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Scale invariant Strichartz estimates on tori and applications

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We prove scale-invariant Strichartz inequalities for the Schrödinger equation on rectangular tori (rational or irrational) in all dimensions. We use these estimates to give a simpler treatment of local well-posedness of the energy-critical nonlinear Schrödinger equation in dimensions three and four.

1. Introduction

The most general flat torus is formed as the quotient of $\mathbb{R}^d$ by a lattice. In this paper, we will only consider rectangular tori, namely, those of the form $\mathbb{R}^d/(L_1\mathbb{Z} \times L_2\mathbb{Z} \times \cdots \times L_d\mathbb{Z})$ with $L_1, \ldots, L_d \in (0, \infty)$. Our goal is to prove certain space-time estimates for solutions of the linear Schrödinger equation on such manifolds.

Notationally, it will be simpler to fix the base space to be $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ and to incorporate the geometry of the torus into the definition of the Laplacian. Put differently, we use coordinates based on the standard torus and then use the Laplace–Beltrami operator associated to the induced metric, that is,

$$\Delta := \sum_{j=1}^{d} \theta_j \frac{\partial^2}{\partial x_j^2}, \quad \text{or equivalently,} \quad \widehat{\Delta f}(k) := -\sum_{j=1}^{d} \theta_j k_j^2 \hat{f}(k).$$

Here $\theta_j = L_j^{-2}$ and we employ the following convention for the Fourier transform:

$$\hat{f}(k) = \int_{\mathbb{T}^d} e^{-2\pi i k x} f(x)\, dx \quad \text{so that} \quad f(x) = \sum_{k \in \mathbb{Z}^d} e^{2\pi i k x} \hat{f}(k).$$

With these notations, the solution $u(t,x)$ to the linear Schrödinger equation with initial data $u_0(x)$ is given by

$$u(t,x) = e^{it\Delta}u_0 = \sum_{k\in\mathbb{Z}^d} \exp\Big\{2\pi i\big[kx - t\textstyle\sum_{j=1}^d \theta_j k_j^2\big]\Big\}\widehat{u_0}(k). \tag{1.1}$$

Note that by making a change of variables in time, there is no loss of generality to assume that $\theta_1,\dots,\theta_d \in (0,1]$.

The main result of this paper is the following:

Theorem 1.1 (Scale-invariant Strichartz estimates). *Fix $d \geq 1$, $\theta_1, \dots, \theta_d \in (0,1]$, $1 \leq N \in 2^{\mathbb{Z}}$, and $p > \frac{2(d+2)}{d}$. Then*

$$\|e^{it\Delta}P_{\leq N} f\|_{L^p_{t,x}([0,1]\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}}\|f\|_{L^2_x}, \tag{1.2}$$

where $\Delta := \theta_1\partial_{x_1}^2 + \cdots + \theta_d\partial_{x_d}^2$.

Unlike $\mathbb{R}^d$, the torus $\mathbb{T}^d$ does not admit a true scaling symmetry; however, for very short times, the linear evolution of highly concentrated initial data will not distinguish the two. For well-posedness questions of nonlinear problems, such concentrated solutions are the principal adversary. Correspondingly, scale-invariant estimates are an essential tool for treating nonlinear problems at the critical regularity.

We should also note that by choosing p close to $\frac{2(d+2)}{d}$ one may make the $\frac{d}{2} - \frac{d+2}{p}$ loss of derivatives as small as one wishes. It is not difficult to verify that the stated estimate fails for the square torus (i.e., $\theta_1 = \cdots = \theta_d = 1$) if one takes $p = \frac{2(d+2)}{d}$; see [2].

Very recently, Bourgain and Demeter (see [5, Theorem 2.4]) proved analogous Strichartz estimates with an arbitrarily small loss of scaling:

Theorem 1.2 (Non-scale-invariant Strichartz estimates). *Fix $d \geq 1$, $\theta_1, \dots, \theta_d \in (0,1]$, $1 \leq N \in 2^{\mathbb{Z}}$, and $p \geq \frac{2(d+2)}{d}$. Then for any $\eta > 0$,*

$$\|e^{it\Delta}P_{\leq N} f\|_{L^p_{t,x}([0,1]\times\mathbb{T}^d)} \lesssim_\eta N^{\frac{d}{2}-\frac{d+2}{p}+\eta}\|f\|_{L^2_x}, \tag{1.3}$$

where $\Delta := \theta_1\partial_{x_1}^2 + \cdots + \theta_d\partial_{x_d}^2$.

This result will be an essential part of the proof of Theorem 1.1. In much earlier work, Bourgain showed that in the case of a square torus, Theorem 1.2 implies Theorem 1.1; see [2, Proposition 3.113].

The space-time Fourier methods used by Bourgain for the square torus are ill-suited to the case of an irrational torus. We will be using the basic dispersive estimate for the propagator (see Lemma 2.2), which pushes all the difficulty into bounding the resulting temporal convolution. This style of argument (which is closer to the usual Euclidean treatment) is indifferent to the rational/irrational character of the θs. In particular, the intricate astigmatism resulting from refocusing at slightly different times in each coordinate direction can be brutishly handled by the arithmetic-geometric mean inequality. Nonetheless, important insights employed by Bourgain in [2] do inform and suffuse our treatment of the subtle temporal convolution.

We now give a brief summary of prior work on Strichartz estimates on square and irrational tori:

- In [2], Bourgain considered only the square torus. He proved Theorems 1.1 and 1.2 in dimensions one and two. He also proved (1.2) for $p > 4$ when $d = 3$ and for $p \geq \frac{2(d+4)}{d}$ when $d \geq 4$.
- The paper [3] of Bourgain was the first to consider irrational tori. It considers only the case $d = 3$ and proves scale-invariant $L_t^p L_x^4$ Strichartz estimates for $p > \frac{16}{3}$.
- Bourgain, [4], and Demeter, [6], gave very different proofs that (1.3) holds for $p = \frac{2(d+3)}{d}$ on all tori.
- The paper [7] of Guo, Oh, and Wang proves several Strichartz estimates on irrational tori. In particular, they obtain (1.2) in the following cases: $d = 2$ and $p > \frac{20}{3}$, $d = 3$ and $p > \frac{16}{3}$, $d = 4$ and $p > 4$, and lastly, $d \geq 5$ and $p = 4$. The also prove that (1.2) holds for $d = 3$ and $p > \frac{14}{3}$ under the additional assumption $\theta_1 = \theta_2$.

As an application of Theorem 1.1 we consider the initial-value problem for the energy-critical nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{\frac{4}{d-2}} u \\ u(0) = u_0 \in H^1(\mathbb{T}^d) \end{cases} \tag{1.4}$$

in spatial dimensions $d \in \{3, 4\}$. Specifically, we show the following:

Theorem 1.3 (Well-posedness for the energy-critical NLS). *Fix $d \in \{3, 4\}$ and let $u_0 \in H^1(\mathbb{T}^d)$. Then there exists a time $T = T(u_0)$ and a unique solution $u \in C_t([0, T); H^1(\mathbb{T}^d)) \cap X^1([0, T))$ to (1.4). Moreover, there exists $\eta_0 = \eta_0(d) > 0$ such that if $\|u_0\|_{H^1(\mathbb{T}^d)} \leq \eta$, then the solution u is global in time.*

The function spaces used to construct the solution in Theorem 1.3, namely those defined in (1.7), are precisely the ones used in [9] to obtain this theorem for the three dimensional square torus. Subsequently, Theorem 1.3 was proved in [7] for the case when $d = 3$ and $\theta_1 = \theta_2$, and for the fully irrational three-torus in [12]. Thus, in the three dimensional case, Theorem 1.3 is not new. Here, we will combine the new estimates provided by Theorem 1.1 with several beautiful ideas introduced in [9] to provide a simpler proof.

In four dimensions, Theorem 1.3 was proved in [10] on the square torus, contingent on certain Strichartz estimates that were subsequently proved in [4, 6]. The results of this paper allow their argument to be adapted to irrational tori; however, we argue in a rather different way that we contend is significantly simpler.

Let us now discuss the principal differences between the arguments presented here to prove Theorem 1.3 and those in previous work. First and foremost, we treat the three and four dimensional cases in a completely parallel manner, though for the sake of readability, we give the details in each case separately. We make no use of multilinear estimates; we only exploit the idea of decomposing into frequency cubes. We do not need to use temporal orthogonality arguments, nor the interpolation theory of U^p and V^p spaces. The authors of [9] indicate that such a simpler proof is possible in the three dimensional setting (see the discussion of their equation (7)), but no such claim is made in their treatment of the four dimensional case [10].

1.1. Notation and useful lemmas

Throughout this text, we will be regularly referring to the spacetime norms

$$\|u\|_{L_t^p L_x^r([0,1]\times\mathbb{T}^d)} := \left(\int_{[0,1]} \left(\int_{\mathbb{T}^d} |u(t,x)|^r \, dx \right)^{p/r} dt \right)^{1/p}, \tag{1.5}$$

with obvious changes if p or r are infinity.

We write $X \lesssim Y$ to indicate that $X \leq CY$ for some constant C, which is permitted to depend on the ambient spatial dimension, d, without further comment.

Let ϕ be a smooth radial cutoff on $\mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. With $N \in 2^{\mathbb{N}}$ we define the Littlewood–Paley projections

$$\widehat{P_1 f}(k) := \widehat{f_1}(k) := \hat{f}(k)\prod_{j=1}^{d}\phi(k_j),$$

$$\widehat{P_{\le N} f}(k) := \widehat{f_{\le N}}(k) := \hat{f}(k)\prod_{j=1}^{d}\phi\big(\tfrac{k_j}{N}\big), \tag{1.6}$$

$$\text{and}\quad \widehat{P_N f}(k) := \widehat{f_N}(k) := \hat{f}(k)\prod_{j=1}^{d}\big[\phi\big(\tfrac{k_j}{N}\big) - \phi\big(\tfrac{2k_j}{N}\big)\big],$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$. Using Littlewood–Paley projectors with this product structure simplifies the proof of Theorem 1.1 slightly.

