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Three-point Lie algebras and Grothendieck's dessins d'enfants

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We define and classify the analogues of the affine Kac-Moody Lie algebras for the ring of functions on the complex projective line minus three points. The classification is given in terms of Grothendieck's dessins d'enfants. We also study the question of conjugacy of Cartan subalgebras for these algebras.

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra, and let $R = \mathbb{C}[t^{\pm 1}]$. The complex Lie algebras $\mathfrak{g} \otimes_{\mathbb{C}} R$ are at the heart of the untwisted affine Kac-Moody Lie algebras. There are, however, other (twisted) affine algebras. They can be realized in term of loop algebras. Relevant to us is that these loop algebras are precisely the Lie algebras over R that are locally for the étale topology of R isomorphic to $\mathfrak{g} \otimes_{\mathbb{C}} R$.

The ring R is the ring of functions of the variety $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1\}$. The ring R' of functions of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ has also led to a fascinating class of Lie algebras $\mathfrak{g} \otimes_{\mathbb{C}} R'$; the so-called three-point algebras. Interesting connections between three-point algebras and the Onsager algebra, which was used in the resolution of the Ising model [HT], have been described in [BT]. Further references and generalizations can be found in [EF] and [PS].

If we think of the three-point algebras $\mathfrak{g} \otimes_{\mathbb{C}} R'$ as the analogues of the untwisted affine Kac-Moody Lie algebras, it is inevitable to ask if there are three-point analogues of the twisted affine Lie algebras. We provide a positive (and natural) answer to this question.

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The main aim of the paper is two-fold. First, we classify $\mathbb{C}$-isomorphism classes of all twisted forms of three-point algebras. The resulting family of algebras is related to Grothendieck's dessins d'enfants. For the details of the classification we refer to Section 7.

Second, we also study the (natural) question of conjugacy of Cartan subalgebras of three-point algebras, which is of importance for their (future) representation theory. In Section 8 we give a cohomological description of conjugacy classes of the so-called generically maximal split tori in reductive group schemes over arbitrary base schemes and in Section 9 we apply this result to conjugacy of Cartan subalgebras in twisted forms of three-point algebras. The main ingredient of our considerations in this part is the correspondence between maximal split tori of a simple group scheme and maximal diagonalizable Lie subalgebras of its Lie algebra given in our paper [CGP, Theorem 7.1].

By looking at the analogue situation in the affine Kac-Moody case, we expect our work to be relevant to possible generalizations of the Wess-Zumino-Witten models and the theory of open strings (where the twisted algebras enter into the picture). Other connections to Physics as well as relevant references can be found in [Sch].

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2. Notation and conventions

Throughout k will denote an algebraically closed field of characteristic 0, and $\mathfrak{g}$ a finite dimensional semisimple Lie algebra over k.

We fix a (necessarily split) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a base Δ of the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. The simply connected (resp. adjoint) Chevalley-Demazure group over k corresponding to $\mathfrak{g}$ will be denoted by $\mathbf{G}^{\rm sc}$ (resp. $\mathbf{G}^{\rm ad}$). We then have the exact sequence of algebraic k-groups

$$1 \to \boldsymbol{\mu} \to \mathbf{G}^{\rm sc} \xrightarrow{\mathrm{Ad}} \mathbf{G}^{\rm ad} \to 1$$

where $\boldsymbol{\mu} = Z(\mathbf{G}^{\rm sc})$ is the centre of $\mathbf{G}^{\rm sc}$. The split torus of $\mathbf{G}^{\rm sc}$ (resp. $\mathbf{G}^{\rm ad}$) corresponding to $\mathfrak{h}$ will be denoted by $\mathbf{T}^{\rm sc}$ (resp. $\mathbf{T}^{\rm ad}$). As usual, we denote the k-group of automorphisms of $\mathbf{G}$ by $\mathbf{Aut}(\mathbf{G})$.

If $\mathbf{Aut}(\mathfrak{g})$ denotes the (linear algebraic) k-group of automorphisms of $\mathfrak{g}$, then $\mathbf{Aut}(\mathfrak{g})^0 \simeq \mathbf{G}^{\mathrm{ad}}$ via the adjoint representation. We have a split exact sequence

$$1 \to \mathbf{G}^{\mathrm{ad}} \to \mathbf{Aut}(\mathfrak{g}) \to \mathbf{Out}(\mathfrak{g}) \to 1 \tag{2.1}$$

of algebraic k-groups. Here $\mathbf{Out}(\mathfrak{g})$ is the finite constant k-group corresponding to the finite (abstract) group of symmetries of the Dynkin diagram of $\mathfrak{g}$.

In what follows we will denote the abstract groups $\mathbf{Aut}(\mathfrak{g})(k)$ and $\mathbf{Out}(\mathfrak{g})(k)$ by $\mathrm{Aut}_k(\mathfrak{g})$ and $\mathrm{Out}_k(\mathfrak{g})$ respectively.

If $\mathbf{X}$ is a k-scheme and $\mathfrak{G}$ a group scheme over $\mathbf{X}$, we will denote throughout $H^1_{fppf}(\mathbf{X},\mathfrak{G})$ by $H^1(\mathbf{X},\mathfrak{G})$. Note that if $\mathfrak{G}$ is smooth then $H^1_{\acute{e}t}(\mathbf{X},\mathfrak{G}) = H^1_{fppf}(\mathbf{X},\mathfrak{G})$ where $H^1_{\acute{e}t}$ stands for étale non-abelian cohomology.

3. Application of a result by Harder

Proposition 3.1. *Let C be a smooth connected affine curve over k and let $\mathfrak{G}$ be a semisimple group scheme over C. Then $H^1(C,\mathfrak{G}) = 1$.*

Proof. Assume first that $\mathfrak{G}$ is simply connected. By Steinberg's theorem (see [St, Theorem 11.2] and also [Se] Ch. III §2, Theorem 1') we know that

$$\mathfrak{G}_{k(C)} = \mathfrak{G} \times_C \mathrm{Spec}\,\big(k(C)\big)$$

has a Borel subgroup (i.e., it is quasi-split). In Harder's [H] terminology, this last means that $\mathfrak{G}$ is *rationally quasi-trivial.* By [H, Satz 3.3] we then have $H^1(C,\mathfrak{G}) = 1$.

For the general case, we consider the simply connected covering $f : \widetilde{\mathfrak{G}} \to \mathfrak{G}$. Its kernel $\boldsymbol{\mu}$ is a finite étale C-group scheme and is central in $\widetilde{\mathfrak{G}}$. We consider the following exact sequence of pointed sets [Gi, IV.4.2.10]

$$H^1(C,\widetilde{\mathfrak{G}}) \to H^1(C,\mathfrak{G}) \to H^2(C,\boldsymbol{\mu}).$$

But as we have seen $H^1(C,\widetilde{\mathfrak{G}}) = 1$, while $H^2(C,\boldsymbol{\mu}) = 0$ by [M, Theorem 15.1] given that $\mathrm{cd}(C) \le \dim(C) = 1$. It follows that $H^1(C,\mathfrak{G}) = 1$. □

Remark 3.2. If $C = \mathbb{A}^1_k$ or $\mathfrak{G}_{m,k}$, we have furthermore $H^1(C,\mathfrak{G}) = 1$ for each reductive group C-scheme $\mathfrak{G}$ [P].

Corollary 3.3. *Let $C = \mathrm{Spec}(R)$ be a smooth connected affine curve over k. Let $\mathbf{G}$ be a semisimple split k-group scheme. Then* *(i) The map* $p : \mathbf{Aut}(\mathbf{G}) \to \mathbf{Out}(\mathbf{G})$ *induces a bijection of pointed sets*

$$p_* : H^1\big(C, \mathbf{Aut}(\mathbf{G})\big) \xrightarrow{\sim} H^1\big(C, \mathbf{Out}(\mathbf{G})\big).$$

Similar considerations apply if we replace $\mathbf{G}$ *by* $\mathfrak{g}$.

