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On two rationality conjectures for cubic fourfolds

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Motivated by the question of rationality of cubic fourfolds, we show that a cubic X has an associated K3 surface in the sense of Hassett if and only if the variety F of lines on X is birational to a moduli space of sheaves on a K3 surface, but that having F birational to $\mathrm{Hilb}^2(\mathrm{K3})$ is more restrictive. We compare the loci in the moduli space of cubics where each condition is satisfied.

It is widely expected that a smooth complex cubic fourfold X is rational if and only if it has an associated K3 surface in the sense of Hassett [8] or Kuznetsov [11]. New work of Galkin and Shinder [7] suggests instead that if X is rational then the variety F of lines on X is birational to the Hilbert scheme of two points on a K3 surface. The purpose of this note is to clarify the relationship between these two conditions. The latter is somewhat stronger.

First let us recall Hassett's Noether–Lefschetz divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds [8, §3.2]. For a very general cubic X, the algebraic lattice $H^{2,2}(X,\mathbb{Z}) := H^{2,2}(X) \cap H^4(X,\mathbb{Z})$ is generated by h^2, the square of the hyperplane class. A *special cubic of discriminant* d is one for which there is a primitive sublattice $K \subset H^{2,2}(X,\mathbb{Z})$ of rank 2 and discriminant d that contains h^2. Such cubics form an irreducible divisor $\mathcal{C}_d \subset \mathcal{C}$, non-empty if and only if

$$(*) \qquad d > 6 \text{ and } d \equiv 0 \text{ or } 2 \pmod 6.$$

Moreover there exists a polarized K3 surface S such that $K^\perp \subset H^4(X,\mathbb{Z})$ is Hodge-isometric to $H^2_{\mathrm{prim}}(S,\mathbb{Z})(-1)$ if and only d satisfies the further condition

$$(**) \qquad d \text{ is not divisible by 4, 9, or any odd prime } p \equiv 2 \pmod 3.$$

Using the Eisenstein integers one can show that (∗∗) is equivalent to saying that d is the norm of a primitive vector in the lattice $A_2 = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, or that d divides $2n^2 + 2n + 2$ for some integer n.

Theorem 1. *The following are equivalent:*

(a) $X \in \mathcal{C}_d$ *for some* d *satisfying* (∗∗).

(b) *The transcendental lattice* $T_X \subset H^4(X, \mathbb{Z})$ *is Hodge-isometric to* $T_S(-1)$ *for some K3 surface* S.

(c) F *is birational to a moduli space of stable sheaves on* S.

By a recent result of Bayer and Macrì [5, Thm. 1.2(c)], this last condition is equivalent to saying that F is isomorphic to a moduli space of Bridgeland-stable objects in the derived category of S. Thus Theorem 1 answers [13, Question 1.2] in the untwisted case.

Hassett [8, Prop. 6.1.3] showed that if the generic $X \in \mathcal{C}_d$ has F isomorphic to $\mathrm{Hilb}^2(S)$ for some K3 surface S then

$$(***) \qquad d \text{ is of the form } \frac{2n^2 + 2n + 2}{a^2} \text{ for some } n, a \in \mathbb{Z},$$

and proved a partial converse [8, Thm. 6.1.4]. Thanks to the global Torelli theorem for hyperkähler manifolds [10, 15, 19] we can now prove a more complete result:

Theorem 2. *The following are equivalent:*

(a) $X \in \mathcal{C}_d$ *for some* d *satisfying* (∗∗∗).

(b) F *is birational to* $\mathrm{Hilb}^2(S)$ *for some K3 surface* S.

In contrast to (∗∗), it is hard to tell at a glance whether a number d satisfies (∗∗∗). On the one hand (∗∗∗) implies (∗∗), but it is strictly stronger: Hassett remarks in [8, §6.1] that 74 satisfies (∗∗) but not (∗∗∗). To address the question systematically, observe that d satisifies (∗∗∗) if and only if there is an integral solution to the Pell-type equation $m^2 - 2da^2 = -3$; just substitute $m = 2n + 1$. If such an equation has any solution then it has one with a below an explicit bound [2, Thm. 4.2.7]. It is then straightforward to have a computer search for solutions up to this bound. Table 1 lists all d up to 200 that satisfy (∗), indicating whether they satisfy (∗∗) and (∗∗∗). I do not know any nice characterization of (∗∗∗) in terms of the A_2 lattice.