Next we recall the definition of the function spaces U^p and V^p and use them to construct the relevant function spaces for our applications. The general theory of U^p and V^p spaces is discussed at some length in [11]; we will confine ourselves here to reviewing the definitions and basic properties in the specific setting that is relevant to our problem. In particular, we only consider finite time intervals of the form $[0, T)$. Let H be a separable Hilbert space over $\mathbb{C}$; in this paper, this will be $\mathbb{C}$ or $H^s(\mathbb{T}^d)$ with $s = 0, 1$. Let $\mathcal{Z}$ be the set of finite partitions $0 = t_0 < t_1 < \ldots < t_K \le T$. We use the convention that $v(T) := 0$ for all functions $v : [0, T) \to H$.

Definition 1.4. Let $1 \le p < \infty$. An U^p-atom is a function $a : [0, T) \to H$ of the form

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)}\phi_{k-1},$$

where $\{t_k\} \in \mathcal{Z}$ and $\{\phi_k\} \subset H$ with $\sum_{k=0}^{K-1} \|\phi_k\|_H^p = 1$. The atomic space $U^p([0, T); H)$ is the space of all functions $u : [0, T) \to H$ of the form

$$u = \sum_{j=1}^{\infty} \lambda_j a_j$$

with $\{\lambda_j\} \in \ell^1(\mathbb{C})$ and a_j being U^p-atoms. The norm on $U^p([0, T); H)$ is given by

$$\|u\|_{U^p} := \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \text{ with } \{\lambda_j\} \in \ell^1(\mathbb{C}) \text{ and } U^p\text{-atoms } a_j\right\}.$$

Definition 1.5. Let $1 \leq p < \infty$. The space $V^p([0,T);H)$ is the space of all functions $v : [0,T) \to H$ such that

$$\|v\|_{V^p} := \sup_{\{t_k\}\in\mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_H^p \right)^{1/p} < \infty.$$

The space $V^p_{rc}([0,T);H)$ denotes the closed subspace of all right-continuous functions $v : [0,T) \to H$ such that $v(0) = 0$.

Remark 1.6. The spaces $U^p([0,T);H)$, $V^p([0,T);H)$, and $V^p_{rc}([0,T);H)$ are Banach spaces and satisfy

$$U^p([0,T);H) \hookrightarrow V^p_{rc}([0,T);H) \hookrightarrow U^q([0,T);H) \hookrightarrow L^\infty([0,T);H)$$

for all $1 \leq p < q < \infty$.

Definition 1.7. Let $s = 0, 1$. Then $U^p_\Delta H^s$ and $V^p_\Delta H^s$ denote the spaces of all functions $u : [0,T) \to H^s(\mathbb{T}^d)$ such that the map $t \to e^{-it\Delta}u(t)$ is in $U^p([0,T);H^s)$ and $V^p([0,T);H^s)$, repectively, with norms given by

$$\|u\|_{U^p_\Delta H^s} := \|e^{-it\Delta}u\|_{U^p([0,T);H^s)} \quad \text{and} \quad \|u\|_{V^p_\Delta H^s} := \|e^{-it\Delta}u\|_{V^p([0,T);H^s)}.$$

We define $X^s([0,T))$ and $Y^s([0,T))$ to be the spaces of all functions $u : [0,T) \to H^s(\mathbb{T}^d)$ such that for every $\xi \in \mathbb{Z}^d$ the map $t \to \widehat{e^{-it\Delta}u(t)}(\xi)$ is in $U^2([0,T);\mathbb{C})$ and $V^2_{rc}([0,T);\mathbb{C})$, respectively, with norms given by

$$\begin{aligned}
\|u\|_{X^s([0,T))} &:= \left(\sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2s} \|\widehat{e^{-it\Delta}u(t)}(\xi)\|_{U^2}^2 \right)^{1/2}, \\
\|u\|_{Y^s([0,T))} &:= \left(\sum_{\xi\in\mathbb{Z}^d} \langle\xi\rangle^{2s} \|\widehat{e^{-it\Delta}u(t)}(\xi)\|_{V^2}^2 \right)^{1/2}.
\end{aligned} \tag{1.7}$$

These are the same spaces used in [9] and subsequent works.

Remark 1.8. From [9, Proposition 2.8], we have the continuous embeddings $U^2_\Delta H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V^2_\Delta H^s$. We also note that

$$\|u\|_{L^\infty_t H^s_x([0,T)\times\mathbb{T}^d)} \lesssim \|u\|_{X^s([0,T))}$$

and, by [9, Proposition 2.11],

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{X^s([0,T))} \lesssim \|F\|_{L^1_t H^s_x([0,T)\times\mathbb{T}^d)}.$$

Using the atomic structure of U^p and Remark 1.8, we can recast the Strichartz estimates from Theorem 1.1 as follows:

$$\|P_{\leq N} u\|_{L^p([0,T)\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|P_{\leq N} u\|_{U^p_\Delta L^2} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|P_{\leq N} u\|_{Y^0([0,T))} \tag{1.8}$$

for all $p > \frac{2(d+2)}{d}$ and $N \geq 1$. In particular, due to the Galilei invariance of solutions to the linear Schrödinger equation,

$$\|P_C u\|_{L^p([0,T)\times\mathbb{T}^d)} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}} \|P_C u\|_{Y^0([0,T))} \quad \text{for all} \quad p > \frac{2(d+2)}{d} \tag{1.9}$$

and for any cube $C \subset \mathbb{R}^d$ of side-length $N \geq 1$.

2. Scale invariant Strichartz estimates

The implicit constants in this section will be allowed to depend on the magnitude of $\{\theta_j\}_{j=1}^d$ and we will not be tracking that dependence. It is worth noting however that the number theoretical properties of $\{\theta_j\}_{j=1}^d$ play no role in our arguments.

In this section, we will write

$$K_N(t,x) := [e^{it\Delta} P_{\leq N}\delta_0](x) = \sum_{k\in\mathbb{Z}^d} \prod_{j=1}^d \phi(k_j/N) e^{2\pi i[x_j k_j - t\theta_j k_j^2]} \tag{2.1}$$

for the convolution kernel associated to the frequency-localized propagator. Here $1 \leq N \in 2^{\mathbb{Z}}$ and $P_{\leq N}$ is the Littewood–Paley projector defined in (1.6).

As observed already in [2], the passage from Theorem 1.2 to Theorem 1.1 requires additional information about the action of the convolution kernel $K_N(t,x)$ only in places where it is large. This will become apparent when we complete the proof of Theorem 1.1 at the end of this section.

From the micro-local perspective, we expect K_N to be large only near conjugate points of the geodesic flow. For the square torus, there are many such conjugate points, one at every rational time; however, the degree of refocusing is governed by the denominator of the rational number concerned. These heuristics are borne out by Lemma 2.2 below, whose statement is best understood in the context of Dirichlet's Lemma on rational approximation.

Lemma 2.1 (Dirichlet). *Given an integer $N \geq 2$ and $\beta \in [0,1]$, there exist integers $1 \leq q < N$ and $0 \leq a \leq q$ so that $(a,q) = 1$ and $|\beta - \frac{a}{q}| \leq \frac{1}{Nq}$.*

Recall that (a,q) denotes the greatest common divisor of a and q; correspondingly, $(a,q) = 1$ asserts that a and q are relatively prime. Note also that $(0,q) = q$. For a proof of Dirichlet's Lemma, see, for example, [8, §3.8].

Lemma 2.2 (Dispersive estimate for K_N). *Choosing integers $0 \leq a_j \leq q_j < N$ so that $(a_j, q_j) = 1$ and $|\theta_j t - \frac{a_j}{q_j}| \leq \frac{1}{q_j N}$, we have*

$$|K_N(t,x)| \lesssim \prod_{j=1}^{d} \frac{N}{\sqrt{q_j}\big(1 + N\big|\theta_j t - \frac{a_j}{q_j}\big|^{1/2}\big)}$$

uniformly for $t \in [0,1]$.

Due to the product structure of (2.1), this d-dimensional estimate is an immediate corollary of the one-dimensional case treated in [2, Lemma 3.18] via an application of Weyl's method.

Incidentally, Lemma 2.2 shows that K_N can only be very large if $\{\theta_j t\}_{j=1}^d$ can all be simultaneously well-approximated by rationals with small denominator. Correspondingly, under a mild Diophantine condition, which holds for Lebesgue almost all d-tuples of parameters θ_j, one may show that K_N is very large only very close to $t = 0$. This leads to a much shorter proof of Theorem 1.1 for such d-tuples.

So far, we have been rather nebulous about what it means for $K_N(t,x)$ to be very large. It turns out that the precise meaning depends on the exponent p from Theorem 1.1 that one is treating. For now, we will use a parameter $0 < \sigma \ll 1$ that will be chosen later and say that $K_N(t,x)$ is large when t belongs to

$$\begin{aligned} \mathcal{T} := \big\{ t \in [0,1] : q_j N^2 \big|\theta_j t - \tfrac{a_j}{q_j}\big| \leq N^{2\sigma} \\ \text{for some } j,\ q_j \leq N^{2\sigma}, \text{ and } (a_j, q_j) = 1 \big\}. \end{aligned}$$

We then define

$$\tilde{K}_N(t,x) := \chi_{\mathcal{T}}(t) K_N(t,x).$$

In view of Lemma 2.2, this construction guarantees that

$$|K_N(t,x) - \tilde{K}_N(t,x)| \lesssim N^{d(1-\sigma)}. \tag{2.2}$$

The centerpiece of our analysis is the following proposition, which establishes space-time estimates for $\tilde{K}_N$.