(ii) All C*–forms of* $\mathbf{G}_C = \mathbf{G} \times_k \mathrm{Spec}(R)$ *are quasi-split. In particular, every semisimple group scheme over* C *is quasi-split.*

Proof. We have the split exact sequence of k (or C, by base change)–groups

$$1 \longrightarrow \mathbf{G}^{\mathrm{ad}} \xrightarrow{\mathrm{int}} \mathbf{Aut}(\mathbf{G}) \xrightarrow{p} \mathbf{Out}(\mathbf{G}) \longrightarrow 1. \tag{3.1}$$

We consider a pinning (épinglage) of $\mathbf{G}$, this includes the choice of a Killing couple $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$ and defines a section

$$h : \mathbf{Out}(\mathbf{G}) \to \mathbf{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T}) \subset \mathbf{Aut}(\mathbf{G})$$

(see [SGA3, XXIV.1]).

Let $\mathbf{F}$ be a representative of a class of $H^1\big(C, \mathbf{Out}(\mathbf{G})\big)$. Consider $\mathbf{G}^{\mathrm{ad}}_{h_*\mathbf{F}}$, namely the semisimple C–group obtained by twisting $\mathbf{G}^{\mathrm{ad}}$ by the $\mathbf{Aut}(\mathbf{G})$-torsor $h_*\mathbf{F}$. It is well known (see [DG] II.4.5.1) that the classes in $H^1\big(C, \mathbf{Aut}(\mathbf{G})\big)$ which map to the class of $\mathbf{F}$ under the map

$$p_* : H^1\big(C, \mathbf{Aut}(\mathbf{G})\big) \to H^1\big(C, \mathbf{Out}(\mathbf{G})\big)$$

are in one-to-one correspondence with elements in the image of

$$H^1(C, \mathbf{G}^{ad}_{h_*\mathbf{F}}) \to H^1\big(C, \mathbf{Aut}(\mathbf{G}^{ad}_{h_*\mathbf{F}})\big).$$

But $H^1(C, \mathbf{G}^{ad}_{h_*\mathbf{F}})$ vanishes by Proposition 3.1. This shows that p_* is injective. It is surjective because (3.1) is split. Similar considerations apply to $\mathfrak{g}$. Indeed if $\mathbf{G}$ is of simply connected type, then $\mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathfrak{g})$ by [SGA3, Exp XXIV §7.3]. This finishes the proof of (i).

(ii) For a twisted form of $\mathbf{G}_C$ to be quasi-split, it suffices that the corresponding adjoint group be quasi-split. That $\mathbf{G}^{\mathrm{ad}}_{h_*\mathbf{F}}$ is quasi-split can be explicitly seen from the fact that $h_*\mathbf{F}$ stabilizes $\mathbf{B}$ and $\mathbf{T}$, so that (ii) follows from (i) (or, alternatively, see [H, Satz 3.1]). □

4. Affine Kac-Moody analogues of the three-point algebras

In what follows $\mathfrak{g}$ denotes a simple Lie algebra over k. Kac-Moody algebras are defined by generators and relations (a là Chevalley-Serre). In the affine case, which is the most interesting outside the classical finite dimensional setup, the algebras can be "realized" in terms of loop algebras (a construction that we will describe shortly). Let $R = k[t^{\pm 1}]$ and $R' = k[t^{\pm 1}]_{(t-1)}$.

The simplest case is that of the untwisted affine algebras. These correspond to the k-algebras $\mathfrak{g} \otimes_k R$. The elements of $\mathfrak{g} \otimes_k R$ can be thought as morphisms (in the algebraic sense) of $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}_k^{\dim_k(\mathfrak{g})}$ which has a natural appeal to many current constructions in Physics. The same is true if we replace $\mathbb{A}^1_k \setminus \{0\} = \mathbb{P}^1_k \setminus \{0, \infty\}$ by $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$. The resulting k-algebras $\mathfrak{g} \otimes_k R'$ are called three-point Lie algebras in the literature (see [B] and also [BT] for references). This raises an inevitable question: What are the analogues, in the case of R', of the affine Kac-Moody Lie algebras which are not of the form $\mathfrak{g} \otimes_k R$? Our point of view is that the affine Kac-Moody algebras can be thought as twisted forms of some $\mathfrak{g} \otimes_k R$ in the étale topology of Spec (R) as we will explain in Theorem 4.2 below.[1]

We now review the realization of the affine algebras as loop algebras.

Fix once and for all a set $(\zeta_n)_{n>0}$ of compatible primitive n-roots of unity (i.e. $\zeta_{nh}^h = \zeta_n$). Having done this, the ingredient for defining loop algebras based on $\mathfrak{g}$ is a finite order automorphism σ of $\mathfrak{g}$. For each $i \in \mathbb{Z}$ consider the eigenspace

$$\mathfrak{g}_i := \{x \in \mathfrak{g} : \sigma(x) = \zeta_m^i x\}$$

where m is a (fixed) period of σ i.e. $\sigma^m = 1$. Let

$$R_m = k[t^{\pm \frac{1}{m}}], \text{ and } R_\infty = \varinjlim_m R_m.$$

The loop algebra $L(\mathfrak{g}, \sigma)$ is then defined as follows

$$L(\mathfrak{g}, \sigma) = \bigoplus \mathfrak{g}_i \otimes t^{\frac{i}{m}} \subset \mathfrak{g} \otimes_k R_m \subset \mathfrak{g} \otimes_k R_\infty.$$

Remark 4.1. $L(\mathfrak{g}, \sigma)$ is a Lie subalgebra of $\mathfrak{g} \otimes_k R_\infty$ which does not depend on the choice of period m of σ. Besides being an (infinite dimensional) Lie algebra over k, the loop algebra $L(\mathfrak{g}, \sigma)$ is also naturally a Lie algebra over $R = k[t^{\pm 1}]$.

[1]The term affine Kac-Moody Lie algebra is being used here to denote the corresponding loop algebra. More precisely. Let $\hat{\mathcal{L}}$ be an affine Kac-Moody Lie algebra in the sense of [Kac], and let $\mathcal{L} = [\hat{\mathcal{L}}, \hat{\mathcal{L}}]/\mathfrak{z}$ where $\mathfrak{z}$ is the centre of the derived algebra $[\hat{\mathcal{L}}, \hat{\mathcal{L}}]$ of $\hat{\mathcal{L}}$. It is well-known that $\mathcal{L}$ is a loop algebra. These are the algebras that we will be considering. In what follows we refer to $\mathcal{L}$ as the "derived algebra of $\hat{\mathcal{L}}$ modulo its centre".

These are the loop algebras appearing in affine Kac-Moody theory. To connect them to non-abelian cohomology the crucial observation, which is easy to verify, is that

$$L(\mathfrak{g},\sigma)\otimes_R R_m \simeq \mathfrak{g}\otimes_k R_m \simeq (\mathfrak{g}\otimes_k R)\otimes_R R_m \tag{4.1}$$

where the isomorphism is of R_m–Lie algebras. Thus loop algebras are a particular kind of twisted forms of the R-Lie algebra $\mathfrak{g}\otimes_k R$. Their isomorphism classes correspond to a subset $H^1_{loop}\big(R,\mathbf{Aut}(\mathfrak{g})\big)$ of $H^1\big(R,\mathbf{Aut}(\mathfrak{g})\big)$ where this last H^1 classifies *all* twisted forms of $\mathfrak{g}\otimes_k R$. We have $H^1_{loop}\big(R,\mathbf{Aut}(\mathfrak{g})\big)=H^1\big(R,\mathbf{Aut}(\mathfrak{g})\big)$, namely all twisted forms are loop algebras. This follows from [P], but a more explicit result dealing with loop algebras as Lie algebras over k will given in Theorem 4.2 below by exploiting the fact that the centroid of any twisted form of $\mathfrak{g}\otimes_k R$ is isomorphic to R (see [GP] Lemma 4.6.3 [2]). It is worth pointing out that for Laurent polynomial rings in more than one variable $H^1_{loop}\big(R,\mathbf{Aut}(\mathfrak{g})\big)$ does not necessarily equal $H^1\big(R,\mathbf{Aut}(\mathfrak{g})\big)$ in general. In other words, there exist twisted forms of $\mathfrak{g}\otimes_k R$ which are not multiloop algebras based on $\mathfrak{g}$. See [GP] for details.

