Hausdorff measure of nodal sets of analytic Steklov eigenfunctions

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Let (Ω, g) be a real analytic Riemannian manifold with real analytic boundary $\partial\Omega$. Let ψ_λ be an eigenfunction of the Dirichlet-to-Neumann operator Λ of $(\Omega, g, \partial\Omega)$ of eigenvalue λ. Let $\mathcal{N}_\lambda$ be its nodal set. Then, there exists a constant $C > 0$ depending only on $(\Omega, g, \partial\Omega)$ so that
$$\mathcal{H}^{n-2}(\mathcal{N}_\lambda) \leq C\lambda.$$
This proves a conjecture of F. H. Lin and K. Bellova.

This article is concerned with the Hausdorff $\mathcal{H}^{n-2}$ measure of the nodal sets

$$\mathcal{N}_\lambda = \{x \in \partial\Omega : \psi_\lambda(x) = 0\} \subset \partial\Omega$$

of Steklov eigenfunctions of eigenvalue λ of a domain $\Omega \subset \mathbb{R}^n$ in the real analytic case. The Steklov eigenvalue problem on a domain Ω is,

(1)
$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = \lambda u(x), & x \in \partial\Omega. \end{cases}$$

It is often assumed that $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain with Euclidean metric, but the problem may be posed on a bounded domain in any Riemannian manifold. The eigenvalue problem may be reduced to the boundary, and ψ_λ is an eigenfunction

(2)
$$\Lambda\psi_\lambda = \lambda\psi_\lambda$$

of the Dirichlet-to-Neumann operator

$$\Lambda f = \frac{\partial u}{\partial \nu}(x)|_{\partial\Omega}.$$

Here, u is the harmonic extension of f,

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u(x) = f(x), & x \in \partial\Omega. \end{cases}.$$

Λ is a positive elliptic self-adjoint pseudo-differential operator of order one on $L^2(\partial\Omega, dS)$ and there exists an orthonormal basis $\{\psi_j\}$ of eigenfunctions

$$\Lambda\psi_j = \lambda_j\psi_j, \quad \psi_j \in C^\infty(\partial\Omega), \quad \int_{\partial\Omega} \psi_j\psi_k dS = \delta_{jk},$$

where dS is the surface measure. We order the eigenvalues in ascending order $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$, counted with multiplicty (see e.g. [DG]).

In a recent article, Bellova-Lin [BL] proved that

$$\mathcal{H}^{n-2}(\mathcal{N}_\lambda) \le C\lambda^6$$

when $\Omega \subset \mathbb{R}^n$ is a bounded *real analytic* Euclidean domain. They suggest that the optimal result is $\mathcal{H}^{n-2}(\mathcal{N}_\lambda) \le C\lambda$. This is the analogue for the Steklov problem of the upper bound half of Yau's conjecture [Y1, Y2] that

(3) $$c_{M,g}\lambda \le \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \le C_{M,g}\lambda.$$

for nodal sets of Laplace eigenfunctions of C^∞ metrics.

The purpose of this article is to prove this upper bound for bounded real analytic domains in general real analytic Riemannian manifolds.

Theorem 1. *Let (Ω, g) be a compact real analytic Riemannian manifold with real analytic boundary $\partial\Omega$. Let ψ_λ be an eigenfunction of the Dirichlet-to-Neumann operator Λ of $(\Omega, g, \partial\Omega)$ of eigenvalue λ, and $\mathcal{N}_\lambda$ be its nodal set as above. Then, there exists a constant $C > 0$ depending only on $(\Omega, g, \partial\Omega)$ so that*

$$\mathcal{H}^{n-2}(\mathcal{N}_\lambda) \le C\lambda.$$

It is not hard to find examples of $(\Omega, g, \partial\Omega)$ and ψ_λ where the upper bound is achieved, for instance on a hemisphere of a round sphere. But it is not clear that it is attained by a sequence of Steklov eigenfunctions on every $(\Omega, g, \partial\Omega)$, or more stringently that it is obtained by every sequence of eigenfunctions. In the setting of real analytic Riemannian manifolds (M, g), it is proved in [DF] that there exists $C > 0$ depending only on the metric g so that $\mathcal{H}^{n-1}(\mathcal{N}_\lambda) \ge C\lambda$ (this is the lower bound half of (3)). Since $\dim \partial\Omega =$

$n - 1$, the analogous lower bound for the real analytic Steklov problem would be $\mathcal{H}^{n-2}(\mathcal{N}_\lambda) \geq C\lambda$, where C depends only on $(\Omega, g, \partial\Omega)$. However the key existence result for Δ-eigenfunctions of eigenvalue λ^2, that every ball of radius $\frac{C}{\lambda}$ contains a zero of φ_λ, does not seem to be known for the Steklov problem (1); here and below Δ is the positive Laplacian. A recent result of Wang-Zhu [WZ] gives a power law lower bound on $\mathcal{H}^{n-2}(\mathcal{N}_\lambda)$ which is consistent with known lower bounds in the C^∞ case. It would be interesting to see if the complex analytic methods of this article can improve the lower bounds in the analytic case.

0.1 Outline of the proof of Theorem 1

The key to proving the sharp upper bound in the generality of Theorem 1 is to use the wave group

$$(4) \qquad U(t) = e^{it\Lambda} : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

generated by Λ. As mentioned above, Λ is a positive elliptic self-adjoint pseudo-differential operator of order one, with principal symbol equal to the metric norm $|\xi|_{g_{\partial\Omega}}$ of the boundary metric (see e.g. [T, LU]). Its wave group is therefore a Fourier integral operator by the methods and results of [Hor, DG]. As in [Z] we study nodal sets by analytically continuing the Schwartz kernel of the wave group to imaginary time $t + i\tau$ with $\tau > 0$, and to the complexification $(\partial\Omega)_\mathbb{C}$ of $\partial\Omega$. The analytic continuation in time and in the first space variable defines the Poisson wave kernel

$$(5) \qquad U_\mathbb{C}(t + i\tau, \zeta, y) = e^{i(t+i\tau)\Lambda}(\zeta, y) : L^2(\partial\Omega) \to L^2((\partial\Omega)_\mathbb{C}).$$

As discussed below, Λ is an analytic pseudo-differential operator on $\partial\Omega$ when $(\Omega, \partial\Omega, g)$ is real analytic, and (5) is a Fourier integral operator with complex phase. (See [Bou2, Sj] for background on analytic pseudo-differential operators).