Proposition 2.3 (Strichartz estimates for $\tilde{K}_N$). *Choose* $2 < p, r \leq \infty$ *such that* $\frac{d}{2} - \frac{2}{p} - \frac{d}{r} > 0$. *Then*

$$\|\tilde{K}_N * F\|_{L_t^p L_x^r([0,1]\times\mathbb{T}^d)} \lesssim N^{2(\frac{d}{2}-\frac{2}{p}-\frac{d}{r})}\|F\|_{L_t^{p'} L_x^{r'}([0,1]\times\mathbb{T}^d)},$$

provided σ *is sufficiently small* (*depending on* (d,p,r) *only*).

As we will see, Proposition 2.3 is a direct consequence of the next two lemmas. The first lemma concerns mapping properties of $\tilde{K}_N$ as a convolution kernel on $\mathbb{T}^d$ (with t fixed); this will follow easily from Lemma 2.2. The second lemma is much more challenging and deals with the resulting temporal convolution. This two-step argument has strong parallels to the standard approach in the Euclidean setting, where one uses the (much simpler) dispersive estimate and then the time convolution is handled very swiftly by an application of the Hardy–Littlewood–Sobolev inequality. Such an approach yields only very poor estimates in the torus setting. It is essential to exploit the non-resonant structure of the temporal convolution kernel which yields substantial gains for large q relative to the Hardy–Littlewood–Sobolev inequality.

In the square torus setting, Bourgain proves a close analogue of Proposition 2.3 with $p = r > \frac{2(d+2)}{d}$ by a rather different argument, which builds upon the Fourier analytic methods used to prove Stein–Tomas restriction theorems. In that case, estimates on the space-time Fourier transform of the relevant convolution kernels can be derived from the analysis of certain equations in integers; in the case of irrational tori, this becomes a morass, dependent on subtle Diophanine approximation properties of the parameters θ_j. This is why a different approach is needed to prove Proposition 2.3 in the case of irrational tori. Nonetheless, in deriving Theorem 1.1 from Proposition 2.3 we do follow the arguments of [2] rather closely.

To state the first lemma used in the proof of Proposition 2.3, we introduce a family of smooth radial cutoffs on $\mathbb{R}$ as follows:

$$\phi_{N^{-2}}(x) := \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| \geq 2 \end{cases}$$

and for all dyadic $T > N^{-2}$ we define $\phi_T(x) := \phi_{N^{-2}}(x) - \phi_{N^{-2}}(2x)$. Exploiting just these definitions, we have

$$(2.3)\qquad \sum_{j=1}^{d}\sum_{Q=1}^{N^{2\sigma}}\sum_{T=N^{-2}}^{N^{2\sigma-2}/Q}\sum_{\substack{(a,q)=1\\ q\sim Q}}\phi_T\bigl(\tfrac{\theta_j t-\frac{a}{q}}{T}\bigr)\geq 1 \quad \text{for all} \quad t\in\mathcal{T}.$$

Here, and in all that follows, Q and T are restricted to lie in $2^{\mathbb{Z}}$ and $q\sim Q$ means that $Q\leq q<2Q$.

Lemma 2.4 (Dispersive estimates for $\tilde{K}_N$). *For $t\in[0,1]$ and $2\leq r\leq\infty$ we have*

$$\|\tilde{K}_N(t)*f\|_{L^r(\mathbb{T}^d)}\lesssim\|f\|_{L^{r'}(\mathbb{T}^d)}\sum_{j=1}^{d}\sum_{Q=1}^{N^{2\sigma}}\sum_{T=N^{-2}}^{N^{2\sigma-2}/Q}(QT)^{\frac{d}{r}-\frac{d}{2}}\sum_{\substack{(a,q)=1\\ q\sim Q}}\phi_T\bigl(\tfrac{\theta_j t-\frac{a}{q}}{T}\bigr).$$

Proof. By the unitarity of the propagator $e^{it\Delta}$, we have

$$\|K_N(t)*f\|_{L^2(\mathbb{T}^d)}=\|f\|_{L^2(\mathbb{T}^d)}.$$

On the other hand, from the kernel estimates of Lemma 2.2, we obtain

$$\|K_N(t)*f\|_{L^\infty(\mathbb{T}^d)}\lesssim\|f\|_{L^1(\mathbb{T}^d)}\prod_{j=1}^{d}\frac{N}{\sqrt{q_j}\bigl(1+N\bigl|\theta_j t-\frac{a_j}{q_j}\bigr|^{1/2}\bigr)},$$

where $0\leq a_j\leq q_j<N$ obey $(a_j,q_j)=1$ and $|\theta_j t-\frac{a_j}{q_j}|\leq\frac{1}{q_jN}$.

Interpolating between these two bounds and using the arithmetic–geometric mean inequality, we derive that for any $2\leq r\leq\infty$,

$$\begin{aligned}\|K_N(t)*f\|_{L^r(\mathbb{T}^d)}&\lesssim\|f\|_{L^{r'}(\mathbb{T}^d)}\prod_{j=1}^{d}\biggl(\frac{N}{\sqrt{q_j}\bigl(1+N\bigl|\theta_j t-\frac{a_j}{q_j}\bigr|^{1/2}\bigr)}\biggr)^{1-\frac{2}{r}}\\ &\lesssim\|f\|_{L^{r'}(\mathbb{T}^d)}\sum_{j=1}^{d}\Bigl(N^{-2}q_j\bigl(1+N^2\bigl|\theta_j t-\tfrac{a_j}{q_j}\bigr|\bigr)\Bigr)^{\frac{d}{r}-\frac{d}{2}}.\end{aligned}$$

The lemma now follows easily from (2.3). □

To continue, for fixed Q we define

$$\mathcal{F}_{1,Q}(t):=\sum_{\substack{(a,q)=1\\ q\sim Q}}\delta\bigl(t-\tfrac{a}{q}\bigr)\quad\text{and}\quad\mathcal{F}_{2,Q}(t):=\sum_{\substack{0\leq a<q\\ q\sim Q}}\delta\bigl(t-\tfrac{a}{q}\bigr).$$

Note that we may write

$$\sum_{\substack{(a,q)=1\\ q\sim Q}} \phi_T\big(\tfrac{\theta_j t-\frac{a}{q}}{T}\big) = \big[\mathcal{F}_{1,Q} * \phi_T\big(\tfrac{\cdot}{T}\big)\big](\theta_j t)$$

and so, by Lemma 2.4,

$$\begin{aligned}(2.4)\quad &\|\tilde{K}_N(t) * F\|_{L^p_t L^r_x([0,1]\times\mathbb{T}^d)}\\ \lesssim &\sum_{j=1}^{d}\sum_{Q=1}^{N^{2\sigma}}\sum_{T=N^{-2}}^{N^{2\sigma-2}/Q} (QT)^{\frac{d}{r}-\frac{d}{2}}\Big\|\big[\mathcal{F}_{1,Q} * \phi_T\big(\tfrac{\cdot}{T}\big)\big](\theta_j t) * \|F(t)\|_{L^{r'}(\mathbb{T}^d)}\Big\|_{L^p_t([0,1])},\end{aligned}$$

for any $2 \le p, r \le \infty$.

To prove Proposition 2.3, we need to estimate the time convolution in the expression above. We are going to do this in two steps. First, we bound convolution with $\mathcal{F}_{1,Q} * \phi_T(\cdot/T)$ as an operator on the torus $\mathbb{T}$; in particular, functions will be understood to be periodic in time. Later, we will reintroduce θ_j and pass to the requisite convolution on the subset $[0, 1]$ of the real line. We now turn to the first part of this program.

Lemma 2.5. *Fix $2 < p \le \infty$. Then for any $\sigma < \min\{\frac{1}{2}, 1-\frac{2}{p}\}$,*

$$\Big\|\mathcal{F}_{1,Q} * \phi_T\big(\tfrac{\cdot}{T}\big) * f\Big\|_{L^p(\mathbb{T})} \lesssim Q^{\frac{2}{p}(1+\varepsilon)} T^{\frac{2}{p}} \|f\|_{L^{p'}(\mathbb{T})} \quad \textit{with} \quad \varepsilon = \tfrac{\sigma(3-2\sigma)}{(1-\sigma)(1-2\sigma)},$$

uniformly for $1 \le Q \le N^{2\sigma}$ and $N^{-2} \le T \le N^{2\sigma-2}/Q$.

As the convolution kernel $\mathcal{F}_{1,Q} * \phi_T$ is positive, we may bound the norm by replacing $\mathcal{F}_{1,Q}$ by $\mathcal{F}_{2,Q}$. The advantage of doing so is that the Fourier transform of $\mathcal{F}_{2,Q}$ is more easily and more efficiently estimated than that of $\mathcal{F}_{1,Q}$; one should compare what follows with [2, Lemma 3.33].

Lemma 2.6 (Fourier transform of $\mathcal{F}_2$). *Let $d_Q(n)$ denote the number of divisors q of n that obey $q \sim Q$. Then*

$$\big|\widehat{\mathcal{F}_{2,Q}}(\omega)\big| \lesssim Q d_Q(\omega) \quad \textit{for all} \quad \omega \neq 0 \tag{2.5}$$

and clearly,

$$\big|\widehat{\mathcal{F}_{2,Q}}(\omega)\big| \lesssim Q^2 \quad \textit{for all} \quad \omega \in \mathbb{Z}.$$

Proof. Recall that $\sum_{a=0}^{q-1} e^{2\pi i a\omega/q} = q$ if q divides ω, but vanishes otherwise. Thus,

$$\widehat{\mathcal{F}_{2,Q}}(\omega) = \sum_{q\sim Q}\sum_{a=0}^{q-1} e^{2\pi i a\omega/q} = \sum_{q\sim Q} q\chi_{\{q\,|\,\omega\}}$$

and the claims immediately follow. □

The proof of Lemma 2.5, will also rely on a distributional estimate for $d_Q(n)$. The bound we need can be found in Lemma 4.28 of [1]; for completeness, we will recapitulate the proof here (with minor modifications).