Table 1: Comparison of numerical conditions.

d	$(**)$	$(***)$	d	$(**)$	$(***)$	d	$(**)$	$(***)$
8			74	x		140		
12			78	x		144		
14	x	x	80			146	x	x
18			84			150		
20			86	x	x	152		
24			90			156		
26	x	x	92			158	x	
30			96			162		
32			98	x		164		
36			102			168		
38	x	x	104			170		
42	x	x	108			174		
44			110			176		
48			114	x	x	180		
50			116			182	x	x
54			120			186	x	x
56			122	x	x	188		
60			126			192		
62	x	x	128			194	x	x
66			132			198		
68			134	x	x	200		
72			138					

Outline

In §1 we review Markman's Mukai lattice for a variety Y of $\mathrm{K3}^{[n]}$-type, which governs the global Torelli theorem for such varieties. We give criteria in terms of this lattice for Y to be birational to a moduli space of sheaves or Hilbert scheme of n points on a K3 surface.

In §2 we review Kuznetsov's K3 category $\mathcal{A}$ associated to X, the special classes $\lambda_1, \lambda_2 \in K_{\mathrm{num}}(\mathcal{A})$, and the Mukai lattice $K_{\mathrm{top}}(\mathcal{A})$ introduced in [1]. We prove that

$$H^2(F, \mathbb{Z})(1) \cong {\lambda_1}^{\perp} \subset K_{\mathrm{top}}(\mathcal{A}). \tag{1}$$

This extends Beauville and Donagi's result [6, Prop. 6] that $H^2_{\mathrm{prim}}(F,\mathbb{Z})(1) \cong H^4_{\mathrm{prim}}(X,\mathbb{Z})(2)$, since the latter is Hodge-isometric to $\langle\lambda_1,\lambda_2\rangle^\perp \subset K_{\mathrm{top}}(\mathcal{A})$. From (1) we deduce that $K_{\mathrm{top}}(\mathcal{A})(-1)$ is the Markman–Mukai lattice of F. All this is consistent with Kuznetsov and Markushevich's result [12, §5] that F is a moduli space of objects in the numerical class $\lambda_1 \in K_{\mathrm{num}}(\mathcal{A})$.

With this lattice theory in hand, we prove Theorems 1 and 2 in §3.

Convention

Since we are speaking about transcendental lattices and moduli spaces of sheaves, we will take all K3 surfaces to be projective unless otherwise stated.

1. The Markman–Mukai lattice of a variety of K3$^{[n]}$-type

A *variety of K3$^{[n]}$-type* is a smooth projective variety Y deformation-equivalent to the Hilbert scheme of n points of a K3 surface, $n \geq 2$. The second cohomology group $H^2(Y,\mathbb{Z})$ carries a quadratic form q, the *Beauville–Bogomolov–Fujiki form*, under which it is a lattice of discriminant $-2n+2$ and signature $(3,20)$. Markman [15, §9] has described an extension of lattices and weight-2 Hodge structures $H^2(Y,\mathbb{Z}) \subset \tilde\Lambda$ with the following properties:

Theorem 3 (Markman[1]).

(a) *As a lattice, $\tilde\Lambda$ is isomorphic to $U^4 \oplus (-E_8)^2$.*

(b) *The orthogonal $H^2(Y,\mathbb{Z})^\perp \subset \tilde\Lambda$ is generated by a primitive vector of square $2n-2$.*

(c) *If Y is a moduli space of sheaves on a K3 surface S with Mukai vector $v \in H^*(S,\mathbb{Z})$ then the extension $H^2(Y,\mathbb{Z}) \subset \tilde\Lambda$ is naturally identified with $v^\perp \subset H^*(S,\mathbb{Z})$.*

(d) *Y_1 and Y_2 are birational if and only if there is a Hodge isometry $\tilde\Lambda_1 \to \tilde\Lambda_2$ taking $H^2(Y_1,\mathbb{Z})$ isomorphically to $H^2(Y_2,\mathbb{Z})$.*

Let $\tilde\Lambda_{\mathrm{alg}} \supset H^{1,1}(Y,\mathbb{Z})$ denote the algebraic part of $\tilde\Lambda$, that is, the integral classes of type $(1,1)$.