In the real analytic case, the Steklov eigenfunctions are real analytic on $\partial\Omega$ and have complex analytic extensions to $(\partial\Omega)_\mathbb{C}$. We then study their complex nodal sets

$$(6) \qquad \mathcal{N}_\lambda^\mathbb{C} = \{\zeta \in (\partial\Omega)_\mathbb{C} : \psi_\lambda^\mathbb{C}(\zeta) = 0\}.$$

To prove Theorem 1, we use Crofton's formula and a multi-dimensional Jensen's formula to give an upper bound for $\mathcal{H}^{n-2}(\mathcal{N}_\lambda)$ in terms of the integral geometry of $\mathcal{N}_\lambda^\mathbb{C}$. The integral geometric approach to the upper bound

is inspired by the classic paper of Donnelly-Fefferman [DG] (see also [Lin]). But, instead of doubling estimates or frequency function estimates, we use the Poisson wave kernel to obtain growth estimates on eigenfunctions, and then use results on pluri-subharmonic functions rather than functions of one complex variable to relate growth of zeros to growth of eigenfunctions. This approach was used in [Z] to prove equidistribution theorems for complex nodal sets when the geodesic flow is ergodic. The Poisson wave kernel approach works for Steklov eigenfunctions as well as Laplace eigenfunctions.

We first use the Poisson wave group (5) to analytically continue eigenfunctions in the form

$$(7) \qquad U_{\mathbb{C}}(i\tau)\psi_j(\zeta) = e^{-\tau\lambda_j}\psi_j^{\mathbb{C}}(\zeta).$$

We then use (7) to determine the growth properties of $\psi_j^{\mathbb{C}}(\zeta)$ in Grauert tubes of the complexification of $\partial\Omega$. The relevant notion of Grauert tube is the standard Grauert tube for $\partial\Omega$ with the metric $g_{\partial\Omega}$ induced by the ambient metric g on M. This is because the principal symbol of Λ is the same as the principal symbol of $\sqrt{\Delta_{\partial\Omega}}$.

Remark: A remark on notation: In [Z] we use M to denote a Riemannian manifold, M_ϵ its Grauert tube of radius ϵ and ∂M_ϵ to denote the boundary of the Grauert tube of radius ϵ. Since $\partial\Omega$ is the Riemannian manifold of interest here, we denote it by M,

$$(8) \qquad (M, g) := (\partial\Omega, g_{\partial\Omega}).$$

Thus the Grauert tube of radius τ of $(\partial\Omega)_{\mathbb{C}}$ is denoted M_τ and its boundary by ∂M_τ, not to be confused with $\partial\Omega$. We also denote $m = \dim M = n - 1$.

Because $U_{\mathbb{C}}(i\tau)$ is a Fourier integral operator with complex phase, it can only magnify the L^2 norm of ψ_j by a power of λ_j. Hence the exponential $e^{\tau\lambda_j}$ dominates the L^2 norm on the boundary of the Grauert tube of radius τ. We prove:

Proposition 2. *Suppose $(\Omega, g, \partial\Omega)$ is real analytic. Let $\{\psi_\lambda\}$ be an eigenfunction of Λ on $M = \partial\Omega$ of eigenvalue λ. Then*

$$\sup_{\zeta \in M_\tau} |\psi_\lambda^{\mathbb{C}}(\zeta)| \le C\lambda^{\frac{m+1}{4}}e^{\tau\lambda}.$$

The proof follows from a standard cosine Tauberian result and the fact that the complexified Poisson kernel is a complex Fourier integral operator of finite order. This simple growth estimate replaces the doubling estimates of

[DF] and [BL]. It is closely related to growth estimates of Δ-eigenfunctions in [Z, Z2, Z3].

For the precise statement that $U_{\mathbb{C}}(t + i\tau)$ is indeed a Fourier integral operator with complex phase, we refer to Theorem 3. It holds for positive elliptic analytic pseudo-differential operators whose principal symbol has convex sublevel sets. Since the principal symbol of Λ is the same as for $\sqrt{\Delta_{\partial\Omega}}$, the proof is only a small modification of that in [L, Z2, St] in the latter case and is omitted.

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1. Geometry and analysis of Grauert tubes

We briefly review the geometry and analysis on Grauert tubes of real analytic Riemannian manifolds. We refer to [Z, Z2, GS1, GS2] for more detailed discussions.