Lemma 2.7. *For any $\alpha, \tau > 0$ we have*

$$\#\{1 \le n \le R:\, d_Q(n) > D\} \lesssim_{\tau,\alpha} D^{-2\alpha} Q^{2\tau} R.$$

Proof. It suffices to treat the case where $2\alpha =: k$ is an integer.

Observe first that for fixed $q_1, \dots, q_k$ we have

$$\begin{aligned}\#\{1 \le n \le R:\, q_j | n \text{ for all } 1 \le j \le k\} &= \#\{1 \le n \le R:\, \operatorname{lcm}(q_1, \dots, q_k) | n\} \\ &\le R/\operatorname{lcm}(q_1, \dots, q_k).\end{aligned}$$

On the other hand, by the trivial sub-polynomial bound (see [8, Theorem 315]) on the total number of divisors function $d(\cdot)$, we have

$$\#\{(q_1, \dots, q_k) : \operatorname{lcm}(q_1, \dots, q_k) = \ell\} \le d(\ell)^k \lesssim_\varepsilon \ell^{k\varepsilon},$$

for any $\varepsilon > 0$. Correspondingly, by Chebyshev's inequality,

$$\begin{aligned}\#\{1 \le n \le R:\, d_Q(n) > D\} &\lesssim D^{-k} \sum_{n=1}^{R} \Big(\sum_{q\sim Q} \chi_{q\mathbb{Z}}(n) \Big)^k \\ &\lesssim D^{-k} \sum_{q_1,\dots,q_k\sim Q} \frac{R}{\operatorname{lcm}(q_1, \dots, q_k)} \\ &\lesssim_\varepsilon D^{-k} \sum_{\ell=1}^{(2Q)^k} \tfrac{R}{\ell} \ell^{k\varepsilon} \lesssim_\varepsilon D^{-k} R Q^{\varepsilon k}\end{aligned}$$

for any $\varepsilon > 0$. The lemma now follows by choosing $\varepsilon < 2\tau/k$. □

We now have all the ingredients we need to complete the proof of Lemma 2.5.

Proof of Lemma 2.5. We first note that for distinct pairs (a_1,q_1) and (a_2,q_2) such that $(a_1,q_1)=1=(a_2,q_2)$ and $q_1\sim Q\sim q_2$ we have

$$\left|\tfrac{a_1}{q_1}-\tfrac{a_2}{q_2}\right|\gtrsim\tfrac{1}{Q^2}\gg T,$$

because $\sigma<\frac{1}{2}$. Thus

$$\left\|\mathcal{F}_{1,Q}*\phi_T\left(\tfrac{\cdot}{T}\right)\right\|_{L^\infty(\mathbb{T})}\le 1,$$

and so

$$\left\|\mathcal{F}_{1,Q}*\phi_T\left(\tfrac{\cdot}{T}\right)*f\right\|_{L^\infty(\mathbb{T})}\lesssim\|f\|_{L^1(\mathbb{T})}. \tag{2.6}$$

Next we will prove a restricted weak type (r_0',r_0) estimate for suitable $r_0\in(2,4)$. The lemma will follow by interpolating between this bound and (2.6).

Fix $r_0>2$ and take $E,F\subseteq\mathbb{T}$. Majorizing $\mathcal{F}_{1,Q}$ by $\mathcal{F}_{2,Q}$ and employing the Plancherel identity and Young's convolution inequality, we obtain

$$\begin{aligned}\langle\chi_E,\mathcal{F}_{1,Q}*\phi_T\left(\tfrac{\cdot}{T}\right)*\chi_F\rangle\lesssim{}&|E|^{\frac12}|F|^{\frac12}\left\|\widehat{\mathcal{F}_{2,Q}}(\omega)T\widehat{\phi_T}(T\omega)\right\|_{\ell^\infty_\omega(\omega\text{ good})}\\&+|E|^{\frac34}|F|^{\frac34}\left\|\widehat{\mathcal{F}_{2,Q}}(\omega)T\widehat{\phi_T}(T\omega)\right\|_{\ell^2_\omega(\omega\text{ bad})},\end{aligned} \tag{2.7}$$

where we declare

$$\omega\in\mathbb{Z}\text{ is }\textit{good}\text{ if and only if }\left|\widehat{\mathcal{F}_{2,Q}}(\omega)\right|\le Q^{1+\delta}A$$

for some small $\delta>0$ and some $A>0$ to be chosen later. By definition,

$$\left\|\widehat{\mathcal{F}_{2,Q}}(\omega)T\widehat{\phi_T}(T\omega)\right\|_{\ell^\infty_\omega(\omega\text{ good})}\lesssim Q^{1+\delta}AT. \tag{2.8}$$

We now turn to estimating the 'bad' frequencies. By Lemma 2.6, for a 'bad' frequency $\omega\neq0$ we must have $d_Q(\omega)\gtrsim AQ^\delta$. Therefore, using the fact that $\widehat{\phi_T}$ has rapid decay uniformly in T and Lemma 2.7, we obtain

$$\begin{aligned}&\left\|\widehat{\mathcal{F}_{2,Q}}(\omega)T\widehat{\phi_T}(T\omega)\right\|^2_{\ell^2_\omega(\omega\text{ bad})}\\&\lesssim T^2Q^4+\sum_{2^{\mathbb{Z}}\ni R\ge T^{-1}}\ \sum_{\substack{0<|\omega|\le R\\ \omega\text{ bad}}}\left|\widehat{\mathcal{F}_{2,Q}}(\omega)\right|^2T^2(RT)^{-100}\\&\lesssim T^2Q^4+\sum_{2^{\mathbb{Z}}\ni R\ge T^{-1}}Q^4A^{-2\alpha}Q^{2\tau-2\alpha\delta}RT^2(RT)^{-100}\\&\lesssim T^2Q^4\left(1+T^{-1}A^{-2\alpha}Q^{2\tau-2\alpha\delta}\right).\end{aligned} \tag{2.9}$$

We choose

$$A := \left(\tfrac{|E||F|}{T^2}\right)^{\frac{1}{2}-\frac{1}{r_0}}, \quad \alpha := \tfrac{4-r_0}{2(r_0-2)}, \quad \delta := \alpha^{-1}, \quad \text{and} \quad \tau := \delta. \tag{2.10}$$

Using that $|E|, |F| \leq 1$ and the restrictions on T and Q, we find

$$T^{-1}A^{-2\alpha} \geq T^{-\frac{2(r_0-2)}{r_0}} \geq Q^{\frac{2(r_0-2)}{r_0\sigma}} \geq Q^2$$

provided $r_0 \geq 2/(1-\sigma)$. Thus, combining (2.7), (2.8), and (2.9) yields

$$\langle \chi_E, \mathcal{F}_{1,Q} * \phi_T\left(\tfrac{\cdot}{T}\right) * \chi_F\rangle \lesssim (|E||F|)^{\frac{1}{r_0'}} Q^{1+\delta} T^{\frac{2}{r_0}}. \tag{2.11}$$

This proves that this convolution operator satisfies a restricted weak type (r_0', r_0) estimate with bound $Q^{1+\delta}T^{2/r_0}$, provided σ and r_0 obey the restriction stated above.

Now for p and σ as in the hypotheses of the lemma, $r_0 = 2/(1-\sigma)$ obeys $2 < r_0 < p$. Interpolating between (2.6) and the restricted weak type (r_0', r_0) estimate above, we deduce that

$$\left\|\mathcal{F}_{1,Q} * \phi_T\left(\tfrac{\cdot}{T}\right) * f\right\|_{L^p(\mathbb{T})} \lesssim Q^{\frac{2}{p}(1+\varepsilon)} T^{\frac{2}{p}} \|f\|_{L^{p'}(\mathbb{T})},$$

which proves the lemma. □

The next result performs the second step in the program laid out above, namely, it allows us to pass from convolution on $\mathbb{T}$ to convolution on $\mathbb{R}$.

