{Y},\mathbb{C})$. For $\alpha,\beta \in H^2(X,\mathbb{R})$ this gives

$$\begin{aligned} q(j^*\alpha, j^*\beta) &= \int_{\widetilde{Y}} \widetilde{\kappa}^{n-2}.j^*(\alpha.\beta) = \int_X j_*\left(\widetilde{\kappa}^{n-2}.j^*(\alpha.\beta)\right) \\ &= \int_X \mu(\kappa^{n-2}).\alpha.\beta = \int_X \kappa^{n-2}.\mu(\alpha).\beta. \end{aligned}$$

So we see that if $\mu(\alpha) = 0$, then $q(j^*\alpha, j^*\beta) = 0$ for all $\beta \in H^2(X,\mathbb{R})$. To conclude that $j^*\alpha = 0$ it would be sufficient to see that q is non-degenerate on $\operatorname{im} j^* \subset H^2(\widetilde{Y},\mathbb{R})$. On the whole of $H^2(\widetilde{Y},\mathbb{R})$ the form q is non-degenerate by the hard Lefschetz theorem and Poincaré duality. As q is positive definite on $(H^{2,0} \oplus H^{0,2})$-part of $H^2(\widetilde{Y},\mathbb{R})$ and as $\operatorname{im} j^*$ is a sub-Hodge structure, it suffices to show that q is non-degenerate on the $(1,1)$-part of $\operatorname{im} j^*$. As q is non-degenerate on $H^{1,1}(\widetilde{Y},\mathbb{R})$ of signature $(1, h^{1,1} - 1)$ and as $q(j^*\kappa, j^*\kappa) > 0$ we see that there is a q-orthogonal decomposition $H^{1,1}(\widetilde{Y},\mathbb{R}) = \mathbb{R}\langle j^*\kappa\rangle \oplus V$ and q is negative definite on V. Clearly, if we decompose any $j^*\alpha = \lambda \cdot$

$j^*\kappa + v$ accordingly, then $v \in \operatorname{im} j^*$ and thus $\operatorname{im} j^* = \mathbb{R}\langle j^*\kappa\rangle \oplus (\operatorname{im} j^* \cap V)$. In particular, q is non-degenerate on $\operatorname{im} j^*$ and the claim follows. $\square$

3. Normal crossing subvarieties

Our next goal is to prove smoothness of the space $\mathrm{Def}^{\mathrm{lt}}(i)$ of locally trivial deformations of $i : Y \hookrightarrow X$, see Theorem 3.8, using a variant of the T^1-lifting principle. Smoothness plays an important role in the proof of our main result, Theorem 4.3. We start with some preliminary considerations on normal crossing varieties.

Definition 3.1. Let $f : \mathcal{Y} \longrightarrow S$ be a proper morphism of schemes. We define $\tau^k_{\mathcal{Y}/S} \subset \Omega^k_{\mathcal{Y}/S}$ to be the subsheaf of sections whose support is contained in the singular locus of f. We put $\widetilde{\Omega}^k_{\mathcal{Y}/S} := \Omega^k_{\mathcal{Y}/S}/\tau^k_{\mathcal{Y}/S}$. Clearly, the exterior differential makes $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ into a complex.

If Y is a normal crossing $\mathbb{C}$-variety, then the natural map $\mathbb{C} \longrightarrow \widetilde{\Omega}^\bullet_Y$ is a resolution. Moreover, if Y is proper, this complex can be used to define the mixed Hodge structure on $H^k(Y, \mathbb{C})$ as it has been done in [Fri83] if Y has simple normal crossings. For a locally trivial deformation $f : \mathcal{Y} \longrightarrow S$ of a simple normal crossing variety over an Artinian base scheme, it has been shown in [Le12] that $\widetilde{\Omega}^\bullet_{\mathcal{Y}/S}$ is a resolution of the constant sheaf $f^{-1}\mathcal{O}_S$ and the Hodge theoretic analogues of Friedman's results have been established. It is possible to extend these results to the normal crossing case, i.e. components are allowed to have self-intersections. For this, let Y be a normal crossing variety. We need a semi-simplicial resolution $\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} Y^{[1]} \rightrightarrows Y^{[0]} \rightarrow Y$ which replaces the canonical one in the simple normal crossing case, see [Fri83, p. 77] and [Le12, 4.4]. For Lagrangian subvarieties the situation is very simple. I am grateful to Claire Voisin for this observation. Its proof is straightforward, cf. [GLR14, Lemma 5.3].

Lemma 3.2. If $Y \subset X$ is a Lagrangian subvariety with normal crossings in a symplectic manifold X, then locally there cannot be more than two components. $\square$

Remark 3.3. We thus obtain a semi-simplicial resolution where $\nu : Y^{[0]} \longrightarrow Y$ is the normalization, $Y^{[1]} := \nu^{-1}(Y^{\mathrm{sing}})$ and the morphisms $Y^{[1]} \rightrightarrows Y^{[0]}$ are the inclusion and its composition with the canonical involution $\tau : Y^{[1]} \longrightarrow Y^{[1]}$ exchanging the two branches. Using this resolution, the Hodge theoretic results from [Le12] carry over to the normal crossing situation.