1.1. Analytic continuation to a Grauert tube

A real analytic manifold M always possesses a complexification $M_{\mathbb{C}}$, i.e. a complex manifold of which M is a totally real submanifold. A real analytic Riemannian metric g on M determines a canonical plurisubharmonic function ρ_g on $M_{\mathbb{C}}$; since the metric is fixed througout, we denote it simply by ρ. Its square-root $\sqrt{\rho}$ is known as the Grauert tube function; it equals $\sqrt{-r_{\mathbb{C}}^2(z, \bar{z})/2}$ where $r_{\mathbb{C}}$ is the holomorphic extension of the distance function. The $(1,1)$ form $\omega = \omega_\rho := i\partial\bar{\partial}\rho$ defines a Kähler metric on $M_{\mathbb{C}}$. The Grauert tubes $M_\epsilon := \{z \in M_{\mathbb{C}} : \sqrt{\rho}(z) < \epsilon\}$ are strictly pseudo-convex domains in $M_{\mathbb{C}}$, whose boundaries are strictly pseudo-convex CR manifolds. We also denote the contact form of ∂M_τ by

$$(9) \qquad \alpha = \frac{1}{i}\partial\rho|_{\partial M_\tau} = d^c\sqrt{\rho}.$$

The complexified exponential map

$$(10) \qquad (x, \xi) \in B_\epsilon^* M \to E(x, \xi) := \exp_x^{\mathbb{C}} \sqrt{-1}\xi \in M_\epsilon$$

defines a symplectic diffeomorphism, where $B_\epsilon^* M \subset T^* M$ is the co-ball bundle of radius ϵ, equipped with the standard symplectic structure, and where M_ϵ is equipped with ω_ρ. The Grauert tube function $\sqrt{\rho}$ pulls back under

E to the metric norm function $|\xi|_g$. We emphase the setting $M_{\mathbb{C}}$ but it is equivalent to using E to endow $B_\epsilon^* M$ with an adapted complex structure. We refer to [GS1, GS2, LS, GLS] for further discussion.

1.2. Geodesic and Hamiltonian flows

The microlocal analysis of the kernels (5) involves the complexification of the geodesic flow. We denote by g^t the (real) homogeneous geodesic flow of (M, g). It is the real analytic Hamiltonian flow on $T^*M \backslash 0_M$ generated by the Hamiltonian $|\xi|_g$ with respect to the standard symplectic form ω. We also consider the Hamiltonian flow of $|\xi|_g^2$, which is real analytic on all of T^*M and denote its Hamiltonian flow by G^t. In general, we denote by Ξ_H the Hamiltonian vector field of a Hamiltonian H and its flow by $\exp t\Xi_H$. Thus, we consider the Hamiltonian flows

$$(11) \qquad g^t = \exp t\Xi_{|\xi|_g}, \quad \text{resp.} \quad G^t = \exp t\Xi_{|\xi|_g^2}.$$

The exponential map is the map $\exp_x : T^*M \to M$ defined by $\exp_x \xi = \pi G^1(x, \xi)$ where π is the standard projection. Since $E^* \sqrt{\rho} = |\xi|$, E^* conjugates the geodesic flow on B^*M to the Hamiltonian flow $\exp t\Xi_{\sqrt{\rho}}$ of $\sqrt{\rho}$ with respect to ω, i.e.

$$(12) \qquad E(g^t(x, \xi)) = \exp t\Xi_{\sqrt{\rho}}(\exp_x i\xi).$$

1.3. Szegö kernel and analytic continuation of the Poisson kernel

We denote by $\mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau)$ the Sobolev spaces of CR holomorphic functions on the boundaries of the strictly pseudo-convex domains M_τ, i.e.

$$(13) \qquad \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\tau) = W^{s+\frac{m-1}{4}}(\partial M_\tau) \cap \mathcal{O}(\partial M_\tau),$$

where W^s is the sth Sobolev space and where $\mathcal{O}(\partial M_\tau)$ is the space of boundary values of holomorphic functions. The inner product on $\mathcal{O}^0(\partial M_\tau)$ is with respect to the Liouville measure or contact volume form

$$(14) \qquad d\mu_\tau := \alpha \wedge \omega^{m-1},$$

on ∂M_τ.

   The study of norms of complexified eigenfunctions is related to the study of the Szegö kernels Π_τ of M_τ, namely the orthogonal projections

(15) $$\Pi_\tau : L^2(\partial M_\tau, d\mu_\tau) \to \mathcal{O}^0(\partial M_\tau, d\mu_\tau)$$

onto the Hardy space of boundary values of holomorphic functions in M_τ which belong to $L^2(\partial M_\tau, d\mu_\tau)$. The Szegö projector Π_τ is a complex Fourier integral operator with a positive complex canonical relation. The real points of its canonical relation form the graph Δ_Σ of the identity map on the symplectic cone $\Sigma_\tau \subset T^*\partial M_\tau$ defined by the spray

(16) $$\Sigma_\tau = \{(\zeta, rd^c\sqrt{\rho}(\zeta)) : r \in \mathbb{R}_+\} \subset T^*(\partial M_\tau)$$

of the contact form $d^c\sqrt{\rho}$. There exists a symplectic equivalence (cf. [GS2])

(17) $$\iota_\tau : T^*M - 0 \to \Sigma_\tau, \quad \iota_\tau(x, \xi) = (E(x, \tau\frac{\xi}{|\xi|}), |\xi|d^c\sqrt{\rho}_{E(x,\tau\frac{\xi}{|\xi|})}).$$

1.4. Analytic continuation of the Poisson wave kernel

The wave group generated by Λ on $M = \partial\Omega$ is the unitary group $U(t) = e^{it\Lambda}$. Its kernel $U(t, x, y)$ solves the 'half-wave equation',

(18) $$\left(\frac{1}{i}\frac{\partial}{\partial t} - \Lambda_x\right)U(t, x, y) = 0, \quad U(0, x, y) = \delta_y(x).$$

Here, Λ_x means that Λ is applied in the x variable. In the real domain it is well known [Hor, DG] that $U(t, x, y)$ is the Schwartz kernel of a Fourier integral operator,

$$U(t, x, y) \in I^{-1/4}(\mathbb{R} \times M \times M, \Gamma)$$

with underlying canonical relation

$$\Gamma = \{(t, \tau, x, \xi, y, \eta) : \tau + |\xi| = 0, g^t(x, \xi) = (y, \eta)\} \subset T^*\mathbb{R} \times T^*M \times T^*M.$$

The Poisson-wave kernel is the analytic continuation $U(t + i\tau, x, y)$ of the wave kernel with respect to time, $t \to t + i\tau \in \mathbb{R} \times \mathbb{R}_+$. For $t = 0$ and for $\tau > 0$, we obtain the Poisson semi-group $U(i\tau) = e^{-\tau\Lambda}$. For general $t + i\tau$,

the Poisson-wave kernel has the eigenfunction expansion,

$$(19) \qquad U(i\tau, x, y) = \sum_j e^{i(t+i\tau)\lambda_j} \psi_{\lambda_j}(x)\psi_{\lambda_j}(y).$$

As stated in Theorem 3 in the introduction, the Poisson-wave kernel $U(t + i\tau, x, y)$ admits an analytic continuation $U_{\mathbb{C}}(t + i\tau, \zeta, y)$ in the first variable to $M_\tau \times M$.