Let $\mathcal{L}$ be a Lie algebra over k. Recall that the centroid $C(\mathcal{L})$ of $\mathcal{L}$ is defined by

$$C(\mathcal{L})=\{\theta\in \mathrm{End}_k(\mathcal{L}) : \theta[x,y]=[x,\theta(y)]=[\theta(x),y] \text{ for all } x,y\in\mathcal{L}\}.$$

If $\mathcal{L}$ is perfect, then $C(\mathcal{L})$ is commutative and we can view $\mathcal{L}$ naturally as a Lie algebra over the (commutative and unital) ring $C(\mathcal{L})$.

Theorem 4.2. *For a Lie algebra $\mathcal{L}$ over k the following conditions are equivalent.*

(i): *$\mathcal{L}$ is the derived algebra modulo its centre of an affine Kac-Moody Lie algebra.*

(ii): *$\mathcal{L}\simeq L(\mathfrak{g},\sigma)$ for some $\mathfrak{g}$ (unique up to isomorphism) and some $\sigma\in$ $\mathrm{Aut}_k(\mathfrak{g})$ of finite order.*

(ii)(bis): *As in* (ii) *but with $\sigma\in \mathrm{Out}_k(\mathfrak{g})\subset \mathrm{Aut}_k(\mathfrak{g})$.*

(iii): *$C(\mathcal{L})\simeq R$, and there exists an étale extension S of $C(\mathcal{L})$ such that*

$$\mathcal{L}\otimes_{C(\mathcal{L})} S\simeq \mathfrak{g}\otimes_k S.$$

[2] That $\mathfrak{g}$ is simple is essential. If $\mathfrak{g}$ is semisimple but not simple, then the centroid of $\mathcal{L}$ is not isomorphic to R.

(iii)(bis): *As in* (iii) *but with S Galois.*

Proof. Except for uniqueness, the equivalence of (i), (ii) and (ii)(bis) is the realization theorem of the affine algebras as described in Ch. 8 of [Kac]. Given that $C(\mathcal{L}) \simeq R$, that (ii) implies (iii)(bis) is easy in view of (4.1) and the fact that R_m/R is Galois. The rest of the equivalences hinges on Corollary 3.3. The uniqueness of $\mathfrak{g}$ and that there is no loss of information in passing from R-algebra isomorphisms to k-algebra isomorphisms is explained in [P]. □

We now explain how cohomology enters into the picture, and how it looks if we replace $R = k[t^{\pm 1}] = k[t]_t$ by $R' = k[t^{\pm 1}]_{t-1} = k[t]_{t(t-1)}$. The Lie algebra $\mathfrak{g} \otimes_k R'$ plays the role of the untwisted affine algebra. The "rest" of the so-called three-point algebras are those whose centroids coincide with R', and which look, locally for the étale topology on $\mathrm{Spec}(R')$, like $\mathfrak{g} \otimes_k R'$.

Definition 4.3. A three-point Lie algebra is a Lie algebra $\mathcal{L}$ over k with the following two properties:

(a) The centroid of $\mathcal{L}$ is isomorphic to R',

(b) There exists an étale extension S' of R' such $\mathcal{L} \otimes_{C(\mathcal{L})} S' \simeq \mathfrak{g} \otimes_k S'$.

Because of Theorem 4.2, this definition is in perfect analogy with the situation that one encounters in the case of affine Kac-Moody Lie algebras.

It follows from the definition that there exists a (natural) well-defined surjective map from the set of R'-isomorphism classes of twisted forms of $\mathfrak{g} \otimes_k R'$ into the set of k-isomorphism classes of three-point Lie algebras which is not necessary injective.[3] Recall that twisted R'-forms of $\mathfrak{g} \otimes_k R'$ are classified by $H^1\big(R', \mathbf{Aut}(\mathfrak{g})\big)$. By considering the split exact sequence of R'–groups corresponding to (2.1) and passing to cohomology we obtain

$$H^1(R', \mathbf{G}^{\mathrm{ad}}) \to H^1\big(R', \mathbf{Aut}(\mathfrak{g})\big) \to H^1\big(R', \mathbf{Out}(\mathfrak{g})\big) \to 1. \tag{4.2}$$

By Corollary 3.3 we in fact have a bijection of pointed sets

$$H^1\big(R', \mathbf{Aut}(\mathfrak{g})\big) \to H^1\big(R', \mathbf{Out}(\mathfrak{g})\big). \tag{4.3}$$

[3]This again uses [GP] Lemma 4.6.3, namely that the centroid on any twisted form of $\mathfrak{g} \otimes_k R'$ is naturally isomorphic to R'. The argument is simple. Using that $\mathfrak{g}$ is simple, one first shows that the centroid of $\mathfrak{g} \otimes_k R'$ is naturally isomorphic to R'. Then the case of twisted forms is handled by descent considerations.

Moreover by [SGA1] XI §5

$$H^1\big(R', \mathbf{Out}(\mathfrak{g})\big) \simeq \mathrm{Hom}_{cont}\,\big(\pi_1(R'), \mathrm{Out}_k(\mathfrak{g})\big)/\text{conjugation} \tag{4.4}$$

where $\pi_1(R')$ is the algebraic fundamental group at the geometric point

$$\overline{x} = \mathrm{Spec}\ \overline{k(t)} \to \mathrm{Spec}\,(R')$$

and $\mathrm{Out}(\mathfrak{g})$ acts on $\mathrm{Hom}_{cont}\,\big(\pi_1(R'), \mathrm{Out}(\mathfrak{g})\big)$ by conjugation.

Remark 4.4. By [Sz] 4.6.12 or the Comparison Theorem $\pi_1(R') \simeq \widehat{\mathbb{Z} * \mathbb{Z}}$. If we fix generators e_1 and e_2 for the two copies of $\mathbb{Z}$, then $H^1\big(R', \mathbf{Out}_k(\mathfrak{g})\big)$ can be thought as conjugacy classes of pairs of elements of $\mathrm{Out}_k(\mathfrak{g})$.

Remark 4.5. It is well known that the abstract group $\mathrm{Out}_k(\mathfrak{g})$ is the symmetric group S_n with $n = 1, 2, 3$. Furthermore, if the pair of elements of S_n assigned to e_1 and e_2 generate a subgroup in S_n acting transitively on $\{1, \dots, n\}$ then the conjugacy class of this pair corresponds to a dessin d'enfant of degree n and conversely (see below for details and references). As we shall see the classification of three-point algebras, as Lie algebras over R', is entirely given in terms of dessins.

5. Dessins d'enfants

In this section we briefly recall the definition of dessins d'enfants and their properties. For more details and applications we refer to the surveys [LZ], [Sch], [W].

A dessin d'enfants is a bipartite connected graph Γ which is embedded into an oriented closed (connected) topological surface X such that it fills the surface, i.e. $X \setminus \Gamma$ is a union of open cells. Two dessins d'enfants (X_1, Γ_1) and (X_2, Γ_2) are called equivalent if there exists a homeomorphism $f : X_1 \to X_2$ such that $f(\Gamma_1) = \Gamma_2$.

Such graphs appear in a nice way to describe all possible coverings $\beta : X \to \mathbb{P}^1(\mathbb{C})$ from a closed (connected) Riemann surface X to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are ramified at most over the points $0, 1$ and ∞. Such pair (X, β) is called a Belyi morphism. The remarkable fact, known as Belyi's theorem, is that a projective smooth connected curve X over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists a finite covering $X \to \mathbb{P}^1$ which is unramified outside $0, 1$ and ∞.