Let $\mathcal{X} \longrightarrow S = \operatorname{Spec} R$ for $R \in \operatorname{Art}_k$ be a deformation of an irreducible symplectic manifold X and let $\omega \in H^0(\Omega^2_{\mathcal{X}/S})$ be a relative symplectic form.

Lemma 3.4. Let $i : Y \hookrightarrow X$ be a normal crossing Lagrangian subvariety. If $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a locally trivial deformation of i over S, then $\mathcal{Y}$ is Lagrangian with respect to the symplectic form ω on $\mathcal{X}$.

Proof. Let $\widetilde{\mathcal{Y}} \longrightarrow S$ be the locally trivial deformation of the normalization of Y obtained from [Le12, Lemma 4.5]. Note that Y is projective by Theorem 1.1, so Lemma [Le12, Lemma 4.5] can be applied. As Y has normal crossings, $\widetilde{\mathcal{Y}} \longrightarrow S$ is smooth and $H^0(\Omega^2_{\mathcal{X}/S})$ and $H^0(\Omega^2_{\widetilde{\mathcal{Y}}/S})$ are free $\mathcal{O}_S$ modules by [Del68, Théorème 5.5]. Therefore, the pullback $H^0(\Omega^2_{\mathcal{X}/S}) \longrightarrow H^0(\Omega^2_{\widetilde{\mathcal{Y}}/S})$ has constant rank by [Le12, Theorem 4.17]. As $\operatorname{rk}(j^* \otimes \mathbb{C}) = 0$ on the central fiber, j^* is identically zero and thus $\mathcal{Y}$ is Lagrangian. □

Lemma 3.5. Let $i : Y \hookrightarrow X$ be a Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \operatorname{Spec} R$ where $R \in \operatorname{Art}_{\mathbb{C}}$ and let $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a locally trivial deformation of i over S. Then the symplectic form $\omega \in H^0(\Omega^2_{\mathcal{X}/S})$ induces a morphism between the exact sequences

$$\begin{array}{ccccccccc} & & \mathcal{I}/\mathcal{I}^2 & \longrightarrow & \Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} & \longrightarrow & \Omega_{\mathcal{Y}/S} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle \omega^{-1}} & & \downarrow {\scriptstyle \omega^{-1}} & & \downarrow {\scriptstyle \omega'} & & \\ 0 & \longrightarrow & T_{\mathcal{Y}/S} & \longrightarrow & T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} & \xrightarrow{\alpha} & N_{\mathcal{Y}/\mathcal{X}} & \longrightarrow & T^1_{\mathcal{Y}/S} \longrightarrow 0. \end{array} \tag{3.1}$$

Proof. Since ω is non-degenerate, the map $\omega^{-1} : \Omega_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}/S}$ is an isomorphism. The composition $\varphi : \mathcal{I}/\mathcal{I}^2 \longrightarrow N_{\mathcal{Y}/\mathcal{X}} = \operatorname{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}})$ is given by $f \mapsto \{f, \cdot\}$ where $\{\cdot, \cdot\}$ is the Poisson bracket associated with ω. So $\varphi = 0$ and the restriction of ω^{-1} to $\mathcal{I}/\mathcal{I}^2$ factors through $T_{\mathcal{Y}/S} = \ker \alpha$. Once we have this, we obtain a morphism $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$, as the first line of (3.1) is exact, by lifting sections to $\Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}$. □

It is well-known that if in the situation of the preceding lemma the morphism $f : \mathcal{Y} \longrightarrow S$ is smooth, then ω gives an isomorphism $\Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$. The following Proposition 3.6 explains what happens for singular Lagrangian subvarieties.

Proposition 3.6. Let $i : Y \hookrightarrow X$ be a Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \operatorname{Spec} R$ where $R \in \operatorname{Art}_{\mathbb{C}}$ and let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over S. Let $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$

be as in (3.1) and let $N'_{\mathcal{Y}/\mathcal{X}}$ be the equisingular normal sheaf defined in (2.3). Then the diagram

$$\begin{array}{ccc} \Omega_{\mathcal{Y}/S} & \xrightarrow{\omega} & N_{\mathcal{Y}/\mathcal{X}} \\ \downarrow & & \uparrow \\ \widetilde{\Omega}_{\mathcal{Y}/S} & \underset{\exists \widetilde{\omega}}{\dashrightarrow} & N'_{\mathcal{Y}/\mathcal{X}} \end{array} \tag{3.2}$$

can be completed and $\widetilde{\omega} : \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}}$ is an isomorphism.