Theorem 3. *Let $U(t)$ be the wave group of the Dirichlet to Neumann operator Λ on $M = \partial\Omega$ as above. Then $\Pi_\epsilon \circ U(i\epsilon) : L^2(M) \to \mathcal{O}(\partial M_\epsilon)$ is a Fourier integral operator with complex phase of order $-\frac{m-1}{4}$ associated to the canonical relation*

$$\Gamma = \{(y, \eta, \iota_\epsilon(y, \eta))\} \subset T^*\partial M_\epsilon \times \Sigma_\epsilon.$$

Moreover, for any s,

$$\Pi_\epsilon \circ U(i\epsilon) : W^s(M) \to \mathcal{O}^{s+\frac{m-1}{4}}(\partial M_\epsilon)$$

is a continuous isomorphism.

This statement is asserted by Boutet de Monvel in [Bou, Bou2] for any real analytic positive elliptic pseudo-differential operator for which the sub-level sets $\{\xi \in T_x^*M : p(x, \xi) \leq 1\}$ are strictly convex, and has been accepted since then as an established fact (see for instance [GS1, GS2]). The proof was only sketched in [Bou, Bou2], and the first complete proofs appeared only recently in [Z2, L, St] for the special case of the wave group of a Riemannian manifold without boundary. Essentially the same proof applies to the wave group of Λ as well because $\sqrt{\Delta_{\partial M}}$ and Λ only differ by an analytic pseudo-differential operator of order zero, as follows from the fact that the principal symbol of Λ,

$$(20) \qquad \sigma_\Lambda : T^*\partial\Omega \to \mathbb{R}, \quad \sigma_\Lambda(x, \xi) = |\xi|_{g_\partial},$$

is the same as for the Laplacian Δ_∂ of the boundary $(\partial\Omega, g_\partial)$. In fact, the complete symbol of Λ is calculated in [LU] (see also [PS]).

2. Growth of complexified eigenfunctions proof of Proposition 2

We further need to generalize sup norm estimates of complexified eigenfunctions in [Z2] to the Λ-eigenfunctions.

As in [Z2, Z3] we prove Proposition 2 by introducing the 'tempered' spectral projections

$$(21) \qquad P_{I_\lambda}^\tau(\zeta, \bar\zeta) = \sum_{j:\lambda_j \in I_\lambda} e^{-2\tau\lambda_j} |\psi_{\lambda_j}^{\mathbb{C}}(\zeta)|^2, \quad (\sqrt{\rho}(\zeta) \le \tau),$$

where I_λ could be a short interval $[\lambda, \lambda+1]$ of frequencies or a long window $[0, \lambda]$. Exactly as in [Z2] but with the wave group of Λ replacing the wave group of $\sqrt{\Delta}$, we prove

$$(22) \quad P_{[0,\lambda]}^\tau(\zeta, \bar\zeta) = (2\pi)^{-m} \left(\frac{\lambda}{\sqrt{\rho}}\right)^{\frac{m-1}{2}} \left(\frac{\lambda}{(m-1)/2+1} + O(1)\right), \quad \zeta \in \partial M_\tau.$$

We then obtain

Corollary 4. *Let ψ_λ be an eigenfunction of Λ as above. Then there exists $C > 0$ so that for all $\sqrt{\rho}(\zeta) = \tau$,*

$$C\lambda^{-\frac{m-1}{2}} e^{\tau\lambda} \le \sup_{\zeta \in M_\tau} |\psi_\lambda^{\mathbb{C}}(\zeta)| \le C\lambda^{\frac{m-1}{4}+\frac{1}{2}} e^{\tau\lambda}.$$

The upper bound is not sharp but is sufficient for our purposes. The lower bound is not used in the nodal analysis.

2.1. Proof of the local Weyl law

We only sketch the proof for the sake of completeness, since it is essentially the same as in [Z, Z2, Z3] and closely follows [DG]. The novelty is that we apply the argument of [DG] to the analytically continued parametrix.

By [Hor, DG] the positive elliptic first order pseudo-differential operator Λ generates a wave group which has a parametrix of the form,

$$(23) \qquad U(t, x, y) = \int_{T_y^* M} e^{it|\xi|_{g_y}} e^{i\langle \xi, \exp_y^{-1}(x)\rangle} A(t, x, y, \xi) d\xi$$

similar to that of the wave kernel of $M = \partial\Omega$, since $\Lambda = \sqrt{\Delta_M} + Q$ where Q is an analytic pseudo-differential operator of order zero. Here, $|\xi|_{g_x}$ is the metric norm function at x, and where $A(t, x, y, \xi)$ is a polyhomogeneous amplitude of order 0 which is supported near the diagonal. The amplitude is different from that of the wave kernel since the transport equations involve Q.