[1]This summary is borrowed from [4, §1].

Proposition 4. *Let Y be a variety of $K3^{[n]}$-type, $n \geq 2$. Then the following are equivalent:*[2]

(a) $\tilde{\Lambda}_{\text{alg}}$ *contains a copy of the hyperbolic plane* $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(b) *The transcendental lattice $T_Y \subset H^2(Y, \mathbb{Z})$ is Hodge-isometric to T_S for some K3 surface S.*

(c) *Y is birational to a moduli space of stable sheaves on S.*

Proof. (c) $\Rightarrow$ (a): This is immediate from Theorem 3, since the algebraic part of $H^*(S, \mathbb{Z})$ contains a copy of U spanned by H^0 and H^4.

(a) $\Rightarrow$ (b): Let $L = U^\perp \subset \tilde{\Lambda}$. As a lattice this is isomorphic to $U^3 \oplus (-E_8)^2$, so by the global Torelli theorem it is Hodge-isometric to $H^2(S, \mathbb{Z})$ for some analytic K3 surface S. In fact S is projective, as follows. By Huybrechts' projectivity criterion [9, Thm. 3.11] there is a $c \in H^{1,1}(Y, \mathbb{Z})$ with $q(c) > 0$. Let v be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$; then $q(v) = 2n - 2 > 0$. Thus c and v span a positive definite sublattice of $\tilde{\Lambda}$. This cannot be contained in U, which is indefinite, so $\langle c, v \rangle \cap L$ contains a class of positive square, so S is projective by Huybrechts' criterion.

Now T_S is the transcendental part of L, which is the transcendental part of $\tilde{\Lambda}$, which is T_Y.

(b) $\Rightarrow$ (c): We have a Hodge isometry $\varphi\colon T_Y \to T_S$, and primitive embeddings $T_Y \subset \tilde{\Lambda} \cong U^4 \oplus (-E_8)^2$ and $T_S \subset H^*(S, \mathbb{Z}) \cong U^4 \oplus (-E_8)^2$. The orthgonal ${T_S}^\perp$ contains a copy of U, so by [18, Prop. 3.8] any two primitive embeddings $T_S \hookrightarrow U^4 \oplus (-E_8)^2$ differ by an automorphism of $U^4 \oplus (-E_8)^2$. Thus the lattice isomorphism $\varphi\colon T_Y \to T_S$ extends to a lattice isomorphism $\tilde{\varphi}\colon \tilde{\Lambda} \to H^*(S, \mathbb{Z})$. Since φ is a Hodge isometry, it takes $H^{2,0}(Y)$ to $H^{2,0}(S)$, so the extension $\tilde{\varphi}$ does as well, so $\tilde{\varphi}$ is a Hodge isometry.

Again let $v \in \tilde{\Lambda}$ be a primitive generator of $H^2(Y, \mathbb{Z})^\perp \subset \tilde{\Lambda}$, and write $\tilde{\varphi}(v) = (r, c, s) \in H^*(S, \mathbb{Z})$. I claim that either $r > 0$, or we can modify v and $\tilde{\varphi}$ to make it so. If $r < 0$, replace v with $-v$. If $r = 0$ and $s \neq 0$, compose $\tilde{\varphi}$ with the Mukai reflection through $(1, 0, 1) \in H^*(S, \mathbb{Z})$, so now $\tilde{\varphi}(v) = (-s, c, 0)$ and we are reduced to the previous case. If $r = s = 0$, note that $c^2 = q(v) = 2n - 2 > 0$, and compose $\tilde{\varphi}$ with multiplication by $\exp(c) = (1, c, n - 1)$, so now $\tilde{\varphi}(v) = (0, c, n - 1)$ and we are reduced to the previous case.