Lemma 2.8. *Let $2 < p \leq \infty$ and assume $g : \mathbb{T} \to [0, \infty)$ is the kernel of a bounded convolution operator from $L^{p'}(\mathbb{T})$ to $L^p(\mathbb{T})$ with norm A. Then for any $\theta \in (0, 1]$ and $R > 0$ we have*

$$\left\|\int_{-R}^{R} g(\theta s) f(t-s)\, ds\right\|_{L^p(\mathbb{R})} \lesssim A\theta^{-\frac{2}{p}}(1+\theta R)\|f\|_{L^{p'}(\mathbb{R})}.$$

Proof. We argue by duality. Pick $k \in \mathbb{Z}$ such that $\theta R \leq k$. Using the hypothesis and the fact that $p > 2$, for $h \in L^{p'}(\mathbb{R})$ we estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \overline{h(t)} \int_{-R}^{R} g(\theta s) f(t-s)\, ds\, dt\right| \\
&\leq \iint_{|t-s|\leq R} |h(t)||f(s)| g(\theta[t-s])\, ds\, dt
\end{aligned}$$

$$
\begin{aligned}
&\le \theta^{-2} \iint_{|t-s|\le k} \left|h\left(\tfrac{t}{\theta}\right)\right|\left|f\left(\tfrac{s}{\theta}\right)\right| g(t-s)\,ds\,dt \\
&\le \theta^{-2} \sum_{\substack{m,n\in\mathbb{Z}\\ |m-n|\le k+1}} \int_0^1\int_0^1 \left|h\left(\tfrac{t+n}{\theta}\right)\right|\left|f\left(\tfrac{s+m}{\theta}\right)\right| g(t-s)\,ds\,dt \\
&\le \theta^{\frac{2}{p'}-2} A \sum_{\substack{m,n\in\mathbb{Z}\\ |m-n|\le k+1}} \|h\|_{L^{p'}([\frac{n}{\theta},\frac{n+1}{\theta}])}\|f\|_{L^{p'}([\frac{m}{\theta},\frac{m+1}{\theta}])} \\
&\le \theta^{-\frac{2}{p}} A(2k+3)\left(\sum_{n\in\mathbb{Z}} \|h\|^{p'}_{L^{p'}([\frac{n}{\theta},\frac{n+1}{\theta}])}\right)^{1/p'}\left(\sum_{m\in\mathbb{Z}} \|f\|^{p}_{L^{p'}([\frac{m}{\theta},\frac{m+1}{\theta}])}\right)^{1/p} \\
&\le \theta^{-\frac{2}{p}} A(2k+3)\|h\|_{L^{p'}(\mathbb{R})}\left(\sum_{m\in\mathbb{Z}} \|f\|^{p'}_{L^{p'}([\frac{m}{\theta},\frac{m+1}{\theta}])}\right)^{1/p'} \\
&\le \theta^{-\frac{2}{p}} A(2k+3)\|h\|_{L^{p'}(\mathbb{R})}\|f\|_{L^{p'}(\mathbb{R})}.
\end{aligned}
$$

This completes the proof of the lemma. □

We now have all the necessary ingredients to prove Strichartz estimates for the kernel $\tilde{K}_N$.

Proof of Proposition 2.3. Fix p, r as in the statement of the proposition. Combining Lemmas 2.4, 2.5, and 2.8 with (2.4), we obtain

$$
\begin{aligned}
&\|\tilde{K}_N(t) * F\|_{L_t^pL_x^r([0,1]\times\mathbb{T}^d)} \\
&\lesssim \sum_{j=1}^{d} \sum_{1\le Q\le N^{2\sigma}} \sum_{T=N^{-2}}^{N^{2\sigma-2}/Q} (QT)^{\frac{d}{r}-\frac{d}{2}} Q^{\frac{2}{p}(1+\varepsilon)} T^{\frac{2}{p}} \|F\|_{L_t^{p'}L_x^{r'}([0,1]\times\mathbb{T}^d)} \\
&\lesssim N^{2(\frac{d}{2}-\frac{2}{p}-\frac{d}{r})}\|F\|_{L_t^{p'}L_x^{r'}([0,1]\times\mathbb{T}^d)},
\end{aligned}
$$

provided we take $\sigma > 0$ sufficiently small so that $\sigma < \min\{\frac{1}{2}, 1-\frac{2}{p}\}$ and $\frac{d}{2} - \frac{2}{p}(1+\varepsilon) - \frac{d}{r} > 0$. This completes the proof of the proposition. □

We are now ready to prove the main theorem. With Proposition 2.3 in place, the rational/irrational nature of the torus plays no further role; indeed, the proof of Theorem 1.1 follows closely the ideas behind Propositions 3.82 and 3.113 in [2].

Proof of Theorem 1.1. Fix $p > p_0 := \frac{2(d+2)}{d}$ and let $f \in L^2(\mathbb{T}^d)$ be normalized via $\|f\|_{L^2(\mathbb{T}^d)} = 1$. By Bernstein's inequality,

$$\|e^{it\Delta} P_{\leq N} f\|_{L^\infty_{t,x}([0,1]\times\mathbb{T}^d)} \leq C N^{\frac{d}{2}} \|e^{it\Delta} P_{\leq N} f\|_{L^\infty_t L^2_x([0,1]\times\mathbb{T}^d)} \leq C N^{\frac{d}{2}}$$

for some $C > 0$. Thus, we may write

$$\begin{aligned} &\|e^{it\Delta} P_{\leq N} f\|_{L^p_{t,x}([0,1]\times\mathbb{T}^d)} \\ &\quad = \int_0^\infty p\lambda^{p-1} \big|\{(t,x) \in [0,1]\times\mathbb{T}^d : \big|(e^{it\Delta} P_{\leq N} f)(x)\big| > \lambda\}\big|\, d\lambda \\ &\quad = \int_0^{CN^{\frac{d}{2}}} p\lambda^{p-1} \big|\{(t,x) \in [0,1]\times\mathbb{T}^d : \big|(e^{it\Delta} P_{\leq N} f)(x)\big| > \lambda\}\big|\, d\lambda. \end{aligned} \tag{2.12}$$

For most values of λ, we exploit the non-scale-invariant Strichartz estimates of Bourgain and Demeter recorded in Theorem 1.2. Specifically, for small $\delta > 0$ to be chosen later, this theorem together with Chebyshev's inequality yields

$$\begin{aligned} &\int_0^{N^{\frac{d}{2}-\delta}} p\lambda^{p-1} \big|\{(t,x) \in [0,1]\times\mathbb{T}^d : \big|(e^{it\Delta} P_{\leq N} f)(x)\big| > \lambda\}\big|\, d\lambda \\ &\quad \lesssim \int_0^{N^{\frac{d}{2}-\delta}} p\lambda^{p-1} \frac{N^{p_0\eta}}{\lambda^{p_0}}\, d\lambda \lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})+p_0\eta-\delta(p-p_0)} \lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})}, \end{aligned} \tag{2.13}$$

provided we take $\eta < \delta(p-p_0)/p_0$. This renders acceptable the contribution of $\lambda \leq N^{\frac{d}{2}-\delta}$ to the RHS(2.12).

It remains to estimate the contribution of large values of λ. To this end, fix $\lambda > N^{\frac{d}{2}-\delta}$ and let

$$\Omega := \{(t,x) \in [0,1]\times\mathbb{T}^d : \big|(e^{it\Delta} P_{\leq N} f)(x)\big| > \lambda\}.$$

By choosing some $\omega \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ appropriately, we have that

$$\Omega_\omega := \{(t,x) \in [0,1]\times\mathbb{T}^d : \operatorname{Re}\big(e^{i\omega} e^{it\Delta} P_{\leq N} f\big)(x) > \tfrac{\lambda}{2}\}$$

satisfies $|\Omega| \leq 4|\Omega_\omega|$. By the definition of Ω_ω and Cauchy–Schwarz,

$$\begin{aligned} \lambda^2 |\Omega_\omega|^2 &\lesssim \bigg| \int_0^1 \int_{\mathbb{T}^d} (e^{it\Delta} P_{\leq N} f)(x) \chi_{\Omega_\omega}(t,x)\, dx\, dt \bigg|^2 \\ &\lesssim \|f\|^2_{L^2(\mathbb{T}^d)} \bigg\| \int_0^1 e^{-it\Delta} P_{\leq N} \chi_{\Omega_\omega}(t)\, dt \bigg\|^2_{L^2(\mathbb{T}^d)} \end{aligned} \tag{2.14}$$

$$\lesssim \int_{\mathbb{T}^d}\int_0^1\int_0^1 \chi_{\Omega_\omega}(t,x)\overline{\left[e^{i(t-s)\Delta}P^2_{\leq N}\chi_{\Omega_\omega}(s)\right]}(x)\,ds\,dt\,dx$$
$$\lesssim \langle \chi_{\Omega_\omega}, K_N\chi_{\Omega_\omega}\rangle_{L^2_{t,x}}.$$

To continue, we fix $r \in (p_0, p)$ and split $K_N = \tilde{K}_N + [K_N - \tilde{K}_N]$. Choosing σ small, we may apply Proposition 2.3 with exponent pair (r, r) to obtain

$$\left|\langle \chi_{\Omega_\omega}, \tilde{K}_N\chi_{\Omega_\omega}\rangle_{L^2_{t,x}}\right| \lesssim |\Omega_\omega|^{\frac{2}{r'}} N^{d-\frac{2(d+2)}{r}}.$$

On the other hand, by (2.2),

$$\left|\langle \chi_{\Omega_\omega}, [K_N - \tilde{K}_N]\chi_{\Omega_\omega}\rangle_{L^2_{t,x}}\right| \lesssim |\Omega_\omega|^2 N^{d(1-\sigma)}.$$

Combining these inequalities with (2.14) we obtain

$$\lambda^2|\Omega_\omega|^2 \lesssim |\Omega_\omega|^{\frac{2}{r'}} N^{d-\frac{2(d+2)}{r}} + |\Omega_\omega|^2 N^{d(1-\sigma)}.$$

We now choose $\delta \ll \frac{d\sigma}{2}$ so that the second term on the right-hand side of the inequality above is much smaller than the left-hand side. Thus we deduce that

$$|\Omega| \leq 4|\Omega_\omega| \lesssim N^{\frac{r}{2}(d-\frac{2(d+2)}{r})}\lambda^{-r}.$$

Recalling the definition of Ω and that $r \in (p_0, p)$, it then follows that

$$\int_{N^{\frac{d}{2}-\delta}}^{CN^{\frac{d}{2}}} p\lambda^{p-1}\left|\{(t,x)\in[0,1]\times\mathbb{T}^d : \left|(e^{it\Delta}P_{\leq N}f)(x)\right| > \lambda\}\right|\,d\lambda$$
$$\lesssim N^{\frac{r}{2}(d-\frac{2(d+2)}{r})}\int_{N^{\frac{d}{2}-\delta}}^{CN^{\frac{d}{2}}} \lambda^{p-1-r}\,d\lambda$$
$$\lesssim N^{p(\frac{d}{2}-\frac{d+2}{p})}.$$

This bounds the contribution of large values of λ to (2.12) in an acceptable manner and so completes the proof of Theorem 1.1. □

3. Bilinear Strichartz estimates

The purpose of this section is to discuss bilinear estimates (in all dimensions) that inform the treatment of the energy-critical NLS. We will not make direct use of the estimates in this section. We will however exploit the main idea in the proof of the following lemma, namely, splitting into frequency cubes adapted to the lower frequency. While the argument in [9] exploits a more complicated trilinear estimate (analogous to Lemma 3.3 below), they remark that it suffices to use a much simpler estimate (see equation (7) in [9]), which can be derived along the same lines as Lemma 3.1 below.