By Theorem 3, the wave group and parametrix may be analytically continued. To obtain uniform asymptotics, we use the analytic continuation of the Hörmander parametrix (23). We choose local coordinates near x and write $\exp_x^{-1}(y) = \Phi(x, y)$ in these local coordinates for y near x, and write the integral $T_y^* M$ as an integral over $\mathbb{R}^m$ in these coordinates. The holomorphic extension of the parametrix to the Grauert tube $|\zeta| < \tau$ at time $t + 2i\tau$ has the form

$$(24) \qquad U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) = \int_{\mathbb{R}^m} e^{(it - 2\tau)|\xi|_{g_y}} e^{i\langle \xi, \Phi(\zeta, \bar{\zeta}) \rangle} A(t, \zeta, \bar{\zeta}, \xi) d\xi,$$

where A is the analytic extensions of the real analytic A and $\Phi(\zeta, \bar{\zeta})$ is the analytic extension of $\exp_y^{-1}(x)$.

We introduce a cutoff function $\chi \in \mathcal{S}(\mathbb{R})$ with $\hat{\chi} \in C_0^\infty$ supported in sufficiently small neighborhood of 0 so that no other singularities of $U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta})$ lie in its support. We also assume $\hat{\chi} \equiv 1$ in a smaller neighborhood of 0. We then change variables $\xi \to \lambda \xi$ and apply the complex stationary phase to the integral,

$$(25) \qquad \int_{\mathbb{R}} \hat{\chi}(t) e^{-i\lambda t} U_{\mathbb{C}}(t + 2i\tau, \zeta, \bar{\zeta}) dt$$

$$= \lambda^m \int_0^\infty \int_{\mathbb{R}} \hat{\chi}(t) e^{-i\lambda t}$$

$$\int_{S^{m-1}} e^{(it - 2\tau)\lambda r} e^{ir\lambda \langle \omega, \Phi(\zeta, \bar{\zeta}) \rangle} A(t, \zeta, \bar{\zeta}, \lambda r \omega) r^{m-1} dr dt d\omega.$$

The resulting integral (25) is a semi-classical Fourier integral distribution with a complex phase, the same phase as in the pure Riemannian case treated in [Z2]. Hence the stationary phase calculation is essentially the same as in section 9.1 of [Z2]. We first integrate over $d\omega$ and find that there are two stationary phase points, one giving an exponentially decaying amplitude of order $e^{-2\lambda \tau r}$ and one for which the critical value is $2\lambda \tau r$. It cancels the term $-2\tau \lambda r$ coming from the factor $e^{(it - 2\tau)\lambda r}$. We then apply stationary phase to the resulting integral over (t, r) with phase $t(r - 1)$. The critical set consists of $r = 1, t = 0$. The phase is clearly non-degenerate with Hessian determinant one and inverse Hessian operator $D_{\theta, t}^2$. Taking into account the factor of λ^{-1} from the change of variables, the stationary phase expansion gives

$$(26) \qquad \sum_j \psi(\lambda - \lambda_j) e^{-2\tau \lambda_j} |\psi_j^{\mathbb{C}}(\zeta)|^2 \sim \sum_{k=0}^\infty \lambda^{\frac{m-1}{2} - k} \omega_k(\tau; \zeta),$$

where the coefficients $\omega_k(\tau, \zeta)$ are smooth for $\zeta \in \partial M_\tau$. The Weyl asymptotics then follows from the standard cosine Tauberian theorem, as in [DG] or [Z2] (loc. cit.).

3. Proof of Theorem 1

We start with the integral geometric approach of [DF] (Lemma 6.3) (see also [Lin] (3.21)). There exists a "Crofton formula" in the real domain which bounds the local nodal hypersurface volume above,

$$(27) \qquad \mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda} \cap U) \le C_L \int_{\mathcal{L}} \#\{\mathcal{N}_{\varphi_\lambda} \cap \ell\} d\mu(\ell).$$

Thus, $\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda} \cap U)$ is bounded above by a constant C_L times the average over all line segments of length L in a local coordinate patch U of the number of intersection points of the line with the nodal hypersurface. The measure $d\mu_L$ is known as the 'kinematic measure' in the Euclidean setting [F] (Chapter 3); see also Theorem 5.5 of [AP]. We will be using geodesic segments of fixed length L rather than line segments, and parametrize them by $S^*M \times [0, L]$, i.e. by their intial data and time. Then $d\mu_\ell$ is essentially Liouville measure $d\mu_L$ on S^*M times dt.

The complexification of a real line $\ell = x + \mathbb{R}v$ with $x, v \in \mathbb{R}^m$ is $\ell_{\mathbb{C}} = x + \mathbb{C}v$. Since the number of intersection points (or zeros) only increases if we count complex intersections, we have

$$(28) \qquad \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda} \cap \ell) d\mu(\ell) \le \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda}^{\mathbb{C}} \cap \ell_{\mathbb{C}}) d\mu(\ell).$$

Note that this complexification is quite different from using intersections with all complex lines to measure complex nodal volumes. If we did that, we would obtain a similar upper bound on the complex hypersurface volume of the complex nodal set. But it would not give an upper bound on the real nodal volume and indeed would the complex volume tends to zero as one shrinks the Grauert tube radius to zero, while (28) stays bounded below.

Hence to prove Theorem 1 it suffices to show

Lemma 5. *We have,*

$$\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda}) \le C_L \int_{\mathcal{L}} \#(\mathcal{N}_{\varphi_\lambda})^{\mathbb{C}} \cap \ell_{\mathbb{C}}) d\mu(\ell) \le C\lambda.$$

We now sketch the proofs of these results using a somewhat novel approach to the integral geometry and complex analysis.

3.1. Background on hypersurfaces and geodesics

The proof of the Crofton formula given below in Proposition 9 involves the geometry of geodesics and hypersurfaces. To prepare for it we provide the relevant background.