Now $\tilde{\varphi}(v)$ is a Mukai vector of positive rank, so for a generic polarization of S the moduli space M of stable sheaves on S with Mukai vector $\tilde{\varphi}(v)$ is

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[2] Mongardi and Wandel have proved a similar result independently in [16, Prop. 2.3].

smooth and non-empty [17]. By construction $\tilde{\varphi}$ is a Hodge isometry from $\tilde{\Lambda}$ to $H^*(S,\mathbb{Z})$ taking $H^2(Y,\mathbb{Z})$ isomorphically to $\tilde{\varphi}(v)^\perp$, so Y is birational to M by Theorem 3. □

Proposition 5. *Let Y be a variety of K3$^{[n]}$-type, $n \geq 2$, and let v be a primitive generator of $H^2(Y,\mathbb{Z})^\perp \subset \tilde{\Lambda}$. Then the following are equivalent:*

(a) *There is a vector $w \in \tilde{\Lambda}_{\mathrm{alg}}$ such that $v.w = -1$ and $w^2 = 0$.*

(b) *Y is birational to* $\mathrm{Hilb}^n(S)$ *for some K3 surface S.*

Proof. (b) ⇒ (a): This is immediate from Theorem 3, since $\mathrm{Hilb}^n(S)$ is the moduli space of sheaves with Mukai vector $v = (1,0,1-n) \in H^*(S,\mathbb{Z})$; take $w = (0,0,1)$.

(a) ⇒ (b): Observe that $e := v + (n-1)w$ and $f := -w$ satisfy $e^2 = f^2 = 0$ and $e.f = 1$, so they span a copy of U in $\tilde{\Lambda}_{\mathrm{alg}}$. Let $L = U^\perp = \langle v, w\rangle^\perp \subset \tilde{\Lambda}$. As in the proof of Proposition 4, there is a projective K3 surface S such that $H^2(S,\mathbb{Z}) \cong L$. Thus we can produce a Hodge isometry from $\tilde{\Lambda} = U \oplus L$ to $H^*(S,\mathbb{Z})$ that takes v to $(1,0,1-n)$, so Y is birational to $\mathrm{Hilb}^n(S)$ by Theorem 3. □

2. The Markman–Mukai lattice of *F*

Recall that X is a smooth cubic fourfold and F is the variety of lines on X. Kuznetsov has observed that the triangulated category

$$\begin{aligned}\mathcal{A} &:= \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle^\perp \subset D^b(\mathrm{Coh}(X)) \\ &:= \{E \in D^b(\mathrm{Coh}(X)) : \mathrm{Ext}^*(\mathcal{O}_X(i), E) = 0 \text{ for } i = 0,1,2\}\end{aligned}$$

is like the derived category of a K3 surface in that it has the same Serre functor and Hochschild homology and cohomology, and has conjectured that X is rational if and only if $\mathcal{A}$ is equivalent to the derived category of an actual K3 surface [11]. By [1], this is essentially equivalent to having $X \in \mathcal{C}_d$ for some d satisfying (∗∗).

Let $K_{\mathrm{num}}(\mathcal{A})$ be the numerical Grothendieck group of $\mathcal{A}$, that is, $K(\mathcal{A})$ modulo the kernel of the Euler pairing. Let $\lambda_1, \lambda_2 \in K_{\mathrm{num}}(\mathcal{A})$ be the classes of the projections of $\mathcal{O}_L(1)$ and $\mathcal{O}_L(2)$ into $\mathcal{A}$, where L is any line on X. The Euler pairing on the sublattice $\langle \lambda_1, \lambda_2\rangle$ is $-A_2 = \left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$.