Given a Belyi pair (X, β) we associate a bipartite graph Γ on X as follows. We may identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. Let $I = [0, 1]$ be the closed segment on the real line $\mathbb{R}$. Then its preimage $\beta^{-1}(I)$ is a connected graph on X. Its vertices are the preimages of 0 and 1. We may colour all preimages of 0 with one colour (say white) and all preimages of 1 with another colour (say black). One checks that $\beta^{-1}(I)$ fills X and hence it is a dessin d'enfants.

It is a striking fact that the above correspondence is one-to-one.

Theorem 5.1. *A Belyi pair (X, β) of degree d is uniquely determined up to equivalence by*

- *a dessin d'enfants with d edges up to equivalence;*
- *a monodromy map $\alpha : F_2 \to S_d$, i.e. a transitive action of the free group F_2 on two generators on a d-element set $\{1, 2, \ldots, d\}$, up to conjugation;*

Proof. See [HS]. □

6. Classification of three-point algebras over R'

Theorem 6.1. *The classification of affine three-point Lie algebras over $R' = k[t]_{t(t-1)}$ is as follows.*

(i): *If $\mathfrak{g}$ is of type A_1, B_ℓ $(\ell \geq 2), C_\ell$ $(\ell \geq 3), G_2, F_4, E_7$ or E_8 then all three-point affine algebras are trivial i.e. isomorphic to $\mathfrak{g} \otimes_k R'$.*

(ii): *If $\mathfrak{g}$ is of type A_ℓ $(\ell > 1)$, D_ℓ $(\ell > 4)$ or E_6 there are four isomorphism classes of three-point affine algebras. The trivial algebra, and three "quadratic" algebras.*

(iii): *If $\mathfrak{g}$ is of type D_4 there are eleven 3-pointed affine algebras. The trivial algebra, three "quadratic" algebras, and seven "trialitarian" algebras of which four are "cyclic cubic" and three are "non-cyclic cubic"–algebras.*

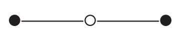
Proof. By general considerations, as explained in §4, three-point Lie algebras as R'-algebras are classified by $H^1\big(R', \mathbf{Aut}(\mathfrak{g})\big)$, and as we have seen the natural map $H^1\big(R', \mathbf{Aut}(\mathfrak{g})\big) \to H^1\big(R', \mathbf{Out}(\mathfrak{g})\big)$ is bijective. In view of Remarks 4.4 and 4.5, the classification is thus given by computing conjugacy classes of pairs of elements of the symmetric groups S_n where $n = 1, 2, 3$ in cases (i), (ii) and (iii) respectively and, if the resulting action is transitive, we provide the corresponding dessin.

(i) It is obvious that there is only one conjugacy class whose dessin is just the interval I, corresponding to the trivial cover of the Riemann sphere.

S_1 pair representative	Dessin	n_0	n_1	n_∞	g
$(1,1)$		1	1	1	0

Here (and everywhere below) n_0, n_1 and n_∞ are the number of points over $0, 1$ and ∞ respectively under the corresponding cover $\beta : X \to \mathbb{P}^1(\mathbb{C})$ and g is the genus of X. Note that if $\beta : X \to \mathbb{P}^1$ has degree n then the genus g of X is equal to $(n - n_0 - n_1 - n_\infty + 2)/2$ and hence n_∞ is determined uniquely by n_0, n_1 and g.

(ii) Let $r = (12)$ be the generator of S_2. There are four conjugacy classes of pairs of elements of S_2. The trivial class $(1,1)$ and three transitive classes with representatives and dessins given by the following table.

S_2 pair representative	Dessin	n_0	n_1	n_∞	g
$(1,r)$		2	1	1	0
$(r,1)$		1	2	1	0
(r,r)		1	1	2	0

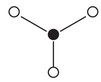
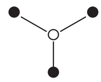
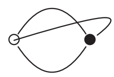
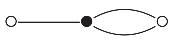
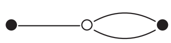
Each conjugacy class in fact consists of a single element and the corresponding covers are given by the étale extension S/R' where $S = R'(\sqrt{t-1})$, $R'(\sqrt{t})$ and $R'(\sqrt{t(t-1)})$ respectively.

(iii) Let $r = (12)$, $s = (23)$ and $c = sr = (123)$. There are eleven conjugacy classes of pairs of elements of S_3. The trivial class $(1,1)$, three "quadratic" classes $(1,r), (r,1), (r,r)$ (whose corresponding algebras are obtained as in (ii) above), and seven transitive classes with representatives and dessins given by the table starting on the next page.

Here the column Trialitarian Type refers to the Galois groups of the corresponding covers (X, β) which are isomorphic to $\mathbb{Z}/3\mathbb{Z}$ or S_3. For explicit descriptions of the corresponding covers we refer to [Z]. $\square$

7. Classification of three-point algebras over k

Just as in the case of Kac-Moody Lie algebras, in infinite dimensional Lie theory one is interested in viewing three-point algebras as algebras over k

S_3 pair representative	Dessin	n_0	n_1	n_∞	g	Trialitarian Type
$(1, c)$		3	1	1	0	cyclic
$(c, 1)$		1	3	1	0	cyclic
(c, c^2)		1	1	3	0	cyclic
(c, c)		1	1	1	1	cyclic
(r, s)		2	2	1	0	non-cyclic
(r, c)		2	1	2	0	non-cyclic
(c, r)		1	2	2	0	non-cyclic

and not over R'. Clearly, if two three-point algebras are isomorphic over R' then they are isomorphic as k-algebras as well. However the converse is not true. Before proceeding to their classification over k we first state and prove a useful criterion for studying the R vs k–isomorphism question for twisted forms of $\mathfrak{g} \otimes_k R$ for an arbitrary R.

Consider then an *arbitrary* (commutative, associative and unital) k-algebra R and the set of all R-isomorphism classes of forms of the R-Lie algebra $\mathfrak{g} \otimes_k R$. These are classified by the pointed set $H^1\big(R, \mathbf{Aut}(\mathfrak{g})\big)$. We have a natural action of the abstract group $\Gamma = \mathrm{Aut}_k(R)$ on the set of isomorphism classes of $\mathbf{Aut}(\mathfrak{g})$-torsors. Namely, every element $\gamma \in \Gamma$ gives rise to the scheme automorphism $\gamma_* : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$. If $\mathcal{T} \to \mathrm{Spec}(R)$ is an $\mathbf{Aut}(\mathfrak{g})$-torsor the base change γ_* produces a scheme ${}^{\gamma}\mathcal{T} = \mathcal{T} \times_{\gamma_*} \mathrm{Spec}(R)$ which is obviously an $\mathbf{Aut}(\mathfrak{g})$-torsor. Thus γ yields a natural bijection

$$\gamma_* : H^1\big(R, \mathbf{Aut}(\mathfrak{g})\big) \to H^1\big(R, \mathbf{Aut}(\mathfrak{g})\big).$$

Similarly, Γ acts on the set of isomorphism classes of twisted R-forms of the simply connected group $\mathbf{G}^{sc}$ attached to $\mathfrak{g}$ and their Lie algebras.

On the level of Lie algebras over R the action of $\gamma \in \Gamma$ can be seen as follows. Let $\mathcal{L}$ be an R-form of $\mathfrak{g} \otimes_k R$. Then we define ${}^{\gamma}\mathcal{L}$ to be $\mathcal{L}$ (as a set) with the same Lie bracket structure, but the R-module structure is given by composing the automorphism $\gamma : R \to R$ with the standard action of R on $\mathcal{L}$. Thus for all $x, y \in \mathcal{L} = {}^\gamma\mathcal{L}$ we have

$$[x, y]_{^\gamma\mathcal{L}} = [x, y]_{\mathcal{L}}$$

and for all $r \in R$ we have

$$r \cdot_{^\gamma\mathcal{L}} x = \gamma(r) \cdot_{\mathcal{L}} x.$$

In particular, it follows that the identity mapping $f_\gamma : {}^\gamma\mathcal{L} \to \mathcal{L}$ is an isomorphism of k-Lie algebras.