As above, we denote by $d\mu_L$ the Liouville measure on S^*M. We also denote by ω the standard symplectic form on T^*M and by α the canonical one form. Then $d\mu_L = \omega^{m-1} \wedge \alpha$ on S^*M. Indeed, $d\mu_L$ is characterized by the formula $d\mu_L \wedge dH = \omega^m$, where $H(x, \xi) = |\xi|_g$. We take the interior product ι_{Ξ_H} with the Hamilton vector field Ξ_H on both sides, and the identity follows from the fact that $\alpha(\Xi_H) = \sum_j \xi_j \frac{\partial H}{\partial \xi_j} = H = 1$ on S^*M, since H is homogeneous of degree one. Henceforth we denote by $\Xi = \Xi_H$ the generator of the geodesic flow.

Let $N \subset M$ be a smooth hypersurface in a Riemannian manifold (M, g). We denote by T_N^*M the of covectors with footpoint on N and S_N^*M the unit covectors along N. We introduce Fermi normal coordinates (s, y_m) along N, where s are coordinates on N and y_m is the normal coordinate, so that $y_m = 0$ is a local defining function for N. We also let σ, ξ_m be the dual symplectic Darboux coordinates. Thus the canonical symplectic form is $\omega_{T^*M} = ds \wedge d\sigma + dy_m \wedge d\xi_m$. Let $\pi : T^*M \to M$ be the natural projection. For notational simplicity we denote π^*y_m by y_m as functions on T^*M. Then y_m is a defining function of T_N^*M.

The hypersurface $S_N^*M \subset S^*M$ is a kind of Poincaré section or symplectic transversal to the orbits of G^t, i.e. is a symplectic transversal away from the (at most codimension one) set of $(y, \eta) \in S_N^*M$ for which $\Xi_{y,\eta} \in T_{y,\eta}S_N^*M$, where as above Ξ is the generator of the geodesic flow. More precisely,

Lemma 6. *The restriction* $\omega|_{S_N^*M}$ *is symplectic on* $S_N^*M \backslash S^*N$.

Indeed, $\omega|_{S_N^*M}$ is symplectic on $T_{y,\eta}S^*N$ as long as $T_{y,\eta}S_N^*M$ is transverse to $\Xi_{y,\eta}$, since $\ker(\omega|_{S^*M}) = \mathbb{R}\Xi$. But S^*N is the set of points of S_N^*M where $\Xi \in TS_N^*M$, i.e. where S_N^*M fails to be transverse to G^t. Indeed, transversality fails when $\Xi(y_m) = dy_m(\Xi) = 0$, and $\ker dy_m \cap \ker dH = TS_N^*M$. One may also see it in Riemannian terms as follows: the generator $\Xi_{y,\eta}$ is the horizontal lift η^h of η to (y, η) with respect to the Riemannian connection on S^*M, where we freely identify covectors and vectors by the metric. Lack of transversality occurs when η^h is tangent to $T_{(y,\eta)}(S_N^*M)$. The latter is the kernel of dy_m. But $dy_m(\eta^h) = dy_m(\eta) = 0$ if and only if $\eta \in TN$.

It follows from Lemma 6 that the symplectic volume form of $S_N^*M \backslash S^*N$ is $\omega^{m-1}|_{S_N^*M}$. The following Lemma gives a useful alternative formula:

Lemma 7. *Define*

$$d\mu_{L,N} = \iota_\Xi d\mu_L \mid_{S_N^*M},$$

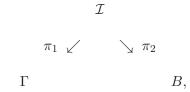*where as above, $d\mu_L$ is Liouville measure on S^*M. Then*

$$d\mu_{L,N} = \omega^{m-1}|_{S_N^*M}.$$

Indeed, $d\mu_L = \omega^{m-1} \wedge \alpha$, and $\iota_\Xi d\mu_L = \omega^{m-1}$.

Corollary 8. $\mathcal{H}^{m-1}(N) = \frac{1}{\beta_m} \int_{S_N^*M} |\omega^{m-1}|.$

3.2. Hausdorff measure and Crofton formula for real geodesic arcs

First we sketch a proof of the integral geometry estimate using geodesic arcs rather than local coordinate line segments. For background on integral geometry and Crofton type formulae we refer to [AB, AP]. As explained there, a Crofton formula arises from a double fibration

$$\mathcal{I}$$

$$\pi_1 \swarrow \qquad \searrow \pi_2$$

$$\Gamma \qquad\qquad\qquad B,$$

where Γ parametrizes a family of submanifolds B_γ of B. The points $b \in B$ then parametrize a family of submanifolds $\Gamma_b = \{\gamma \in \Gamma : b \in B_\gamma\}$ and the top space is the incidence relation in $B \times \Gamma$ that $b \in B_\gamma$.

We would like to define Γ as the space of geodesics of (M, g), i.e. the space of orbits of the geodesic flow on S^*M. Heuristically, the space of geodesics is the quotient space $S^*M/\mathbb{R}$ where $\mathbb{R}$ acts by the geodesic flow G^t (i.e. the Hamiltonian flow of H). Of course, for a general (i.e. non-Zoll) (M, g) the 'space of geodesics' is not a Hausdorff space and so we do not have a simple analogue of the space of lines in $\mathbb{R}^n$. Instead we consider the space $\mathcal{G}_T$ of geodesic arcs of length T. If we only use partial orbits of length T, no two partial orbits are equivalent and the space of geodesic arcs $\gamma_{x,\xi}^T$ of length T is simply parametrized by S^*M. Hence we let $B = S^*M$ and also $\mathcal{G}_T \simeq S^*M$. The fact that different arcs of length T of the same geodesic are distinguished leads to some redundancy.

In the following, let L_1 denote the length of the shortest closed geodesic of (M, g).