Lemma 3.1 (Bilinear Strichartz estimate). *Fix $d \geq 3$ and $T \leq 1$. Then for every $1 \leq N_2 \leq N_1$ we have*

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}([0,T)\times\mathbb{T}^d)} \lesssim N_2^{\frac{d-2}{2}} \|u_{N_1}\|_{Y^0([0,T))} \|v_{N_2}\|_{Y^0([0,T))}. \tag{3.1}$$

The implicit constant does not depend on T.

Remark 3.2. In the Euclidean setting one has the following stronger estimate:

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim N_2^{\frac{d-1}{2}} N_1^{-\frac{1}{2}} \|u_{N_1}\|_{Y^0} \|v_{N_2}\|_{Y^0}.$$

No such estimate holds on the torus. Indeed, choosing u and v to be linear solutions with characters as initial data, one can see that no negative power of the higher frequency can appear on the RHS(3.1).

Proof. To prove (3.1), we decompose $\mathbb{R}^d = \cup_j C_j$, where each C_j is a cube of side-length N_2. We write P_{C_j} for the (sharp) Fourier projection onto this cube. As the spatial Fourier support of $(P_{C_j} u_{N_1}) v_{N_2}$ is contained in a fixed dilate of the cube C_j, for each j, we deduce that

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}([0,T)\times\mathbb{T}^d)} \lesssim \Big(\sum_j \big\| (P_{C_j} u_{N_1}) v_{N_2} \big\|^2_{L^2_{t,x}([0,T)\times\mathbb{T}^d)} \Big)^{1/2}.$$

Using the Strichartz inequality (1.9), we estimate

$$\begin{aligned} \big\| (P_{C_j} u_{N_1}) v_{N_2} \big\|_{L^2_{t,x}([0,T)\times\mathbb{T}^d)} &\lesssim \|P_{C_j} u_{N_1}\|_{L^4_{t,x}([0,T)\times\mathbb{T}^d)} \|v_{N_2}\|_{L^4_{t,x}([0,T)\times\mathbb{T}^d)} \\ &\lesssim N_2^{\frac{d-2}{2}} \|P_{C_j} u_{N_1}\|_{Y^0} \|v_{N_2}\|_{Y^0}. \end{aligned}$$

Observing that

$$\|u\|_{Y^0([0,T))} \lesssim \Big(\sum_j \big\| P_{C_j} u_{N_1} \big\|^2_{Y^0([0,T))} \Big)^{1/2} \lesssim \|u\|_{Y^0([0,T))},$$

we immediately derive (3.1). □

By further exploiting the ideas in [9], one can obtain a stronger bilinear Strichartz estimate. We will not use this result in this paper and simply record the estimate for comparison. In the case $d = 4$, what follows is essentially [10, Proposition 2.8]. Their argument can be adapted to dimensions $d \geq 3$ because of the $L^4_{t,x}$ Strichartz estimate given in Theorem 1.1.

Lemma 3.3 (Improved bilinear Strichartz estimate). *Fix $d \geq 3$ and $T \leq 1$. Then there exists $\delta > 0$ such that for every $1 \leq N_2 \leq N_1$ we have*

$$\|u_{N_1} v_{N_2}\|_{L^2_{t,x}([0,T)\times\mathbb{T}^d)} \lesssim N_2^{\frac{d-2}{2}} \big(\tfrac{N_2}{N_1} + \tfrac{1}{N_2}\big)^{\delta} \|u_{N_1}\|_{Y^0([0,T))} \|v_{N_2}\|_{Y^0([0,T))}.$$

4. Well-posedness for the energy-critical NLS

The main estimates needed to prove local well-posedness for the energy-critical NLS are contained in the following proposition.

Proposition 4.1. *Fix $d \in \{3,4\}$ and $F(u) = \pm|u|^{\frac{4}{d-2}}u$. Then for any $0 < T \leq 1$,*

$$\left\| \int_0^t e^{i(t-s)\Delta} F(u(s))\, ds \right\|_{X^1([0,T])} \lesssim \|u\|^{\frac{d+2}{d-2}}_{X^1([0,T])} \tag{4.1}$$

and

$$\left\| \int_0^t e^{i(t-s)\Delta} \big[F(u+w)(s) - F(u)(s)\big]\, ds \right\|_{X^1([0,T])} \lesssim \|w\|_{X^1([0,T])} \big(\|u\|_{X^1([0,T])} + \|w\|_{X^1([0,T])}\big)^{\frac{4}{d-2}}. \tag{4.2}$$

The implicit constants do not depend on T.

Proof. As (4.1) follows from (4.2) by taking $u \equiv 0$, we will only treat the latter. Throughout the proof of the proposition all spacetime norms will be taken on $[0,T] \times \mathbb{T}^d$.

Fix $N \geq 1$ and observe that $P_{\leq N}[F(u+w) - F(u)] \in L^1([0,T]; H^1(\mathbb{T}^d))$. By duality (see Proposition 2.11 in [9]),

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-s)\Delta} P_{\leq N}\big[F(u+w)(s) - F(u)(s)\big]\, ds \right\|_{X^1([0,T])} \\
&\leq \sup_{\|\tilde{v}\|_{Y^{-1}([0,T])}=1} \left| \int_0^T \int_{\mathbb{T}^d} P_{\leq N}\big[F(u+w)(t) - F(u)(t)\big] \overline{\tilde{v}(t,x)}\, dx\, dt \right|.
\end{aligned}$$

Let $v := \overline{P_{\leq N}\tilde{v}}$. We will prove that

$$\begin{aligned}(4.3)\qquad &\Big|\int_0^T\int_{\mathbb{T}^d}\big[F(u+w)(t)-F(u)(t)\big]v(t,x)\,dx\,dt\Big|\\ &\lesssim \|v\|_{Y^{-1}([0,T])}\|w\|_{X^1([0,T])}\big(\|u\|_{X^1([0,T])}+\|w\|_{X^1([0,T])}\big)^{\frac{4}{d-2}}.\end{aligned}$$

Estimate (4.2) follows from this by letting $N\to\infty$.

A little combinatorics shows that (4.3) follows from an estimate of the form

$$\begin{aligned}(4.4)\qquad &\sum_{N_0\geq 1}\ \sum_{N_1\geq\cdots\geq N_{\frac{d+2}{d-2}}\geq 1}\Big|\int_0^T\int_{\mathbb{T}^d} v_{N_0}(t,x)\prod_{j=1}^{\frac{d+2}{d-2}} u^{(j)}_{N_j}(t,x)\,dx\,dt\Big|\\ &\lesssim \|v\|_{Y^{-1}}\prod_{j=1}^{\frac{d+2}{d-2}}\|u^{(j)}\|_{X^1([0,T])},\end{aligned}$$

by choosing $u^{(j)}$ varying over the collection $\{u,\bar{u},w,\bar{w}\}$. The remainder of the proof is dedicated to the verification of (4.4).

The treatments of the cases $d=3$ and $d=4$ are completely parallel; in both cases, we make principal use of the Strichartz estimate with exponent $\frac{2(d+1)}{d-1}$. However, we chose to write out the argument in each dimension separately, because we feel it is more readable.

Case I: $d=3$. In order to have a non-zero contribution to LHS(4.4), the two highest frequencies must be comparable. We distinguish two subcases.

Case I.1: $N_0\sim N_1\geq\cdots\geq N_5$. We exploit the main idea of the proof of Lemma 3.1. As there, let P_{C_j} denote the family of Fourier projections onto a tiling of cubes of size N_2. We write $C_j\sim C_k$ if the sum set overlaps the Fourier support of $P_{\leq 2N_2}$. Observe that given C_k there are a bounded number of $C_j\sim C_k$. Using Hölder, Bernstein, and Cauchy–Schwarz, we estimate

$$\begin{aligned}&\sum_{N_0\sim N_1\geq\cdots\geq N_5}\Big|\int_0^T\int_{\mathbb{T}^d} v_{N_0}(t,x)u^{(1)}_{N_1}(t,x)\ldots u^{(5)}_{N_5}(t,x)\,dx\,dt\Big|\\ &\lesssim \sum_{N_0\sim N_1\geq\cdots\geq N_5}\ \sum_{C_j\sim C_k}\|P_{C_j}v_{N_0}\|_{L^4_{t,x}}\|P_{C_k}u^{(1)}_{N_1}\|_{L^4_{t,x}}\\ &\qquad\qquad \|u^{(2)}_{N_2}\|_{L^4_{t,x}}\|u^{(3)}_{N_3}\|_{L^4_{t,x}}\|u^{(4)}_{N_4}\|_{L^\infty_{t,x}}\|u^{(5)}_{N_5}\|_{L^\infty_{t,x}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{N_0\sim N_1\geq\cdots\geq N_5}\sum_{C_j\sim C_k}\frac{N_0N_4^{\frac12}N_5^{\frac12}}{N_1N_2^{\frac14}N_3^{\frac34}}\|P_{C_j}v_{N_0}\|_{Y^{-1}}\|P_{C_k}u^{(1)}_{N_1}\|_{Y^1}\\
&\qquad\qquad \|u^{(2)}_{N_2}\|_{Y^1}\|u^{(3)}_{N_3}\|_{Y^1}\|u^{(4)}_{N_4}\|_{Y^1}\|u^{(5)}_{N_5}\|_{Y^1}\\
&\lesssim \|u^{(4)}\|_{Y^1}\|u^{(5)}\|_{Y^1}\sum_{N_0\sim N_1}\sum_{C_j\sim C_k}\|P_{C_j}v_{N_0}\|_{Y^{-1}}\|P_{C_k}u^{(1)}_{N_1}\|_{Y^1}\\
&\quad \sum_{N_2\geq N_3}\left(\tfrac{N_3}{N_2}\right)^{\frac14}\|u^{(2)}_{N_2}\|_{Y^1}\|u^{(3)}_{N_3}\|_{Y^1}\\
&\lesssim \prod_{j=2}^{5}\|u^{(j)}\|_{Y^1}\sum_{N_0\sim N_1}\|v_{N_0}\|_{Y^{-1}}\|u^{(1)}_{N_1}\|_{Y^1}\\
&\lesssim \|v\|_{Y^{-1}}\prod_{j=1}^{5}\|u^{(j)}\|_{Y^1}.
\end{aligned}
$$

This settles Case I.1 because $X^1 \hookrightarrow Y^1$.