Proposition 7.1. *Let $\mathcal{L}$ and $\mathcal{L}'$ be R-forms of $\mathfrak{g} \otimes_k R$. Then $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic as k-Lie algebras if and only if there exists $\gamma \in \Gamma = \mathrm{Aut}_k(R)$ such that $\mathcal{L}' \simeq {}^\gamma\mathcal{L}$ over R.*

Proof. This follows essentially from the fact that the centroids $C(\mathcal{L})$ and $C(\mathcal{L}')$ of $\mathcal{L}$ and $\mathcal{L}'$ are naturally isomorphic to R, as we now explain. For $r \in R$ consider the homothety $\chi_r \in \mathrm{End}_k(\mathcal{L})$ given by $\chi_r : x \mapsto rx$. Clearly $\chi_r \in C(\mathcal{L})$. This gives a natural morphism of associative and commutative k-algebras $R \to C(\mathcal{L})$ which is known to be an isomorphism (see [P2, Lemma 3.4] and [GP, Lemma 4.6]).

We now turn to the proof of the Proposition. Assume that $f : \mathcal{L} \to \mathcal{L}'$ is a k-algebra isomorphism. It is immediate to verify that the map $C(f) : C(\mathcal{L}') \to \mathrm{End}_k(\mathcal{L})$ defined by $\chi \mapsto f^{-1} \circ \chi \circ f$ is in fact a k-algebra isomorphism of $C(\mathcal{L}')$ onto $C(\mathcal{L})$. Under our identification $C(\mathcal{L}') \simeq R \simeq C(\mathcal{L})$ we thus have $\gamma := C(f) \in \mathrm{Aut}_k(R)$.

The reader can easily verify from the definition that for the isomorphism $f_\gamma : {}^\gamma\mathcal{L} \to \mathcal{L}$ above one has $C(f_\gamma) = \gamma^{-1}$. Thus, if we consider the composition

$$f' : {}^\gamma\mathcal{L} \xrightarrow{f_\gamma} \mathcal{L} \xrightarrow{f} \mathcal{L}'$$

then $C(f') = 1$. But this means that f' is R-linear.

The converse is clear since we have already observed that $f_\gamma : {}^\gamma\mathcal{L} \to \mathcal{L}$ is an isomorphism of k-Lie algebras. $\square$

Remark 7.2. The above Proposition has nothing to do with Lie algebras. We could replace $\mathfrak{g}$ by any finite dimensional central simple algebra over k.

We now come back to the ring R'. For such a ring we have $\mathrm{Aut}_k(R') \simeq S_3$. Indeed, an arbitrary k-automorphism $\gamma : R' \to R'$ induces an automorphism $\mathrm{Spec}(R') \to \mathrm{Spec}(R')$ and hence an automorphism of the fraction field $k(t)$ of R'. On the other hand, by [Hart, Chapter 1, Exercise 6.6] one has $\mathrm{Aut}_k\, k(t) \simeq \mathrm{Aut}(\mathbb{P}^1_k) \simeq \mathrm{PGL}_2(k)$ and thus we obtain a mapping from $\mathrm{Aut}_k(R')$ into the permutation group S_3 of the points $\{0, 1, \infty\}$ on $\mathbb{P}^1_k$. Also, it is known [Hart, Chapter IV, Exercise 2.2] that any permutation of $\{0, 1, \infty\}$ is induced by a unique automorphism of $\mathbb{P}^1_k$. This induces an automorphism of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$, hence of $\mathrm{Spec}(R')$.

We now pass to the classification of k–isomorphism classes of three-point Lie algebras.

Theorem 7.3. *The classification of three-point Lie algebras over k is as follows.*

(i): *If $\mathfrak{g}$ is of type A_1, B_ℓ $\ell \geq 2, C_\ell$ $\ell \geq 3, G_2, F_4, E_7$ or E_8 then all three-point Lie algebras are trivial i.e. isomorphic to $\mathfrak{g} \otimes_k R'$.*

(ii): *If $\mathfrak{g}$ is of type A_ℓ $\ell > 1$, D_ℓ $\ell > 4$ or E_6 there are two isomorphism classes of three-point Lie algebras. The trivial algebra, and one quadratic algebra arising for the three quadratic algebras over R'. More precisely, the three non-isomorphic quadratic algebras over R' are isomorphic as k–algebras.*

(iii): *If $\mathfrak{g}$ is of type D_4 there are five three-point Lie algebras.*

1) *The trivial algebra.*
2) *One quadratic algebra arising from the three quadratic algebras over R'.*
3) *One cyclic cubic algebra arising from the three cyclic cubic algebras over R' corresponding to dessins of genus* 0.
4) *One cyclic cubic algebra corresponding to the cyclic cubic R'-algebra whose dessin is of genus* 1.
5) *One non-cyclic cubic algebra arising from the three non-cyclic cubic algebras over R'.*

Proof. (i) Follows for Theorem 6.1(i) because there exists a unique R'-form of $\mathfrak{g} \otimes_k R'$. Indeed, in this case we have $\mathrm{Out}(\mathfrak{g}) = 1$.

(ii) and (iii): The problem is equivalent to classifying the set of R'-forms of $\mathfrak{g} \otimes_k R'$ up to the action of $\mathrm{Aut}_k(R') = S_3$. Let $\mathcal{L}$ be an R'-form of $\mathfrak{g} \otimes_k R'$ and let $\gamma \in S_3$. The form $\mathcal{L}$ corresponds to some class $[\xi] \in H^1\big(R', \mathbf{Out}(\mathfrak{g})\big)$ and hence to a homomorphism $F_2 \to S_d$, $d = 2, 3$, which we may assume to be transitive (if it is not, we replace S_d by $S_{d'}$ with $d' < d$.) By Theorem 5.1 this leads us to a unique Belyi pair $\beta : X \to \mathbb{P}^1(\mathbb{C})$.

Let $\gamma \in S_d$. The form ${}^\gamma\mathcal{L}$ corresponds to the class of the torsor ${}^\gamma\xi$ and hence to another Belyi pair ${}^\gamma\beta : {}^\gamma X \to \mathbb{P}^1(\mathbb{C})$. This new Belyi pair $({}^\gamma X, {}^\gamma\beta)$ can be explicitly described as follows: as we noticed before the permutation γ of the set $\{0, 1, \infty\}$ gives rise to a unique automorphism $\gamma_* : \mathbb{P}^1 \to \mathbb{P}^1$ and then we can take ${}^\gamma X$ to be equal to $X \times_{\gamma_*} \mathbb{P}^1$ and ${}^\gamma\beta$ to be a natural projection ${}^\gamma X = X \times_{\gamma_*} \mathbb{P}^1 \to \mathbb{P}^1$. As a consequence of this construction we conclude that the numbers n_0, n_1, n_∞ for the pair $({}^\gamma X, {}^\gamma\beta)$ are obtained from those of the Belyi pair (X, β) by permuting them with the use of γ. Looking at the tables 2 and 3 we observe that the cover is fully determined by the number of preimages of 0, 1, and ∞. In case (ii) three of the twisted forms of $\mathfrak{g} \otimes_k R'$ presented in table 2 are in the same S_3-orbit, and that in case (iii) there are five S_3-orbits as claimed in the Theorem. □

Remark 7.4. (i) An étale extension of R' of degree 2 is isomorphic, up to $\mathrm{Aut}_k\,(R')$-base change, to either $R' \times R'$ or $R'[\sqrt{t}]$.
(ii) An étale extension of R' of degree 3 is isomorphic, up to $\mathrm{Aut}_k\,(R')$-base change, to one (and only one) of the algebras of the following list:

(a) $R' \times R' \times R'$;

(b) $R'[\sqrt{t}] \times R'$ (the quadratic case);

(c) $R'[\sqrt[3]{t}]$ (the cyclic cubic case of genus 0);

(d) $R'[\sqrt[3]{t(t-1)}]$ (the cyclic cubic case of genus 1);

(e) $R'[X]/(X^3 + 3X^2 - 4t)$ (the non-cyclic cubic case of genus 0).