Proof. As Y is Lagrangian, it is of pure dimension and thus $\mathcal{O}_Y$ and also $\mathcal{O}_{\mathcal{Y}}$ have no embedded primes. Locally, the sheaf $N_{\mathcal{Y}/\mathcal{X}}$ can be embedded in a locally free sheaf and thus it does not have any embedded primes either. Hence, $\tau^1_{\mathcal{Y}/S}$ maps to zero and $\widetilde{\omega}$ exists. But as ω is an isomorphism at smooth points of f, the support of $\ker\omega$ is contained in the singular locus of f, hence $\ker\omega \subset \tau^k_{\mathcal{Y}/S}$ and $\widetilde{\omega}$ is injective. Moreover, $\widetilde{\Omega}_{\mathcal{Y}/S}$ maps onto $\ker(N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S})$ by (3.1), hence is identified with $N'_{\mathcal{Y}/\mathcal{X}}$. □

3.7. The T^1-lifting Principle

To prove smoothness of $\mathrm{Def}^{\mathrm{lt}}(i)$ we will use Ran's T^1-lifting principle [Ra92Def, Kaw92, Kaw97], we refer to [Le11] for a gentle introduction. The basic idea is that in order to prove smoothness of a deformation functor it suffices to show that the corresponding T^1-modules are locally free for every infinitesimal deformation over a local Artinian scheme. This is achieved by means of Hodge theory.

Theorem 3.8. *Let Y be a Lagrangian normal crossing subvariety of an irreducible symplectic manifold X. Then the complex space $\mathrm{Def}^{\mathrm{lt}}(i)$ is smooth at 0. Moreover, M_Y is smooth and $\mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M_Y$ is a submersion.*

Proof. We have to show that the $H^1(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle)$ are free. The sheaf $T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle$ was defined in (2.2). Let $i : Y \hookrightarrow X$ be the inclusion and let $\mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$. Consider the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle) &\longrightarrow H^0(T_{\mathcal{X}/S}) \\ &\longrightarrow H^0(N'_{\mathcal{Y}/\mathcal{X}}) \longrightarrow H^1(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle) \longrightarrow \cdots \end{aligned} \tag{3.3}$$

obtained from the sequence (2.2). We transform this sequence using the isomorphism $T_{\mathcal{X}/S} \cong \Omega_{\mathcal{X}/S}$, Lemma 3.4, Proposition 3.6 and $H^0(\Omega_{\mathcal{X}/S}) = 0$ to obtain an exact sequence

$$(3.4) \qquad 0 \longrightarrow H^0(\widetilde{\Omega}_{\mathcal{Y}/S}) \longrightarrow H^1(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle) \\ \longrightarrow H^1(\Omega_{\mathcal{X}/S}) \longrightarrow H^1(\widetilde{\Omega}_{\mathcal{Y}/S}) \longrightarrow \cdots$$

Recall from Definition 2.2 that we have a factorization $p : \mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M_Y \hookrightarrow M$. However, in general it is not clear whether $\mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M_Y$ is surjective, let alone submersive. This is because M_Y is by definition a complex subspace but in general, the set-theoretic image is only a constructible set. By Theorem [Del68, Théorème 5.5] we know that $H^k(\Omega_{\mathcal{X}/S})$ is free. By Theorem 1.1 we know that Y is a projective variety, so Theorem [Le12, Theorem 4.13] applies and $H^k(\widetilde{\Omega}_{\mathcal{Y}/S})$ is free. Note that the results of [Le12] carry over literally to the normal crossing case as was explained in Remark 3.3. Then by Theorem [Le12, Theorem 4.22] also the cokernel (and hence the kernel) of $H^k(\Omega_{\mathcal{X}/S}) \longrightarrow H^k(\widetilde{\Omega}_{\mathcal{Y}/S})$ is free. From sequence (3.4) we deduce that all $H^k(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle)$ are free and that all morphisms in (3.4) have constant rank. In particular, all morphisms in (3.3) have constant rank. The T^1-lifting principle implies that $\mathrm{Def}^{\mathrm{lt}}(i)$ is smooth.

So the canonical morphism $p : (\mathrm{Def}^{\mathrm{lt}}(i), 0) \longrightarrow (M, 0)$ is just a holomorphic map between (germs of) complex manifolds. To prove the theorem it suffices to show that the differential Dp has constant rank in a neighbourhood of 0. This holds if the stalk of $\operatorname{coker}(p_* : T_{\mathrm{Def}^{\mathrm{lt}}(i)} \longrightarrow p^*T_M)$ at 0 is free. Freeness may be tested after completion, and then by the local criterion for flatness [Ser06, Thm A.5] we may test it for the truncations modulo powers of the maximal ideal. In other words we have to verify, given as above a locally trivial deformation $\mathcal{Y} \hookrightarrow \mathcal{X}$ of i over $S = \operatorname{Spec} R$ with $R \in \mathrm{Art}_{\mathbb{C}}$, that the map $H^1(T_{\mathcal{X}/S}\langle\mathcal{Y}\rangle) \longrightarrow H^1(T_{\mathcal{X}/S})$ has constant rank. This was already noted in the first part of the proof. □

4. Codimension formula

Let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian subvariety in an irreducible symplectic manifold. In this section we show that if Y has normal crossings, then the inclusion $M_Y \subset M'_Y$ from Theorem 2.6 is an equality. In this way, we obtain a formula for the codimension of M_Y.