Proposition 9. *Let $N \subset M$ be any smooth hypersurface[1], and let $S_N^* M$ denote the unit covers to M with footpoint on N. Then for $0 < T < L_1$,*

$$\mathcal{H}^{m-1}(N) = \frac{1}{\beta_m T} \int_{S^* M} \#\{t \in [-T, T] : G^t(x, \omega) \in S_N^* M\} d\mu_L(x, \omega),$$

where β_m is $2(m-1)!$ times the volume of the unit ball in $\mathbb{R}^{m-2}$.

Proof. By Corollary 8, the Hausdorff measure of N is given by

$$(29) \qquad \mathcal{H}^{m-1}(N) \quad = \quad \frac{1}{\beta_m} \int_{S_N^* M} |\omega^{m-1}|.$$

We use the Lagrange (or more accurately, Legendre) immersion,

$$\iota : S^* M \times \mathbb{R} \to S^* M \times S^* M, \quad \iota(x, \omega, t) = (x, \omega, G^t(x, \omega)),$$

where as above, G^t is the geodesic flow (11). We also let $\pi : T^* M \to M$ be the standard projection. We restrict ι to $S^* M \times [-T, T]$ and define the incidence relation

$$\mathcal{I}_T = \{((y, \eta), (x, \omega), t) \subset S^* M \times S^* M \times [-T, T] : (y, \eta) = G^t(x, \omega)\},$$

which is isomorphic to $[-T, T] \times S^* M$ under ι. We form the diagram

$$\mathcal{I}_T \simeq S^* M \times [-T, T]$$

$$\pi_1 \swarrow \qquad\qquad\qquad\qquad \searrow \pi_2$$

$$S^* M \simeq \mathcal{G}_T \qquad\qquad\qquad\qquad\qquad S^* M,$$

using the two natural projections, which in the local parametrization take the form

$$\pi_1(t, x, \xi) = G^t(x, \xi), \quad \pi_2(t, x, \xi) = (x, \xi).$$

As noted above, the bottom left $S^* M$ should be thought of as the space of geodesic arcs. The fiber

$$\pi_1^{-1}(y, \eta) = \{(t, x, \xi) \in [-T, T] \times S^* M : G^t(x, \xi) = (y, \eta)\} \simeq \gamma_{(y, \eta)}^T$$

---

[1]The same formula is true if N has a singular set Σ with $\mathcal{H}^{m-2}(\Sigma) < \infty$

may be identified with the geodesic segment through (y, η) and the fiber $\pi_2^{-1}(x, \omega) \simeq [-T, T]$.

We 'restrict' the diagram above to $S_N^* M$:

$$\mathcal{I}_T \simeq S_N^* M \times [-T, T]$$

(30) $\pi_1 \swarrow$ $\searrow \pi_2$

$(S_N^* M)_T$ $S_N^* M,$

where

$$(S_N^* M)_T = \pi_1 \pi_2^{-1}(S_N^* M) = \bigcup_{|t| < T} G^t(S_N^* M).$$

We define the Crofton density φ_T on $S_N^* M$ corresponding to the diagram (30) [AP] (section 4) by

(31) $$\varphi_T = (\pi_2)_* \pi_1^* d\mu_L.$$

Since the fibers of π_2 are 1-dimensional, φ_T is a differential form of dimension $2 \dim M - 2$ on $S^* M$. To make it smoother, we can introduce a smooth cutoff χ to $(-1, 1)$, equal to 1 on $(-\frac{1}{2}, \frac{1}{2})$, and use $\chi_T(t) = \chi(\frac{t}{T})$. Then $\pi_1^*(d\mu_L \otimes \chi_T dt)$ is a smooth density on $\mathcal{I}_T$.

Lemma 10. *The Crofton density* (31) *is given by,* $\varphi_T = T d\mu_{L,N}$

Proof. In (30) we defined the map $\pi_1 : (y, \eta, t) \in S_N^* M \times [-T, T] \to G^t(y, \eta) \in (S^* M)_\epsilon$. We first claim that $\pi_1^* d\mu_L = d\mu_{L,N} \otimes dt$. This is essentially the same as Lemma 7. Indeed, $d\pi_1(\frac{\partial}{\partial t}) = \Xi$, hence $\iota_{\frac{\partial}{\partial t}} \pi_1^* d\mu_L|_{(t, y, \eta)} = (G^t)^* \omega^{m-1} = \omega^{m-1}|_{T_{y, \eta} S_N^* M}$.

Combining Lemma 10 with (29) gives

(32) $$\int_{S_N^* M} \varphi_T = \int_{\pi_2^{-1}(S_N^* M)} d\mu_L = T \beta_m \mathcal{H}^{m-1}(N).$$

$\square$

We then relate the integral on the left side to numbers of intersections of geodesic arcs with N. The relation is given by the co-area formula: if $f : X \to Y$ is a smooth map of manifolds of the same dimension and if Φ is a smooth density on Y, and if $\#\{f^{-1}(y)\} < \infty$ for every regular value y,

then

$$\int_X f^*\Phi = \int_Y \#\{f^{-1}(y)\}\,\Phi.$$

If we set set $X = \pi_2^{-1}(S_N^*M)$, $Y = S^*M$, and $f = \pi_1|_{\pi_2^{-1}(S_N^*M)}$ then the co-area formula gives,

$$(33)\quad \int_{\pi_2^{-1}(S_N^*M)} \pi_1^*d\mu_L = \int_{S^*M} \#\{t \in [-T,T] : G^t(x,\omega) \in S_N^*M\}d\mu_L(x,\omega).$$

Combining (32) and (33) gives the result stated in Proposition 9,

$$(34)\quad T\beta_m\mathcal{H}^{m-1}(N) = \int_{S^*M} \#\{t \in [-T,T] : G^t(x,\omega) \in S_N^*M\}d\mu_L(x,\omega).$$

$\square$

3.3. Proof of Lemma 5

The next step is to complexify.