Remark 7.5. Let $n > 0$ and consider the semisimple Lie algebra $\mathfrak{g}_n = \mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2$ (n copies). Then $\mathbf{Out}(\mathfrak{g})(k) = S_n$. As in §4 we have bijections

$$H^1\big(R',\, \mathbf{Aut}(\mathfrak{g})\big) \to H^1\big(R',\, \mathbf{Out}(\mathfrak{g})\big)$$

and

$$H^1\big(R', \mathbf{Out}(\mathfrak{g})\big) \simeq \mathrm{Hom}_{cont}\,\big(\pi_1(R'), \mathrm{Out}_k(\mathfrak{g})\big)/\text{conjugation}.$$

By Theorem 5.1 we obtain a correspondence between dessins d'enfants of degree $\leq n$ and isomorphisms classes of twisted forms of the R'-Lie algebra $\mathfrak{g}_n \otimes_k R'$. Note, however, that the methods of this section cannot be applied to the classification of these algebras over k. The reason is that for $n > 1$, the natural map from R' to the centroid of $\mathfrak{g}_n$ is not an isomorphism.

We use this opportunity to clarify a misunderstanding regarding loop algebras. Assume $\mathfrak{g}$ is semisimple and that σ is an automorphism of finite order of $\mathfrak{g}$. It is not true that the isomorphism class of the loop algebra $L(\mathfrak{g}, \sigma)$, as a Lie algebra over k, depends only on the "outer part" of σ. The assertion is true up to R-isomorphism classes as shown in [P], but not up to k-isomorphism.

Here is a concrete example. Let σ be the automorphism of $\mathfrak{g}_2$ that switches the two copies of $\mathfrak{sl}_2$. The outer part of σ is the diagram automorphism that switches the two disjoint nodes of the corresponding Coxeter-Dynkin diagram. The Lie algebras $L(\mathfrak{g}_2, \mathrm{id})$ and $L(\mathfrak{g}_2, \sigma)$ are not isomorphic as R-Lie algebras, but are isomorphic as k-Lie algebras.

8. Cohomological description of conjugacy classes of split tori

Let $\mathfrak{G}$ be a reductive group scheme over a base scheme X and let $\mathfrak{S}$ be a split subtorus of $\mathfrak{G}$. According to [SGA3, XI.5.9], the fppf sheaf $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})/\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$ is representable by a X–group scheme $\mathbf{W}(\mathfrak{S})$ called the Weyl group scheme. The action of $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ on $\mathfrak{S}$ gives rise to a monomorphism $i : \mathbf{W}(\mathfrak{S}) \to \mathbf{Aut}(\mathfrak{S})$ which is an open immersion so that $\mathbf{W}(\mathfrak{S})$ is quasi-finite, separated and étale over X (*loc. cit.*).

Lemma 8.1. $\mathbf{W}(\mathfrak{S})$ *is a finite group scheme over X which is Zariski-locally constant.*

Proof. We shall prove firstly that $\mathbf{W}(\mathfrak{S})$ is finite over X. The centralizer $\mathfrak{L} := \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$ is a Levi subgroup of a parabolic subgroup $\mathfrak{P}$ of $\mathfrak{G}$ [SGA3, XXVI.6.2]. We denote by $\mathfrak{R}$ the radical of $\mathfrak{L}$, namely its maximal central subtorus [SGA3, XXII.4.3.6]. Observe that $\mathfrak{L}$ is a critical X–subgroup of $\mathfrak{G}$ [SGA3, XXVI.1.13], i.e. we have

$$\mathfrak{L} = \mathbf{Z}_{\mathfrak{G}}(\mathfrak{R}).$$

The Weyl group scheme $\mathbf{W}(\mathfrak{R})$ is finite according to [SGA3, XXII.5.10.9]. We shall link $\mathbf{W}(\mathfrak{R})$ and $\mathbf{W}(\mathfrak{S})$ by using the fact that $\mathbf{N}_{\mathfrak{G}}(\mathfrak{L}) = \mathbf{N}_{\mathfrak{G}}(\mathfrak{R})$. Since $\mathfrak{L} = \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$, we have $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \subset \mathbf{N}_{\mathfrak{G}}(\mathfrak{L}) = \mathbf{N}_{\mathfrak{G}}(\mathfrak{R})$ and both are closed subgroups of $\mathfrak{G}$ [SGA3, XI.5.9]. In particular $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ is a closed subgroup of $\mathbf{N}_{\mathfrak{G}}(\mathfrak{R})$. By moding out by $\mathfrak{L} = \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S}) = \mathbf{Z}_{\mathfrak{G}}(\mathfrak{R})$, we get a closed immersion of X–groups $\mathbf{W}(\mathfrak{S}) \to \mathbf{W}(\mathfrak{R})$ (it is a closed immersion by fppf descent [EGAIV, Prop 2.7.1(xii)]) which is in particular a finite morphism. Since $\mathbf{W}(\mathfrak{R})$ is finite over X, we conclude that $\mathbf{W}(\mathfrak{S})$ is finite over X.

From what has been shown heretofore we get that $\mathbf{W}(\mathfrak{S})$ is finite étale over X. Consider the open immersion $i : \mathbf{W}(\mathfrak{S}) \to \mathbf{Aut}(\mathfrak{S})$. To show that $\mathbf{W}(\mathfrak{S})$ is Zariski-locally constant, we can obviously assume that X is affine, hence that is $\mathbf{W}(\mathfrak{S})$ as well.

We use now that $\mathfrak{S}$ is split, that is $\mathfrak{S} \simeq \mathfrak{G}_m^r$ for some $r \geq 0$. According to [SGA3, VII.1.5], the fppf sheaf $\mathbf{Aut}(\mathfrak{G}_{\mathfrak{m}}^{\mathfrak{r}})$ is representable by $\mathbf{GL}_r(\mathbb{Z})_X$ where

$$\mathbf{GL}_r(\mathbb{Z})_X = \bigsqcup_{\sigma \in \mathrm{GL}_r(\mathbb{Z})} X_\sigma$$

stands for the constant X–group scheme attached to the abstract group $\mathrm{GL}_r(\mathbb{Z})$. The morphism i induces a decomposition in open subsets

$$\mathbf{W}(\mathfrak{S}) = \bigsqcup_{\sigma \in \mathrm{GL}_r(\mathbb{Z})} \mathbf{W}(\mathfrak{S})_\sigma$$

where $\mathbf{W}(\mathfrak{S})_\sigma$ is the inverse image of X_σ under i. Since $\mathbf{W}(\mathfrak{S})$ is affine, almost all $\mathbf{W}(\mathfrak{S})_\sigma$ are empty so that there exists a finite subset Σ of $\mathrm{GL}_r(\mathbb{Z})$ such that $\mathbf{W}(\mathfrak{S}) = \bigsqcup_{\sigma\in\Sigma} \mathbf{W}(\mathfrak{S})_\sigma$. Since i is an open immersion, each morphism $\mathbf{W}(\mathfrak{S})_\sigma \to X$ is a clopen X–immersion.

For each $\sigma \in \Sigma$, we denote by U_σ^+ the (isomorphic) image of $\mathbf{W}(\mathfrak{S})_\sigma$ in X and by $U_\sigma^- = \bigcup\limits_{\tau\neq\sigma} U_\tau^+$. For each function $\epsilon : \Sigma \to \{+,-\}$, we define the open subset $U^\epsilon = \bigcap\limits_{\sigma\in\Sigma} U_\sigma^{\epsilon(\sigma)}$ of X. Then $(U^\epsilon)_\epsilon$ (for ϵ running over the functions $\Sigma \to \{+,-\}$) is an open cover of X such that $\mathbf{W}(\mathfrak{S}) \times_X U^\epsilon$ is a finite constant U^ϵ-scheme for each such function ϵ. We conclude that $\mathbf{W}(\mathfrak{S})$ is a finite X–group scheme which is locally constant for the Zariski topology. □

The fppf sheaf $\mathbf{Y} = \mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ is representable [SGA3, XI.5.3.bis] and for each scheme X' over X, the elements of the set $\mathbf{Y}(X')$ are in one-to-one correspondence with X'-subtori $\mathfrak{S}'$ of $\mathfrak{G}_{X'}$ which are fppf-locally conjugate to $\mathfrak{S}_{X'}$. Since $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ is smooth, locally conjugation with respect to étale topology is equivalent to conjugacy in the fppf topology.