A Mukai lattice for $\mathcal{A}$ was introduced in [1, Def. 2.2]:

$$K_{\mathrm{top}}(\mathcal{A}) := \{\kappa \in K_{\mathrm{top}}(X) : \chi([\mathcal{O}_X(i)], \kappa) = 0 \text{ for } i = 0, 1, 2\}.$$

Here $K_{\mathrm{top}}(X)$ is the Grothendieck group of topological vector bundles and χ is the Euler pairing, which is integer-valued and extends the Euler pairing on $K_{\mathrm{num}}(X)$. It has a Hodge structure of K3 type pulled back via the Chern character or the Mukai vector

$$K_{\mathrm{top}}(\mathcal{A}) \otimes \mathbb{C} \hookrightarrow \bigoplus H^{2i}(X, \mathbb{C})(i).$$

In [1] this was called a weight-two Hodge structure, but it should really be called weight-zero. We will need the following properties:

Theorem 6 (Addington, Thomas [1, §§2.3–2.4]).

(a) *As a lattice,* $K_{\mathrm{top}}(\mathcal{A})$ *is isomorphic to* $U^4 \oplus {E_8}^2$.

(b) *The algebraic part of* $K_{\mathrm{top}}(\mathcal{A})$ *is isomorphic to* $K_{\mathrm{num}}(\mathcal{A})$.

(c) $\langle \lambda_1, \lambda_2 \rangle^\perp \subset K_{\mathrm{top}}(\mathcal{A})$ *is Hodge-isometric to* $H^4_{\mathrm{prim}}(X, \mathbb{Z})(2)$.

(d) $X \in \mathcal{C}_d$ *if and only if there is a primitive sublattice* $M \subset K_{\mathrm{num}}(\mathcal{A})$ *of rank 3 and discriminant* d *that contains* λ_1 *and* λ_2.

Proposition 7. *Let* $P \subset F \times X$ *be the universal line and* $p\colon P \to F$ *and* $q\colon P \to X$ *the two projections. Then the map* φ *from* ${\lambda_1}^\perp \subset K_{\mathrm{top}}(\mathcal{A})$ *to* $H^2(F, \mathbb{Z})(1)$ *defined by* $\varphi(\kappa) = c_1(p_* q^* \kappa)$ *is a Hodge isometry.*

Proof. Both ${\lambda_1}^\perp$ and $H^2(F, \mathbb{Z})(1)$ are lattices of rank 23 and discriminant 2. It is enough to show that φ is a Hodge isometry when tensored with $\mathbb{Q}$; a priori this only implies that φ embeds ${\lambda_1}^\perp$ as a finite-index sublattice of $H^2(F, \mathbb{Z})(1)$, but since they have the same discriminant the index must in fact be 1.

By the Riemann–Roch formula [3, §3], $\varphi(\kappa)$ is the degree-2 part of

$$p_*(q^*(\mathrm{ch}(\kappa)) \cup \mathrm{td}(T_p)), \tag{2}$$

where T_p is the relative tangent bundle of the $\mathbb{P}^1$-bundle $p\colon P \to F$. First we calculate $\mathrm{td}(T_p)$. Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class. Let S be the restriction to F of the tautological sub-bundle on $\mathrm{Gr}(2, 6)$. Then $g := -c_1(S) \in H^2(F, \mathbb{Z})$ is the hyperplane class in the Plücker embedding. The universal line P is the projectivization $\mathbb{P}S$, and $\mathcal{O}_{\mathbb{P}S}(1) = q^*\mathcal{O}_X(1)$. Since T_p is line bundle, we can take determinants in the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}S} \to \mathcal{O}_{\mathbb{P}S}(1) \otimes p^*S \to T_p \to 0$$

to get $T_p = \mathcal{O}_{\mathbb{P}S}(2) \otimes p^* \det S$. Thus

$$\text{td}(T_p) = 1 + \tfrac{1}{2}(2q^*h - p^*g) + \tfrac{1}{12}(2q^*h - p^*g)^2 + \cdots . \tag{3}$$

The orthogonal to λ_1 in $\langle \lambda_1, \lambda_2 \rangle$ is generated by $\lambda_1 + 2\lambda_2$. Since we are tensoring with $\mathbb{Q}$, we have orthogonal direct sums

$$\lambda_1{}^\perp = \langle \lambda_1 + 2\lambda_2 \rangle \oplus \langle \lambda_1, \lambda_2 \rangle^\perp \tag{4}$$

$$H^2(F, \mathbb{Q}) = \langle g \rangle \oplus H^2_{\text{prim}}(F, \mathbb{Q}). \tag{5}$$