Lemma 4.1. Suppose Y has normal crossings. Then

$$\ker\left(H^1(\Omega_X) \xrightarrow{j^*} H^1(\Omega_{\widetilde{Y}})\right) = \ker\left(H^1(\Omega_X) \xrightarrow{i^*} H^1(\widetilde{\Omega}_Y)\right),$$

where $\nu : \widetilde{Y} \longrightarrow Y$ is the normalization.

Proof. As $j^* = \nu^* \circ i^*$ the inclusion $\supset$ is obvious. For the other direction it suffices to show that ν^* is injective on $\operatorname{im} i^*$. By Theorem 1.1 the subvariety Y is projective, hence by [Del71, Del74] there is a functorial mixed Hodge structure on $H^k(Y, \mathbb{C})$ for every k. We denote by $F^\bullet$ the Hodge filtration on $H^2(Y)$ and by $W_\bullet$ the weight filtration. As a special case of [Le12, Cor 4.16], we deduce that

$$H^1(\widetilde{\Omega}_Y) = \mathrm{Gr}^1_F H^2(Y) = F^1H^2(Y)/F^2H^2(Y).$$

Let $\cdots \Rrightarrow Y^1 \rightrightarrows Y^0 \to Y$ be the canonical semi-simplicial resolution in the simple normal crossing case, see e.g. [Le12, 4.8], or the one from Remark 3.3 in the normal crossing case. Note that $\widetilde{Y} = Y^0$. Consider the weight spectral sequence associated with the first graded objects of the Hodge filtration given by

$$E_1^{r,s} = H^s(Y^r, \Omega^1_{Y^r}) \Rightarrow H^{r+s}(Y, \widetilde{\Omega}^1_Y) \tag{4.1}$$

By [PS08, Thm 3.12 (3)] it degenerates on the same level as the weight spectral sequence, which is known to degenerate at E_2. The differential $d_1 : E_1^{0,1} \longrightarrow E_1^{0,1}$ is given by $\delta : H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1})$ and degeneration at E_2 tells us that

$$\begin{aligned}\mathrm{Gr}^W_2 \mathrm{Gr}^1_F H^2(Y) &= F^1H^2(Y)/(W_1F^1H^2(Y) + F^2H^2(Y)) = E_\infty^{0,1} = E_2^{0,1}\\ &= \ker\left(H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1})\right).\end{aligned}$$

In other words, as $W_2\mathrm{Gr}^1_F H^2(Y) = \mathrm{Gr}^1_F H^2(Y) = H^1(\widetilde{\Omega}_Y)$ there is an exact sequence

$$0 \longrightarrow W_1\mathrm{Gr}^1_F H^2(Y) \longrightarrow H^1(\widetilde{\Omega}_Y) \xrightarrow{\nu^*} H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1}),$$

so that $\ker \nu^* = W_1\mathrm{Gr}^1_F H^2(Y)$. But the Hodge structure on $H^2(X, \mathbb{C})$ has pure weight two because X is smooth. In particular, $W_1\mathrm{Gr}^1_F H^2(X) = 0$.

Morphisms of mixed Hodge structures are strict with respect to both filtrations, so we have

$$0 = i^*(W_1 \mathrm{Gr}^1_F H^2(X)) = \mathrm{im}\, i^* \cap W_1 \mathrm{Gr}^1_F H^2(Y) = \mathrm{im}\, i^* \cap \ker \nu^*$$

hence ν^* is injective on $\mathrm{im}\, i^*$ and we deduce $\ker i^* = \ker j^*$ completing the proof. □

The following lemma generalizes [Vo92, Lem 2.3] to the normal crossing case.

Lemma 4.2. Suppose Y has normal crossings. Then we have $T_{M'_Y,0} = T_{M_Y,0}$ for the Zariski tangent spaces at $0 \in M_Y \cap M'_Y$.

Proof. By Lemma 2.5 the tangent space of M'_Y at 0 is

$$T_{M'_Y,0} = \ker\left(j^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\widetilde{\Omega}_Y)\right).$$

By Lemma 4.1 this equals $\ker\left(i^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\Omega_{\widetilde{Y}})\right)$, where $\widetilde{Y} \longrightarrow Y$ is the normalization. On the other hand, M_Y is the smooth image of $p : \mathrm{Def}^{\mathrm{lt}}(i) \longrightarrow M$ so that

$$\begin{aligned} T_{M_Y,0} &= \mathrm{im}\left(p_* : T_{\mathrm{Def}^{\mathrm{lt}}(i),0} \longrightarrow T_{M,0}\right) \\ &= \mathrm{im}\left(H^1(X, T_{\mathcal{X}/S}\langle \mathcal{Y} \rangle) \longrightarrow H^1(X, T_X)\right) \\ &= \ker\left(H^1(X, T_X) \xrightarrow{\alpha} H^1(Y, N'_{Y/X})\right) \end{aligned}$$

where the third equality holds because the sequence (3.3) is exact.