The orbits of $\mathfrak{G}(X)$ on $\mathbf{Y}(X)$ can be described by means of the exact sequence

$$\mathbf{Y}(X) \xrightarrow{\phi} H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \longrightarrow H^1(X, \mathfrak{G})$$

arising from an exact sequence

$$1 \longrightarrow \mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow \mathfrak{G} \longrightarrow \mathbf{Y} \longrightarrow 1.$$

More precisely, we have a natural bijection [Gi, III.3.2.4]

$$\mathfrak{G}(X) \setminus \mathbf{Y}(X) \xrightarrow{\sim} \ker\Big(H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1(X, \mathfrak{G})\Big).$$

To summarize,

$$\ker\Big(H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1(X, \mathfrak{G})\Big)$$

classifies the $\mathfrak{G}(X)$–conjugacy classes of subtori $\mathfrak{S}'$ of $\mathfrak{G}$ which are locally-étale $\mathfrak{G}$-conjugate to $\mathfrak{S}$. For each such $\mathfrak{S}'$, we denote by $\gamma(\mathfrak{S}') \in H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big)$ its $\mathfrak{G}(X)$-conjugacy class. In terms of torsors, $\gamma(\mathfrak{S}')$ is the class of the strict transporter $\mathrm{Transpstr}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')$ which is an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$-torsor [Gi]. We recall here its definition: for each $X' \to X$

$$\begin{aligned} &\mathrm{Transpstr}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')(X') \\ &= \Big\{ g \in \mathfrak{G}(X') \mid g\, \mathfrak{S}(X'')\, g^{-1} = \mathfrak{S}'(X'') \ \ \forall\, X'' \to X' \Big\}. \end{aligned}$$

The converse map is given as follows ([Gi, Lemme 2.1]). Consider an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$-torsor $\mathbf{E}$ equipped with a trivialization

$$u : \mathfrak{G} \xrightarrow{\sim} \mathbf{E} \wedge^{\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})} \mathfrak{G}.$$

where $\wedge$ is the contracted product. Then $u^{-1}(\mathbf{E} \wedge^{\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})} \mathfrak{S})$ is a subtorus of $\mathfrak{G}$.

Furthermore, each such $\mathfrak{S}'$ is Zariski-locally $\mathfrak{G}$-conjugated to $\mathfrak{S}$ if and only if

$$\gamma(\mathfrak{S}') \in H^1_{Zar}\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \subset H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big).$$

It follows that the set

$$\ker\Big(H^1_{Zar}\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1_{Zar}(X, \mathfrak{G})\Big)$$

classifies the $\mathfrak{G}(X)$–conjugacy classes of tori which are locally conjugate to $\mathfrak{S}$ for the Zariski topology.

Now we consider the following exact sequence of X-group-schemes

$$1 \to \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S}) \to \mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \to \mathbf{W}(\mathfrak{S}) \to \mathbf{1}.$$

There is a natural action of $\mathbf{W}(\mathfrak{S})(X)$ on $H^1(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S}))$ and it induces a bijection [Gi, III.3.3.1]

$$H^1\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big) / \mathbf{W}(\mathfrak{S})(X) \xrightarrow{\sim} \ker\Big(H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1\big(X, \mathbf{W}(\mathfrak{S})\big)\Big).$$

We consider then the subset $H^1_{Zar}(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S}))\,/\,\mathbf{W}(\mathfrak{S})(X)$ of

$$H^1_{Zar}(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) \subset H^1(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})).$$

Proposition 8.2. *Assume that the base scheme X is connected.*

(1) *The pointed set*

$$\ker\Big(H^1\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big)\,/\,\mathbf{W}(\mathfrak{S})(X) \to H^1(X, \mathfrak{G})\Big)$$

classifies split subtori of $\mathfrak{G}$ which are locally conjugate to $\mathfrak{S}$ for the étale topology.

(2) *We have*

$$\begin{aligned}&\ker\Big(H^1_{Zar}\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1_{Zar}(X, \mathfrak{G})\Big)\\ &\overset{\sim}{\longrightarrow} \ker\Big(H^1\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1(X, \mathfrak{G})\Big).\end{aligned}$$

(3) *The pointed set*

$$\ker\Big(H^1_{Zar}\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big)\,/\,\mathbf{W}(\mathfrak{S})(X) \to H^1_{Zar}(X, \mathfrak{G})\Big)$$

classifies $\mathfrak{G}(X)$-conjugacy classes of split subtori of $\mathfrak{G}$ which are locally conjugate to $\mathfrak{S}$ for the Zariski topology or equivalently locally for the étale topology.

The connectedness assumption is used to ensure that $\mathbf{W}(\mathfrak{S})$ is a finite constant X-group (Lemma 8.1). The proof of the proposition will be given below. It uses the fact that $\mathbf{W}(\mathfrak{S})$ is an open subgroup of the constant X-group $\mathbf{Aut}(\mathfrak{S})$.

Lemma 8.3. (1) *The map $H^1\big(X, \mathbf{W}(\mathfrak{S})\big) \to H^1\big(X, \mathbf{Aut}(\mathfrak{S})\big)$ has trivial kernel.*

(2) *The compositum*

$$\begin{aligned}\big(\mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big)(X) &\longrightarrow H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big)\\ &\longrightarrow H^1\big(X, \mathbf{W}(\mathfrak{S})\big) \longrightarrow H^1\big(X, \mathbf{Aut}(\mathfrak{S})\big)\end{aligned}$$

maps a subtorus $\mathfrak{S}'$ of $\mathfrak{G}$ which is étale-locally conjugate to $\mathfrak{S}$ to the class of the $\mathbf{Aut}(\mathfrak{S})$-torsor $\mathrm{Isom}_{gr}(\mathfrak{S}, \mathfrak{S}')$.

Proof. (1) Since $\mathfrak{S}$ is split, $\mathbf{Aut}(\mathfrak{S})$ is the constant X–group associated to $\mathrm{GL}_r(\mathbb{Z})$. Hence $\mathbf{W}(\mathfrak{S})$ is the constant $\mathbf{X}$–group associated to a finite subgroup Γ of $\mathrm{GL}_r(\mathbb{Z})$. We consider an exact sequence of étale $\mathbf{X}$-sheaves

$$1 \to \mathbf{\Gamma}_X \to \mathbf{GL}_r(\mathbb{Z})_X \to (\mathbf{GL}_r(\mathbb{Z})/\Gamma)_X \to 1.$$

It gives rise to a long exact sequence of pointed sets

$$\begin{aligned} 1 \to \Gamma^{\pi_0(X)} \to \mathbf{GL}_r(\mathbb{Z})^{\pi_0(X)} &\to (\mathbf{GL}_r(\mathbb{Z})/\Gamma)^{\pi_0(X)} \\ &\to H^1(X, \mathbf{\Gamma}_X) \xrightarrow{\lambda} H^1(X, \mathbf{GL}_r(\mathbb{Z})_X). \end{aligned}$$

It follows that λ has trivial kernel, because $\pi_0(X)$ acts trivially on $\Gamma, \mathbf{GL}_r(\mathbb{Z})$ and $\mathbf{GL}_r(\mathbb{Z})/\Gamma$.

(2) The $\mathfrak{G}(X)$–conjugacy class of $\mathfrak{S}'$ is nothing but the class of the strict transporter $\mathrm{Transpstr}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')$ which is an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$-torsor. The change of groups $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \to \mathbf{W}(\mathfrak{S}) \to \mathbf{Aut}(\mathfrak{S})$ applied to that transporter yields indeed the $\mathbf{Aut}(\mathfrak{S})$-torsor $\mathrm{Isom}_{gr}(\mathfrak{S}, \mathfrak{S}')$. □

We now proceed with the proof of the Proposition.