By [1, Prop. 2.3], the Chern character[3] gives a Hodge isometry from the second summand of (4) to $H^4_{\text{prim}}(X, \mathbb{Q})(2)$. By [6, Prop. 6], $p_* q^*$ gives a Hodge isometry from this to the second summand of (5). Since the degree-0 part of $\text{td}(T_p)$ is 1, we see that for $\alpha \in H^4(X, \mathbb{Q})$, the degree-2 part of $p_*(q^*\alpha \cup \text{td}(T_p))$ is just $p_* q^* \alpha$. Thus φ gives a Hodge isometry from the second summand of (4) to the second summand of (5).

For the first summands of (4) and (5), observe that the Euler square of $\lambda_1 + 2\lambda_2$ is -6, and by [8, §2.1] we have $q(g) = -6$ as well (the minus sign comes because we have twisted down to weight zero). Thus it is enough to show that

$$\varphi(\lambda_1 + 2\lambda_2) = g. \tag{6}$$

To calculate $\text{ch}(\lambda_1 + 2\lambda_2)$, recall that λ_i is the class of the left mutation of $\mathcal{O}_L(i)$ past $\mathcal{O}_X(2)$, $\mathcal{O}_X(1)$, and $\mathcal{O}_X$, where L is any line on X, so a straightforward calculation gives

$$\begin{aligned} \lambda_1 &= [\mathcal{O}_L(1)] - [\mathcal{O}_X(1)] + 4[\mathcal{O}_X] \\ \lambda_2 &= [\mathcal{O}_L(2)] - [\mathcal{O}_X(2)] + 4[\mathcal{O}_X(1)] - 6[\mathcal{O}_X] \end{aligned}$$

and thus

$$\text{ch}(\lambda_1 + 2\lambda_2) = -3 + 3h - \tfrac{1}{2}h^2 + \cdots .$$

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[3] In fact [1, Prop. 2.3] says that the Mukai vector gives such a Hodge isometry, but since $\text{td}(X)$ is a polynomial in h, multiplying by $\sqrt{\text{td}(X)}$ does not affect $H^4_{\text{prim}}(X, \mathbb{Q})$.

By [8, §2.1] we have $p_*q^*h^2 = g$. We also have $p_*q^*h = 1$: to see this, take a smooth hyperplane section $X \cap H$ and take its preimage under q; this is the blow-up of F along the surface of lines contained in the cubic threefold $X \cap H$, hence is generically 1-to-1 over F. Combining these facts with (2) and (3) we get (6). ☐

Corollary 8. *The embedding $H^2(F,\mathbb{Z}) \subset K_{\mathrm{top}}(\mathcal{A})(-1)$ given by the previous proposition can be identified with Markman's embedding $H^2(F,\mathbb{Z}) \subset \tilde{\Lambda}$ discussed in §1.*

Proof. If $n = 2$ or if $n-1$ is a prime power then for any Y of $\mathrm{K3}^{[n]}$-type, any two primitive embeddings of $H^2(Y,\mathbb{Z})$ into $U^4 \oplus (-E_8)^2$ differ by an automorphism of the latter [14, §4.1]. ☐

3. Proofs of Theorems 1 and 2

Theorem 1. *The following are equivalent:*

(a) *$X \in \mathcal{C}_d$ for some d satisfying (**).*

(b) *The transcendental lattice $T_X \subset H^4(X,\mathbb{Z})$ is Hodge-isometric to $T_S(-1)$ for some K3 surface S.*

(c) *F is birational to a moduli space of stable sheaves on S.*

Proof. By [1, Thm. 3.1], condition (a) holds if and only if $K_{\mathrm{num}}(\mathcal{A})$ contains a copy of $U \cong -U$. Moreover we have $T_X \cong T_F(-1)$. Thus the theorem follows from Corollary 8 and Proposition 4. ☐

To prove Theorem 2 we will have to work in a basis:

Lemma 9. *If $X \in \mathcal{C}_d$ then there is a $\tau \in K_{\mathrm{num}}(\mathcal{A})$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant d with Euler pairing*