Case I.2: $N_0 \lesssim N_1 \sim N_2 \geq N_3 \geq N_4 \geq N_5$. In this subcase, we do not need to decompose into cubes and only use the Strichartz inequalities proved in Theorem 1.1:

$$
\begin{aligned}
&\sum_{N_0\lesssim N_1\sim N_2\geq\cdots\geq N_5}\left|\int_0^T\int_{\mathbb{T}^d}v_{N_0}(t,x)u^{(1)}_{N_1}(t,x)\dots u^{(5)}_{N_5}(t,x)\,dx\,dt\right|\\
&\lesssim \sum_{N_0\lesssim N_1\sim N_2\geq\cdots\geq N_5}\|v_{N_0}\|_{L^4_{t,x}}\|u^{(1)}_{N_1}\|_{L^4_{t,x}}\|u^{(2)}_{N_2}\|_{L^4_{t,x}}\|u^{(3)}_{N_3}\|_{L^4_{t,x}}\|u^{(4)}_{N_4}\|_{L^\infty_{t,x}}\|u^{(5)}_{N_5}\|_{L^\infty_{t,x}}\\
&\lesssim \sum_{N_0\lesssim N_1\sim N_2\geq\cdots\geq N_5}\frac{N_0^{\frac54}N_4^{\frac12}N_5^{\frac12}}{N_1^{\frac34}N_2^{\frac34}N_3^{\frac34}}\|v_{N_0}\|_{Y^{-1}}\|u^{(1)}_{N_1}\|_{Y^1}\|u^{(2)}_{N_2}\|_{Y^1}\\
&\qquad\qquad \|u^{(3)}_{N_3}\|_{Y^1}\|u^{(4)}_{N_4}\|_{Y^1}\|u^{(5)}_{N_5}\|_{Y^1}\\
&\lesssim \|v\|_{Y^{-1}}\prod_{j=3}^{5}\|u^{(j)}\|_{Y^1}\sum_{N_1\sim N_2}\left(\tfrac{N_1}{N_2}\right)^{\frac12}\|u^{(1)}_{N_1}\|_{Y^1}\|u^{(2)}_{N_2}\|_{Y^1}\\
&\lesssim \|v\|_{Y^{-1}}\prod_{j=1}^{5}\|u^{(j)}\|_{Y^1}.
\end{aligned}
$$

This completes the proof of the proposition in the case of three space dimensions.

Case II: $d = 4$. Again we distinguish two subcases: either $N_0 \sim N_1 \geq N_2 \geq N_3$ or $N_0 \lesssim N_1 \sim N_2 \geq N_3$.

Case II.1: $N_0 \sim N_1 \geq N_2 \geq N_3$. Arguing exactly as in Case I.1, we obtain

$$\begin{aligned}
&\sum_{N_0\sim N_1\geq N_2\geq N_3} \Big| \int_0^T \int_{\mathbb{T}^d} v_{N_0}(t,x) u^{(1)}_{N_1}(t,x) u^{(2)}_{N_2}(t,x) u^{(3)}_{N_3}(t,x)\,dx\,dt \Big| \\
&\lesssim \sum_{N_0\sim N_1\geq N_2\geq N_3} \sum_{C_j\sim C_k} \big\|P_{C_j} v_{N_0}\big\|_{L^{10/3}_{t,x}} \big\|P_{C_k} u^{(1)}_{N_1}\big\|_{L^{10/3}_{t,x}} \big\|u^{(2)}_{N_2}\big\|_{L^{10/3}_{t,x}} \big\|u^{(3)}_{N_3}\big\|_{L^{10}_{t,x}} \\
&\lesssim \sum_{N_0\sim N_1} \sum_{C_j\sim C_k} \tfrac{N_0}{N_1} \big\|P_{C_j} v_{N_0}\big\|_{Y^{-1}} \big\|P_{C_k} u^{(1)}_{N_1}\big\|_{Y^1} \sum_{N_2\geq N_3} \big(\tfrac{N_3}{N_2}\big)^{\frac{2}{5}} \big\|u^{(2)}_{N_2}\big\|_{Y^1} \big\|u^{(3)}_{N_3}\big\|_{Y^1} \\
&\lesssim \|v\|_{Y^{-1}} \prod_{j=1}^3 \|u^{(j)}\|_{Y^1}.
\end{aligned}$$

Case II.2: $N_0 \lesssim N_1 \sim N_2 \geq N_3$. We argue in the same manner as Case I.2:

$$\begin{aligned}
&\sum_{N_0\lesssim N_1\sim N_2\geq N_3} \Big| \int_0^T \int_{\mathbb{T}^d} u^{(1)}_{N_1}(t,x) u^{(2)}_{N_2}(t,x) u^{(3)}_{N_3}(t,x) v_{N_0}(t,x)\,dx\,dt \Big| \\
&\lesssim \sum_{N_0\lesssim N_1\sim N_2\geq N_3} \big\|v_{N_0}\big\|_{L^{10/3}_{t,x}} \big\|u^{(1)}_{N_1}\big\|_{L^{10/3}_{t,x}} \big\|u^{(2)}_{N_2}\big\|_{L^{10/3}_{t,x}} \big\|u^{(3)}_{N_3}\big\|_{L^{10}_{t,x}} \\
&\lesssim \sum_{N_0\lesssim N_1\sim N_2\geq N_3} \frac{N_0^{\frac{6}{5}} N_3^{\frac{2}{5}}}{N_1^{\frac{4}{5}} N_2^{\frac{4}{5}}} \big\|v_{N_0}\big\|_{Y^{-1}} \big\|u^{(1)}_{N_1}\big\|_{Y^1} \big\|u^{(2)}_{N_2}\big\|_{Y^1} \big\|u^{(3)}_{N_3}\big\|_{Y^1} \\
&\lesssim \|v\|_{Y^{-1}} \|u^{(3)}\|_{Y^1} \sum_{N_1\sim N_2} \big(\tfrac{N_1}{N_2}\big)^{\frac{2}{5}} \|u^{(1)}_{N_1}\|_{Y^1} \|u^{(2)}_{N_2}\|_{Y^1} \\
&\lesssim \|v\|_{Y^{-1}} \prod_{j=1}^3 \|u^{(j)}\|_{Y^1}.
\end{aligned}$$

This completes the proof of the proposition when $d = 4$. □

Proof of Theorem 1.3. With Proposition 4.1 in place, the remainder of the proof mimics the arguments in [9]. We first consider the case of *small initial data.* Fix $d \in \{3, 4\}$ and let $u_0 \in H^1(\mathbb{T}^d)$ satisfy

$$\|u_0\|_{H^1(\mathbb{T}^d)} \leq \eta \leq \eta_0$$

for a small $\eta_0 = \eta_0(d)$ to be chosen later.

We first note that by conservation of mass and energy, it suffices to construct the solution to the initial-value problem (1.4) on the time interval

$[0,1]$. Indeed, by Sobolev embedding,

$$\|f\|_{L^{\frac{2d}{d-2}}(\mathbb{T}^d)} \lesssim_d \|f\|_{H^1(\mathbb{T}^d)}$$

and so, in both the defocusing and the focusing cases we have

$$M(u)+E(u) = \int_{\mathbb{T}^d} \tfrac{1}{2}|u_0(x)|^2 + \tfrac{1}{2}|\nabla u_0(x)|^2 \pm \tfrac{d-2}{2d}|u_0(x)|^{\frac{2d}{d-2}}\, dx \sim \|u_0\|^2_{H^1(\mathbb{T}^d)},$$

provided $\eta_0(d)$ is chosen sufficiently small. Using a continuity argument together with the conservation of mass and energy, we deduce that this equivalence holds at all times of existence, namely,

$$M(u) + E(u) \sim \|u(t)\|^2_{H^1(\mathbb{T}^d)}.$$

Thus, a simple iteration argument allows us to extend the local-in-time solution to a global-in-time solution.