Proof. (1) Consider the diagram of exact sequences of pointed sets

$$\begin{array}{ccccc} & & \big(\mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big)(X) & & 1 \\ & & \varphi\downarrow & & \downarrow \\ H^1\big(X, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big) & \xrightarrow{i_*} & H^1\big(X, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})\big) & \longrightarrow & H^1\big(X, \mathbf{W}(\mathfrak{S})\big) \\ & & & & \downarrow \\ & & & & H^1(X, \mathbf{Aut}(\mathfrak{S})). \end{array}$$

Let $\mathfrak{S}'$ be a subtorus of $\mathfrak{G}$ which is étale-locally conjugate to $\mathfrak{S}$. According to Lemma 8.3.(2), $\mathfrak{S}'$ is split if and only if the image of $\gamma(\mathfrak{S}')$ in $H^1\big(X, \mathbf{Aut}(\mathfrak{S})\big)$ vanishes. Since $H^1\big(X, \mathbf{W}(\mathfrak{S})\big) \to H^1\big(X, \mathbf{Aut}(\mathfrak{S})\big)$ has trivial kernel by the first part of that Lemma, it follows that $\mathfrak{S}'$ is split if and only if the image of $\gamma(\mathfrak{S}')$ in $H^1\big(X, \mathbf{W}(\mathfrak{S})\big)$ vanishes, that is if and only if $\gamma(\mathfrak{S}')$ admits a reduction to $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$.

(2) The statement is local so we may assume that $X = \mathrm{Spec}(R)$ with R a local ring. We need to show that the kernel $\ker\Big(H^1\big(R, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big) \to H^1(R, \mathfrak{G})\Big)$ vanishes. But this is [SGA3, XXVI 5.10.(i)].

(3) By (2), the pointed set

$$\ker\Big(H^1_{Zar}\big(X,\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big)/\mathbf{W}(\mathfrak{S})(X)\to H^1_{Zar}(X,\mathfrak{G})\Big)$$

maps bijectively to $\ker\Big(H^1(X,\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S}))/\mathbf{W}(\mathfrak{S})(X)\to H^1(X,\mathfrak{G})\Big)$ so it classifies $\mathfrak{G}(X)$-conjugacy classes of split subtori of $\mathfrak{G}$ which are étale-locally conjugated to $\mathfrak{S}$ by (2). But for such a subtorus $\mathfrak{S}'$, its conjugacy class $\gamma(\mathfrak{S}')$ belongs then to $H^1_{Zar}(X,\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}))$ so that $\mathfrak{S}'$ is locally conjugated to $\mathfrak{S}$ for the Zariski topology. □

The following strengthens Proposition 10.1 of [CGP].

Proposition 8.4. *Assume that X is a connected scheme and $\mathfrak{S}$ is a split subtorus of $\mathfrak{G}$ such that $\mathfrak{S}$ is generically maximal split in $\mathfrak{G}$.*

(1) *The torus $\mathfrak{S}$ is a maximal split subtorus of $\mathfrak{G}$ and $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$ is a Levi subgroup of a parabolic subgroup of $\mathfrak{G}$ which is generically minimal.*

(2) *The generically maximal split split subtori of $\mathfrak{G}$ are the subtori locally conjugated to $\mathfrak{S}$ for the Zariski topology. Their $\mathfrak{G}(X)$-conjugacy class are classified by the pointed set* $\ker\Big(H^1_{Zar}\big(X,\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big)/\mathbf{W}(\mathfrak{S})(X)\to H^1_{Zar}\big(X,\mathfrak{G}\big)\Big)$.

Proof. (1) The first assertion is obvious. It is known that $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$ is a Levi subgroup of a parabolic subgroup $\mathbf{P}$ of $\mathfrak{G}$ [SGA3, XXVI.6.2]. Since $\mathfrak{S}$ remains maximal split at the generic point of X, it follows that $\mathbf{P}\times_X\kappa(X)$ is a minimal parabolic subgroup of $\mathfrak{G}_{\kappa(X)}$ [SGA3, XXVI.6].

(2) If $\mathfrak{S}'$ is a split torus of $\mathfrak{G}$ which is generically maximal split then it is Zariski-locally maximal split. In particular, $\mathfrak{S}'$ is Zariski-locally $\mathfrak{G}$-conjugate to $\mathfrak{S}$ by Demazure's conjugacy theorem [SGA3, XXVI.6.16]. Conversely, if $\mathfrak{S}'$ is $\mathfrak{G}$-conjugate to $\mathfrak{S}$ in the Zariski topology, it is generically maximal split because $\mathfrak{S}$ is. The second assertion follows from Proposition 8.2(3). □

We apply the above general considerations to the particular case of $X=\mathrm{Spec}(R')$, a simple adjoint group scheme $\mathfrak{G}$ over R' and a split torus $\mathfrak{S}$ of $\mathfrak{G}$ which is generically maximal split. By Proposition 3.1 we have $H^1(X,\mathfrak{G})=1$. Thus $H^1_{Zar}(X,\mathfrak{G})=1$ and therefore

$$\ker\Big(H^1_{Zar}\big(X,\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big)\to H^1_{Zar}\big(X,\mathfrak{G})\big)\Big)=H^1_{Zar}\big(X,\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})\big).$$

Since by Corollary 3.3 our groups are always quasi-split, the centralizer $\widetilde{\mathfrak{S}}=\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$ is always a maximal torus of $\mathfrak{G}$. Since $\mathfrak{G}$ is of adjoint type, $\widetilde{\mathfrak{S}}$ is the direct product of a split R'-torus and a Weil restriction $R_{S'/R'}(\mathbb{G}_{m,S'})$

of a one-dimensional split S'-torus where S'/R' is an étale extension of R' of degree 2 or 3.

By Shapiro's lemma $H^1(R', \widetilde{\mathfrak{S}})$ vanishes whenever S' is isomorphic to the coordinate ring of the affine line minus a finite number of points. This is the case whenever our dessin is of genus 0, namely in all cases except for case (c, c) in type D_4. In this exceptional case the situation is quite the opposite. The scheme S' is a non-empty open affine subscheme of an elliptic curve over an algebraically closed field, hence $H^1(S', \mathbb{G}_{m,S'}) = H^1_{Zar}(\mathbf{X}, \widetilde{\mathfrak{S}})$ is infinite. It follows that there are infinitely many conjugacy classes of maximal split tori inside the corresponding twisted adjoint R'–group $\mathfrak{G}$ of type D_4.

9. Conjugacy questions for three-point Lie algebras

Let $\mathcal{L}$ be a twisted form of $\mathfrak{g} \otimes_k R'$. The role of split Cartan subalgebras for the infinite dimensional k-Lie algebra $\mathcal{L}$ is played by maximal abelian k-diagonalizable subalgebras. Recall that a subalgebra $\mathbf{k}$ of $\mathcal{L}$ is *k-diagonalizable* if there exists a k-basis v_λ, $\lambda \in \Lambda$, of $\mathcal{L}$ consisting of eigenvectors for the adjoint action of $\mathbf{k}$, i.e. for all $\lambda \in \Lambda$ and $x \in \mathbf{k}$, there exists $\lambda(x) \in k$ such that $[x, v_\lambda] = \lambda(x) v_\lambda$. A k-diagonalizable subalgebra of $\mathcal{L}$ is necessarily abelian. The MADs are those k-diagonalizable subalgebras which are maximal with respect to inclusion.

Using the correspondence between maximal split R'-tori of $\mathfrak{G}$ and MADs of $\mathcal{L} = \mathrm{Lie}(\mathfrak{G})$ given in [CGP, Theorem 7.1] and an obvious fact that a maximal split torus in $\mathfrak{G}$ is generically maximal split (because by Corollary 3.3 there are no R'-anisotropic semisimple group schemes) we obtain the following.

Theorem 9.1. *If $\mathcal{L}$ is not of dessin type (c, c) there exists a single conjugacy class of MADs under the adjoint action of $\mathfrak{G}(R)$ on $\mathcal{L}$. If $\mathcal{L}$ is of type (c, c) the number of conjugacy classes of MADs is infinite.* □

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