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2k \end{pmatrix} \qquad \text{when } d = 6k, \text{ or}$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix} \qquad \text{when } d = 6k+2.$$

Proof. By Theorem 6(d), we can choose a $\tau \in K_{\mathrm{num}}(\mathcal{A})$ such that $\langle \lambda_1, \lambda_2, \tau \rangle$ is a primitive sublattice of discriminant d. Write the Euler pairing as

$$\begin{pmatrix} -2 & 1 & a \\ 1 & -2 & * \\ a & * & * \end{pmatrix}$$

for some $a \in \mathbb{Z}$. Replace τ with $\tau - a\lambda_2$; then the Euler pairing becomes

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 3b+c \\ 0 & 3b+c & * \end{pmatrix}$$

for some b and some $-1 \leq c \leq 1$. Replace τ with $\tau + b(\lambda_1 + 2\lambda_2)$; then the Euler pairing becomes

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & c \\ 0 & c & 2k \end{pmatrix}$$

for some k, since $K_{\mathrm{num}}(\mathcal{A})$ is an even lattice. If $c = 0$ this has determinant $6k$. If $c = 1$ this has determinant $6k + 2$. If $c = -1$, replace τ with $-\tau$ to get back to the previous case. □

Theorem 2. *The following are equivalent:*

(a) $X \in \mathcal{C}_d$ *for some* d *satisfying* $(***)$.

(b) F *is birational to* $\mathrm{Hilb}^2(S)$ *for some K3 surface* S.

Proof. We will show that condition (a) holds if and only if there is a $w \in K_{\mathrm{num}}(\mathcal{A})$ such that

$$\chi(\lambda_1, w) = 1 \quad \text{and} \quad \chi(w, w) = 0. \tag{7}$$

Then the theorem follows from Corollary 8 and Proposition 5.

If there is such a w, let $L = \langle \lambda_1, \lambda_2, w \rangle \subset K_{\mathrm{num}}(\mathcal{A})$. By hypothesis, the Euler pairing on L is

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & n \\ 1 & n & 0 \end{pmatrix}$$

for some $n \in \mathbb{Z}$, so $\mathrm{disc}(L) = 2n^2 + 2n + 2$. Let M be the saturation of L, let a be the index of L in M, and let $d = \mathrm{disc}(M)$. Then $\mathrm{disc}(L) = a^2 d$, and $X \in \mathcal{C}_d$ by Theorem 6(d).

Conversely, suppose $X \in \mathcal{C}_d$ for some d satisfying (∗∗∗). Choose integers n and a such that

$$da^2 = 2n^2 + 2n + 2.$$

Recall that d is even. Since $2n^2 + 2n + 2$ satisfies (∗∗) we see that a is a product of primes $p \equiv 1 \pmod 3$, and in particular $a \equiv 1 \pmod 3$. We consider three cases.

Case 1: $n \equiv 1 \pmod 3$. In this case we find that $d \equiv 0 \pmod 6$. Write $d = 6k$. By Lemma 9 there is a $\tau \in K_{\mathrm{num}}(\mathcal{A})$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2k \end{pmatrix}.$$

Let $m = (n-1)/3$, which is an integer; then we find that

$$w := m\lambda_1 + (2m+1)\lambda_2 + a\tau$$

satisfies (7).

Case 2: $n \equiv 2 \pmod 3$. In this case we find that $d \equiv 2 \pmod 6$. Write $d = 6k + 2$. By Lemma 9 there is a $\tau \in K_{\mathrm{num}}(\mathcal{A})$ such that the Euler pairing on $\langle \lambda_1, \lambda_2, \tau \rangle$ is

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 2k \end{pmatrix}.$$

Let $m = (a - n - 2)/3$, which is an integer; then we find that

$$w := m\lambda_1 + (2m+1)\lambda_2 + a\tau$$

satisfies (7).

Case 3: $n \equiv 0 \pmod 3$. Again we find that $d \equiv 2 \pmod 6$. Argue as in the previous case but with $m = (a + n - 1)/3$. □

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