By (3.1) and Proposition 3.6 we have a commutative diagram

$$\begin{array}{ccc} H^1(X, \Omega_X) & \xrightarrow{j^*} & H^1(Y, \widetilde{\Omega}_Y) \\ \uparrow \omega' & & \downarrow \tilde{\omega} \\ H^1(X, T_X) & \xrightarrow{\alpha} & H^1(Y, N'_{Y/X}) \end{array}$$

where the vertical maps are isomorphisms and this completes the proof. □

Theorem 4.3. *Let $i : Y \hookrightarrow X$ be a normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, let $\nu : \widetilde{Y} \longrightarrow Y$ be the* *normalization and denote $j = i \circ \nu$. Then M_Y is smooth at* 0 *of codimension*

$$\operatorname{codim}_M M_Y = \operatorname{rk}\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right) \tag{4.2}$$

in M.

Proof. By Theorems 2.6 and 3.8 we have $M_Y \subset M'_Y$ and it suffices to show equality. This is deduced from $\dim M_Y \leq \dim M'_Y \leq \dim T_{M'_Y,0} = \dim T_{M_Y,0}$, where the last equality comes from Lemma 4.2, again by invoking smoothness of M_Y. □

Corollary 4.4. Let $K := \ker\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right)$, let q be the Beauville-Bogomolov quadratic form and consider the period domain

$$Q := \{\alpha \in \mathbb{P}(H^2(X,\mathbb{C})) \mid q(\alpha) = 0, q(\alpha + \bar{\alpha}) > 0\}$$

of X. Then the period map $\wp : M \longrightarrow Q$ identifies M_Y with $\mathbb{P}(K) \cap Q$ locally at $[X] \in M$.

Proof. As the period map identifies M with Q it suffices to show that $\wp(M_Y) = \mathbb{P}(K) \cap Q$. By [Huy99, 1.14], $\mathbb{P}(K) \cap Q$ is the locus where $K^\perp \subset H^2(X,\mathbb{C})$ remains of type $(1,1)$ and its codimension is $\dim K^\perp$. Note that $K^\perp \subset H^{1,1}(X)$ is defined over $\mathbb{Z}$ and therefore is spanned by the Chern classes of a collection of line bundles on X. By Lemma 3.4 the subspace $K^\perp$ remains of type $(1,1)$ over M_Y. Hence, $\wp(M_Y) \subset \mathbb{P}(K) \cap Q$. Moreover, we have

$$\begin{aligned}\operatorname{codim}_Q \wp(M_Y) &= \operatorname{codim}_M M_Y = \operatorname{rk}\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right)\\ &= b_2(X) - \dim K = \dim K^\perp\\ &= \operatorname{codim}_Q \mathbb{P}(K) \cap Q.\end{aligned}$$

So both sets are equal. □

5. Applications to Lagrangian fibrations

In this section we give some applications of Theorems 2.6 and 4.3 to Lagrangian fibrations. Our main goal is to determine $\operatorname{codim}_M M_Y$. Let X be an irreducible symplectic manifold. Recall that a *Lagrangian fibration* is a morphism $f : X \longrightarrow B$ with connected fibers to a normal projective variety B such that the general fiber of f is a Lagrangian subvariety.

Lagrangian fibrations are an important tool to study irreducible symplectic manifolds. It is conjectured that an arbitrary irreducible symplectic manifold can always be deformed to one that admits a Lagrangian fibration. Moreover, Matsushita has shown in a series of papers [Mat99, Mat00, Mat01, Mat03] that every fibration of an irreducible symplectic manifold is a Lagrangian fibration and that the base B resembles the projective space $\mathbb{P}^n$. The holomorphic Liouville-Arnol'd theorem shows that every smooth fiber is a complex torus, hence singular fibers enter the focus.

Hwang-Oguiso [HO09] classified generic singular fibers of a Lagrangian fibration. For such a fiber they defined the *characteristic cycle*, a (maybe infinite) cycle Θ of curves on the fiber, and they have shown that it is either a Kodaira singular fiber of an elliptic surface or an infinite chain of smooth rational curves intersecting transversally (so-called I_∞-type). Locally, the fiber is isomorphic to $\Theta \times \mathbb{C}^{n-1}$ and the intersection graph of the fiber is a quotient of the graph of the characteristic cycle. The datum of the intersection graph of the fiber together with its local singularities is what we call *fiber type*.

In view of these classification results, Theorem 4.3 applies to the majority of (reductions of) generic singular fibers of a Lagrangian fibration. Only fibers with characteristic cycle of Kodaira types II, III and IV are not normal crossing varieties; for those we have Theorem 2.6. Here it is important that we consider locally trivial deformations. It entails that the fiber type in the Hwang-Oguiso sense does not change so that we obtain an invariant of this fiber type, see Theorem 5.7. Note that this is not in general the case for the characteristic cycle, see [HO10, Proposition 5.3] for an example.