To construct the solution to (1.4) on the time interval $[0,1]$, we use a contraction mapping argument. More precisely, we will show that the mapping

$$\Phi(u)(t) := e^{it\Delta}u_0 \mp i \int_0^t e^{i(t-s)\Delta} F(u(s))\, ds \tag{4.5}$$

is a contraction on the ball

$$B := \left\{u \in X^1([0,1]) \cap C_t H^1_x([0,1]\times\mathbb{T}^d) :\ \|u\|_{X^1([0,1])} \le 2\eta\right\}$$

under the metric

$$d(u,v) := \|u - v\|_{X^1([0,1])}.$$

Using Proposition 4.1, we see that for $u \in B$,

$$\begin{aligned}\|\Phi(u)\|_{X^1([0,1])} &\le \|e^{it\Delta}u_0\|_{X^1([0,1])} + \left\|\int_0^t e^{i(t-s)\Delta}F(u(s))\, ds\right\|_{X^1([0,1])} \\ &\le \|u_0\|_{H^1(\mathbb{T}^d)} + C\|u\|^{\frac{d+2}{d-2}}_{X^1([0,1])} \le \eta + C(2\eta)^{\frac{d+2}{d-2}} \le 2\eta,\end{aligned}$$

provided η_0 is chosen sufficiently small. This proves Φ maps the ball B to itself.

To see that Φ is a contraction under the metric d, we apply Proposition 4.1 to $u, v \in B$ to get

$$\begin{aligned} d(\Phi(u), \Phi(v)) &\leq \Big\| \int_0^t e^{i(t-s)\Delta}\big[F(u(s)) - F(v(s))\big]\, ds \Big\|_{X^1([0,1])} \\ &\lesssim \|u - v\|_{X^1([0,1])} \big(\|u\|_{X^1([0,1])} + \|v\|_{X^1([0,1])} \big)^{\frac{4}{d-2}} \\ &\lesssim d(u,v)(4\eta)^{\frac{4}{d-2}} \\ &\leq \tfrac{1}{2} d(u,v), \end{aligned}$$

provided η_0 is chosen sufficiently small.

This completes the discussion of small initial data. We now turn to the statement in Theorem 1.3 concerning *large initial data.*

Let $u_0 \in H^1(\mathbb{T}^d)$ with

$$\|u_0\|_{H^1(\mathbb{T}^d)} \leq A$$

for some $0 < A < \infty$. Let $\delta > 0$ be a small number to be chosen later (depending on A) and let $N = N(u_0) \geq 1$ be such that

$$\|P_{>N} u_0\|_{H^1(\mathbb{T}^d)} \leq \delta.$$

We will show that the mapping $\Phi(u)$ defined in (4.5) is a contraction on the ball

$$\begin{aligned} B := \big\{ u \in X^1([0,T]) \cap C_t H^1_x([0,T] \times \mathbb{T}^d) : \\ \|u\|_{X^1([0,T])} \leq 2A,\ \|u_{>N}\|_{X^1([0,T])} \leq 2\delta \big\} \end{aligned}$$

under the metric

$$d(u,v) := \|u - v\|_{X^1([0,T])},$$

provided T is chosen sufficiently small (depending on A, δ, and N). For the remainder of the proof, all space-time norms will be on $[0,T] \times \mathbb{T}^d$.

First we verify that Φ maps B to itself. Using Remark 1.8, Proposition 4.1, and Bernstein, for $u \in B$ we estimate

$$\begin{aligned} \|\Phi(u)\|_{X^1} &\leq \|e^{it\Delta} u_0\|_{X^1} + \Big\| \int_0^t e^{i(t-s)\Delta} F(u_{\leq N}(s))\, ds \Big\|_{X^1} \\ &\quad + \Big\| \int_0^t e^{i(t-s)\Delta}\big[F(u(s)) - F(u_{\leq N}(s))\big]\, ds \Big\|_{X^1} \\ &\leq \|u_0\|_{H^1(\mathbb{T}^d)} + C\|F(u_{\leq N})\|_{L^1_t H^1_x} + C\|u_{>N}\|_{X^1} \|u\|_{X^1}^{\frac{4}{d-2}} \end{aligned}$$

$$\begin{aligned}
&\le A + CT\|u_{\le N}\|_{L_t^\infty H_x^1}\|u_{\le N}\|_{L^\infty_{t,x}}^{\frac{4}{d-2}} + C(2\delta)(2A)^{\frac{4}{d-2}}\\
&\le A + CTN^2(2A)^{\frac{d+2}{d-2}} + C(2\delta)(2A)^{\frac{4}{d-2}} \le 2A,
\end{aligned}$$

provided δ is chosen small enough depending on A, and T is chosen small enough depending on A and N.

We decompose

$$\begin{aligned}
&F(u) = F_1(u) + F_2(u)\\
&\text{where}\quad F_1(u) = O\big(u_{>N}^2 u^{\frac{6-d}{d-2}}\big) \quad\text{and}\quad F_2(u) = O\big(u_{\le N}^{\frac{4}{d-2}} u\big).
\end{aligned}$$

Here, O aggregates terms of similar structure, where factors may additionally have complex conjugates and/or further Littlewood–Paley projections. Arguing similarly to the above, we estimate

$$\begin{aligned}
&\|P_{>N}\Phi(u)\|_{X^1}\\
&\le \|e^{it\Delta}P_{>N}u_0\|_{X^1} + \left\|\int_0^t e^{i(t-s)\Delta}F_1(u(s))\,ds\right\|_{X^1}\\
&\quad + \left\|\int_0^t e^{i(t-s)\Delta}F_2(u(s))\,ds\right\|_{X^1}\\
&\le \|P_{>N}u_0\|_{H^1(\mathbb{T}^d)} + C\|u_{>N}\|_{X^1}^2\|u\|_{X^1}^{\frac{6-d}{d-2}} + C\|F_2(u)\|_{L_t^1H_x^1}\\
&\le \delta + C(2\delta)^2(2A)^{\frac{6-d}{d-2}}\\
&\quad + CT\left[\|\nabla u\|_{L_t^\infty L_x^2}\|u_{\le N}\|_{L^\infty_{t,x}}^{\frac{4}{d-2}} + \|u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} N\|u_{\le N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}}^{\frac{4}{d-2}}\right]\\
&\le \delta + C(2\delta)^2(2A)^{\frac{6-d}{d-2}} + CTN^2(2A)^{\frac{d+2}{d-2}}\\
&\le 2\delta,
\end{aligned}$$

provided δ is chosen small enough depending on A, and T is chosen small enough depending on A, δ, and N.

Next, we prove that Φ is a contraction. We again decompose $F = F_1 + F_2$ and observe that

$$F_1(u) - F_1(v) = O\Big((u-v)(u_{>N}+v_{>N})\big(u^{\frac{6-d}{d-2}} + v^{\frac{6-d}{d-2}}\big)\Big)$$

and

$$\begin{aligned}
F_2(u) - F_2(v) = {}& O\Big((u-v)\big(u_{\le N}+v_{\le N}\big)^{\frac{4}{d-2}}\Big)\\
&+ O\Big((u_{\le N}-v_{\le N})(u+v)\big(u_{\le N}+v_{\le N}\big)^{\frac{6-d}{d-2}}\Big).
\end{aligned}$$

Employing Remark 1.8, Proposition 4.1, and Bernstein as before, for $u, v \in B$ we estimate

$$
\begin{aligned}
& d\big(\Phi(u), \Phi(v)\big) \\
& \lesssim \|u-v\|_{X^1}\big(\|u_{>N}\|_{X^1} + \|v_{>N}\|_{X^1}\big)\big(\|u\|_{X^1} + \|v\|_{X^1}\big)^{\frac{6-d}{d-2}} \\
& \quad + \|F_2(u) - F_2(v)\|_{L_t^1 H_x^1} \\
& \lesssim (4\delta)(4A)^{\frac{6-d}{d-2}} d(u,v) + T\|\nabla(u-v)\|_{L_t^\infty L_x^2}\big(\|u_{\leq N}\|_{L_{t,x}^\infty} + \|v_{\leq N}\|_{L_{t,x}^\infty}\big)^{\frac{4}{d-2}} \\
& \quad + T\|u-v\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} N\big(\|u_{\leq N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}} + \|v_{\leq N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}}\big)^{\frac{4}{d-2}} \\
& \quad + T\big(\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla v\|_{L_t^\infty L_x^2}\big)\|u_{\leq N} \\
& \quad - v_{\leq N}\|_{L_{t,x}^\infty}\big(\|u_{\leq N}\|_{L_{t,x}^\infty} + \|v_{\leq N}\|_{L_{t,x}^\infty}\big)^{\frac{6-d}{d-2}} \\
& \quad + T\big(\|u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} + \|v\|_{L_t^\infty L_x^{\frac{2d}{d-2}}}\big)N\|u_{\leq N} - v_{\leq N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}} \\
& \quad \times \big(\|u_{\leq N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}} + \|v_{\leq N}\|_{L_t^\infty L_x^{\frac{4d}{d-2}}}\big)^{\frac{6-d}{d-2}} \\
& \lesssim \big[(4\delta)(4A)^{\frac{6-d}{d-2}} + TN^2(4A)^{\frac{4}{d-2}}\big] d(u,v) \\
& \leq \tfrac{1}{2} d(u,v),
\end{aligned}
$$

provided δ is chosen small enough depending on A, and T is chosen small enough depending on A and N.

By the contraction mapping theorem, this allows us to construct a unique solution u to (1.4) in the ball B. To see that uniqueness holds in the larger class $X^1([0,T]) \cap C_t H_x^1([0,T] \times \mathbb{T}^d)$, we need only observe that if $v \in X^1([0,T]) \cap C_t H_x^1([0,T] \times \mathbb{T}^d)$ is a second solution to (1.4) with data $v(0) = u_0$, then there exists $N_0 \geq 1$ such that

$$\|v_{>N_0}\|_{X^1([0,T])} \leq 2\delta.$$

Choosing the larger of N and N_0, we find a new ball B that contains both u and v. In this way, the contraction mapping argument guarantees $u = v$ on a possibly smaller interval $[0, T']$. Iterating this argument yields uniqueness in the larger class. □

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