5.1. Deforming fibrations

We show first that if we deform a fiber of a fibration, then also the fibration deforms, see Lemma 5.2. Let $f : X \longrightarrow B$ be a Lagrangian fibration and assume that B is projective. Matsushita showed in [Mat09, Corollary 1.2] that there is a unique hypersurface $\mathrm{Def}(X, f) \subset M$ with a relative Lagrangian fibration extending f

(5.1)
$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\quad F \quad} & P \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & \mathrm{Def}(X, f) & \end{array}$$

where $\pi : \mathfrak{X} \longrightarrow \mathrm{Def}(X, f)$ is the restriction of the universal family to $\mathrm{Def}(X, f)$ and $P \longrightarrow \mathrm{Def}(X, f)$ is a projective morphism. In particular, $F_t : \mathfrak{X}_t \longrightarrow P_t$ is a Lagrangian fibration and $F_0 = f$. Moreover, $\mathrm{Def}(X, f)$ is smooth. Let T be a smooth fiber of f and let $M_T \subset M$ be as in Theorem 3.8. Then $M_T = \mathrm{Def}(X, f)$ by [Mat09, Proposition 2.1(3)]. The following lemma tells us that if the reduced fiber is preserved as a subvariety, then also the fibration is preserved.

Lemma 5.2. Let $f : X \longrightarrow B$ be a Lagrangian fibration, let $t \in B$, and let $Y = (X_t)_{\mathrm{red}}$ be the reduction of a fiber. Consider the deformation $F : \mathfrak{X} \longrightarrow P$ of f over the smooth hypersurface $\mathrm{Def}(X, f) \subset \mathrm{Def}(X)$ as in (5.1). Then we have $M_Y \subset \mathrm{Def}(X, f)$. Moreover, locally trivial deformations of Y remain fibers of F.

Proof. As explained in 5.1, it is sufficient to show that $M_Y \subset M_T$ by Matsushita's result [Mat09, Proposition 2.1(3)]. Let $Y = \cup_{i \in I} Y_i$ be a decomposition into irreducible components. As in (2.7) we have $M_Y \subset \cap_i M_{[Y_i]}$ and for a smooth fiber T of f we have $\sum_i n_i [Y_i] = [T]$ and so $\cap_i M_{[Y_i]} \subset M_{[\sum_i n_i Y_i]} = M_{[T]} = M_T$, where the last equality is Voisin's theorem. Put together this gives $M_Y \subset M_T = \mathrm{Def}(X, f)$. As Lagrangian fibrations are equidimensional, the last claim follows from the Rigidity Lemma [KM98, Lem 1.6]. □

5.3. Codimension estimates

Let X be an irreducible symplectic manifold and let $f : X \longrightarrow B$ be a Lagrangian fibration. We put $Y = (X_t)_{\mathrm{red}}$ for $t \in D := \{t \in B : X_t \text{ is singular}\}$. The analytic subset D is called the *discriminant locus* of f. We know by [Hwa08, Prop 4.1] and [HO09, Prop 3.1] that D is nonempty and of pure codimension one.

Let $D_0 \ni t$ be an irreducible component of D and let $X_0 := X \times_B D_0 = f^{-1}(D_0)$. Let $Y = \cup_{i \in I} Y_i$ and $X_0 = \cup_{j \in J} X_j$ be decompositions into irreducible components and consider the surjective map $j : I \longrightarrow J$ mapping $i \in I$ to the unique $j = j(i) \in J$ with $Y_i \subset X_j$.

I am very grateful to Keiji Oguiso for explaining the following lemma.

Lemma 5.4. Let $f : X \longrightarrow B$ be a Lagrangian fibration of a projective irreducible symplectic manifold X. Let $X_0 = \bigcup_{j \in J} X_j$ where $J = \{1, \ldots, r\}$ and let $i : Y = (X_t)_{\mathrm{red}} \hookrightarrow X$ for $t \in D_0 \subset B$ be the reduction of a general singular fiber contained in X_0. Then

$$\operatorname{rk}\left(H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C})\right) \geq r,$$

where $\nu : \widetilde{Y} \longrightarrow Y$ is the normalization and $j = \nu \circ i$. More precisely, the subspace of $H^2(X,\mathbb{C})$ generated by the classes of the divisors X_j maps onto a subspace of dimension $\geq r-1$ not containing the class of the ample divisor.

Proof. Let $C \subset B$ be a curve obtained by the intersecting $n-1$ general very ample divisors on B and consider the fiber product $X_C = X \times_B C$. As B is normal, X_C is smooth. As $t \in D_0$ is general, there is such a curve C with $t \in C$. Let H be a very ample divisor on X and let $H_1, \ldots, H_{n-1} \in |H|$ be general. Then the intersection $S = X_C \cap H_1 \cap \cdots \cap H_{n-1}$ is a smooth surface by Bertini's theorem. By construction it comes with a morphism $g : S \longrightarrow C$.

Consider the diagram

$$\begin{array}{ccc} H^2(X,\mathbb{C}) & \xrightarrow{j^*} & H^2(\widetilde{Y},\mathbb{C}) \\ \varrho \downarrow & & \downarrow \varrho_Y \\ H^2(S,\mathbb{C}) & \xrightarrow{j_S^*} & H^2(\widetilde{F},\mathbb{C}) \end{array} \tag{5.2}$$

where $F = Y \cap H_1 \cap \cdots \cap H_{n-1} \subset S$ and $\widetilde{F} \longrightarrow F$ is the normalization. Note that $\widetilde{Y}$ is smooth by [HO09, Thm 1.3] and $\widetilde{F}$ is smooth, as F is a curve. Let $Y = \bigcup_{i=1}^{s} Y_i$ and $F = \bigcup_{\lambda=1}^{q} F_\lambda$ be decompositions into irreducible components where $s = \#I$. We put $F(i) := Y_i \cap H_1 \cap \cdots \cap H_{n-1} = \bigcup_{\lambda \in \Lambda_i} F_\lambda$, where $\Lambda_i \subset \Lambda := \{1, \ldots, q\}$ is the subset of all λ such that $F_\lambda \subset Y_i$. If the H_k are general enough, Λ is the disjoint union of the Λ_i.

We will show that the subspace $V \subset H^2(X,\mathbb{C})$ spanned by the X_j and H maps surjectively onto an r-dimensional subspace in $H^2(\widetilde{F},\mathbb{C})$. This would imply the claim by diagram (5.2).

Write $X_0 = \sum_j n_j X_j$ and $X_t = \sum_i n_{j(i)} Y_i$ as cycles, where as above $j(i)$ is the unique $j \in J$ with $Y_i \subset X_j$. Recall that $\Lambda = \coprod_i \Lambda_i$ is a disjoint union. So $n_\lambda := n_{j(i)}$ for $\lambda \in \Lambda_i$ is well-defined and we have $F = \sum_\lambda n_\lambda F_\lambda$. As $F = \bigcup_{\lambda=1}^{q} F_\lambda$, we obtain $\widetilde{F} = \bigcup_{\lambda=1}^{q} \widetilde{F}_\lambda$ where $\widetilde{F}_\lambda$ is the normalization of F_λ. Thus,

$$H^2(\widetilde{F},\mathbb{C}) \cong \bigoplus_{\lambda=1}^{q} H^2(\widetilde{F}_\lambda,\mathbb{C}) \cong \mathbb{C}^q.$$

If we denote the intersection pairing on S by $(\cdot,\cdot)_S$, then under this isomorphism $j_S^* : H^2(S,\mathbb{C}) \longrightarrow H^2(\widetilde{F},\mathbb{C})$ is given by

$$\alpha \mapsto ((\alpha, F_1)_S, \ldots, (\alpha, F_q)_S)\,.$$

Let $\{x_\lambda \mid \lambda \in \Lambda\} \subset H^2(\widetilde{F},\mathbb{C})^\vee$ be the dual basis of the basis of $H^2(\widetilde{F},\mathbb{C})$ obtained corresponding to the standard basis of $\mathbb{C}^q \cong H^2(\widetilde{F},\mathbb{C})$. By Zariski's Lemma [BHPV, Ch III, Lem 8.2] the subspace $W \subset H^2(S,\mathbb{C})$ spanned by the classes of the F_λ maps surjectively to the hyperplane of $\mathbb{C}^q$ given by $\sum_\lambda n_\lambda x_\lambda = 0$, So the subspace of $H^2(S,\mathbb{C})$ spanned by the classes of the F_λ and $H|_S$ maps surjectively onto $\mathbb{C}^q$. We have $\varrho_Y(j^*X_j) = j_S^*\varrho(X_j) = \big((\varrho(X_j),F_\lambda)_S\big)_\lambda$. As the Λ_i are mutually disjoint, so are the $\Lambda_j := \bigcup_{j(i)=j}\Lambda_i$. We see from $(\varrho(X_j),F_\lambda)_S = \sum_{\mu\in\Lambda_j}(F_\mu,F_\lambda)_S$ that the subspace of $H^2(X,\mathbb{C})$ generated by the X_j surjects onto a subspace of $\mathbb{C}^q$ of dimension $\geq r-1$. The claim follows as the image of V does not contain $j_S^*(H|_S)$. □

Corollary 5.5. In the situation of the preceding lemma $\operatorname{codim} M_Y \geq r$.

Proof. This follows from Theorem 2.6 and Lemma 5.4. □

Note that there is no need for a normal crossing hypothesis in the corollary as we only prove an estimate and no equality. The codimension of M_Y is thus bounded by the number of irreducible components of $X_0 = f^{-1}(D_0)$ whereas the number of irreducible components of Y does not a priori play a role. Hence, a very interesting and important question is the following

Question 5.6. Let $Y = \cup_{i\in I}Y_i$ and $X_0 = \cup_{j\in J}X_j$ as in the beginning of Section 5.3. Is then $\#I = \#J$? Do we always have $\operatorname{codim}_M M_Y = \#J$ for normal crossing Y?

There is no obvious reason, why these numbers should be equal, but in all examples we know they are equal. Recall that general singular fibers have been classified by Hwang-Oguiso according to their characteristic 1-cycle: this is an effective 1-cycle on a fiber $Y \subset X$, possibly an infinite sum of curves. It was shown to be of Kodaira type or of type I_∞, see [HO09, Theorem 1.4] and [HO11, Theorem 2.4]. The type of a singular fiber will be the type of its characteristic 1-cycle.

Theorem 5.7. *Let X be an irreducible symplectic manifold and let $f : X \longrightarrow B$ be a Lagrangian fibration. Then X can be deformed, keeping the fibration, to an irreducible symplectic manifold X' with a Lagrangian fibration*

$f' : X' \longrightarrow B'$ such that outside a codimension 2 subset $Z \subset B'$, all singular fibers of f' over the complement of Z have a characteristic cycle of Kodaira type I, II, III or IV and such that for every irreducible component D_0 of the discriminant divisor the preimage $X'_{D_0} = (f')^{-1}(0)$ is irreducible.

Proof. Let D_0 be an irreducible component of the discriminant divisor and let $Y = (X_t)_{\mathrm{red}}$ for $t \in D_0$ be a general singular fiber. By Lemma 5.2, the space M_Y is contained in $\mathrm{Def}(X, f)$. As the fiber type of a singular fiber is generically constant along an irreducible component of the discriminant divisor, it suffices to show that $M_Y \subsetneq \mathrm{Def}(X, f)$ if X_t is not of type I-IV. But for all other Kodaira fibers, there are irreducible components of X_t with different multiplicities. Fiber components with different multiplicities lie in different components of $X_0 = f^{-1}(D_0)$, hence $\mathrm{codim}\, M_Y \geq 2$ by Corollary 5.5. By [Mat09, Corollary 1.2], $\mathrm{codim}\,\mathrm{Def}(X, f) = 1$ so that $M_Y \subsetneq \mathrm{Def}(X, f)$ and we conclude the proof. □

Example 5.8. In the case of K3 surfaces, the situation becomes easier. For elliptic K3 surfaces it is not difficult to see that $\mathrm{codim}\, M_Y = \#I = \#J$ of irreducible components of the reduction Y of a fiber, if the latter has normal crossings, and $\mathrm{codim}\, M_Y \geq \#I$ in all other cases, see [Le11, Thm VII.3.8]. The analogue of the Hodge-de Rham spectral sequence for $\widetilde{\Omega}_Y$ does not degenerate at E_1 if Y does not have normal crossings, but one can show that for infinitesimal deformations $\mathcal{Y} \longrightarrow S$ the $H^q(\widetilde{\Omega}^p_{\mathcal{Y}/S})$ are free $\mathcal{O}_S$-modules if one uses the differentials in the spectral sequence. With this at hand, one deduces as in the normal crossing case that $\mathrm{Def}^{\mathrm{lt}}(i)$ and M_Y are smooth. So $M_Y \subset M'_Y$ in all cases. Consequently, using Theorem 5.7 any elliptic K3 surface can be deformed as a fibration to an elliptic K3 with only nodal and cuspidal curves as singularities. There are examples, where a cuspidal rational curve Y deforms into two nodal curves so that we have $M_Y \subsetneq M'_Y$.

5.9. Vista

There are several results assuming the general singular fibers to be of a special kind, see [HO10], [Saw08], [Saw12], [Thi08]. If we knew that complicated general singular fibers only show up in higher codimension in M, we could always deform to such special situations.

Acknowledgements

The large part of this work is the author's thesis. It is a pleasure to thank my advisor Manfred Lehn for his constant support, for suggesting many interesting problems and for thoroughly asking questions. Furthermore, I would like to thank Duco van Straten for some extremely helpful remarks, Yasunari Nagai and Keiji Oguiso for explaining Lagrangian fibrations, Klaus Hulek for helpful discussions on abelian fibrations, Jean-Pierre Demailly, Daniel Greb, Sam Grushevsky, Dmitry Kaledin, Stefan Müller-Stach, Thomas Peternell, Sönke Rollenske, Justin Sawon, Gerard van der Geer, Claire Voisin and Kang Zuo for valuable discussions. I am also grateful to Justin Sawon for sending me his preprint [Saw15].

While working on this project, I benefited from the support of the DFG through the SFB/TR 45 "Periods, moduli spaces and arithmetic of algebraic varieties" and the DFG research grant Le 3093/1-1. During the revision of this article I was supported by the DFG through the research grant Le 3093/2-1.

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Fakultät für Mathematik
Technische Universität Chemnitz
Reichenhainer Strasse 39, 09126 Chemnitz, Germany
E-mail address: christian.lehn@mathematik.tu-chemnitz.de

Received March 30, 2015