Proof. We complexify the Lagrange immersion ι from a line (segment) to a strip in $\mathbb{C}$: Define

$$F : S_\epsilon \times S^*M \to M_{\mathbb{C}}, \quad F(t + i\tau, x, v) = \exp_x(t + i\tau)v, \quad (|\tau| \leq \epsilon)$$

By definition of the Grauert tube, ψ is surjective onto M_ϵ. For each $(x,v) \in S^*M$,

$$F_{x,v}(t + i\tau) = \exp_x(t + i\tau)v$$

is a holomorphic strip. Here, $S_\epsilon = \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\}$. We also denote by $S_{\epsilon,L} = \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon, |t| \leq L\}$.

Since $F_{x,v}$ is a holomorphic strip,

$$F_{x,v}^*(\frac{1}{\lambda}dd^c \log |\psi_j^{\mathbb{C}}|^2) = \frac{1}{\lambda}dd_{t+i\tau}^c \log |\psi_j^{\mathbb{C}}|^2(\exp_x(t + i\tau)v)$$

$$= \frac{1}{\lambda}\sum_{t+i\tau:\psi_j^{\mathbb{C}}(\exp_x(t+i\tau)v)=0} \delta_{t+i\tau}.$$

Put:

(35)
$$\mathcal{A}_{L,\epsilon}(\frac{1}{\lambda}dd^c \log |\psi_j^{\mathbb{C}}|^2)$$

$$= \frac{1}{\lambda} \int_{S^*M} \int_{S_{\epsilon,L}} dd^c_{t+i\tau} \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v)d\mu_L(x,v).$$

A key observation of [DF, Lin] is that

(36)
$$\#\{\mathcal{N}_\lambda^{\mathbb{C}} \cap F_{x,v}(S_{\epsilon,L})\} \geq \#\{\mathcal{N}_\lambda^{\mathbb{R}} \cap F_{x,v}(S_{0,L})\},$$

since every real zero is a complex zero. It follows then from Proposition 9 (with $N = \mathcal{N}_\lambda$) that

$$\mathcal{A}_{L,\epsilon}(\frac{1}{\lambda}dd^c \log |\psi_j^{\mathbb{C}}|^2) = \frac{1}{\lambda} \int_{S^*M} \#\{\mathcal{N}_\lambda^{\mathbb{C}} \cap F_{x,v}(S_{\epsilon,L})\} d\mu(x,v)$$

$$\geq \frac{1}{\lambda} \mathcal{H}^{m-1}(\mathcal{N}_{\psi_\lambda}).$$

Hence to obtain an upper bound on $\frac{1}{\lambda}\mathcal{H}^{m-1}(\mathcal{N}_{\psi_\lambda})$ it suffices to prove that there exists $M < \infty$ so that

(37)
$$\mathcal{A}_{L,\epsilon}\left(\frac{1}{\lambda}dd^c \log |\psi_j^{\mathbb{C}}|^2\right) \leq M.$$

To prove (37), we observe that since $dd^c_{t+i\tau} \log |\psi_j^{\mathbb{C}}|^2(\exp_x(t+i\tau)v)$ is a positive $(1,1)$ form on the strip, the integral over S_ϵ is only increased if we integrate against a positive smooth test function $\chi_\epsilon \in C_c^\infty(\mathbb{C})$ which equals one on $S_{\epsilon,L}$ and vanishes off $S_{2\epsilon,L}$. Integrating by parts the dd^c onto χ_ϵ, we have

$$\mathcal{A}_{L,\epsilon}(\frac{1}{\lambda}dd^c \log |\psi_j^{\mathbb{C}}|^2)$$

$$\leq \frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} dd^c_{t+i\tau} \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t+i\tau)v)\chi_\epsilon(t+i\tau)d\mu_L(x,v)$$

$$= \frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} \log |\psi_j^{\mathbb{C}}|^2(\exp_x(t+i\tau)v)dd^c_{t+i\tau}\chi_\epsilon(t+i\tau)d\mu_L(x,v).$$

Now write $\log|x| = \log_+|x| - \log_-|x|$. Here $\log_+|x| = \max\{0, \log|x|\}$ and $\log_-|x| = \max\{0, -\log|x|\}$. Then we need upper bounds for

$$\frac{1}{\lambda} \int_{S^*M} \int_{\mathbb{C}} \log_\pm |\psi_j^{\mathbb{C}}|^2(\exp_x(t+i\tau)v)dd^c_{t+i\tau}\chi_\epsilon(t+i\tau)d\mu_L(x,v).$$

For $\log_+$ the upper bound is an immediate consequence of Proposition 2. For $\log_-$ the bound is subtler: we need to show that $|\varphi_\lambda(z)|$ cannot be too small on too large a set. As we know from Gaussian beams, it is possible that $|\varphi_\lambda(x)| \leq Ce^{-\delta\lambda}$ on sets of almost full measure in the real domain; we need to show that nothing worse can happen.

The map (10) is a diffeomorphism and since $B_\epsilon^* M = \bigcup_{0 \leq \tau \leq \epsilon} S_\tau^* M$ we also have that

$$E : S_{\epsilon,L} \times S^* M \to M_\tau, \quad E(t + i\tau, x, v) = \exp_x(t + i\tau)v$$

is a diffeomorphism for each fixed t. Hence by letting t vary, E is a smooth fibration with fibers given by geodesic arcs. Over a point $\zeta \in M_\tau$ the fiber of the map is a geodesic arc

$$\{(t + i\tau, x, v) : \exp_x(t + i\tau)v = \zeta, \ \tau = \sqrt{\rho}(\zeta)\}.$$

Pushing forward the measure $dd_{t+i\tau}^c \chi_\epsilon(t + i\tau)d\mu_L(x, v)$ under E gives a measure $d\mu$ on M_τ. We claim that

$$(38) \qquad \mu := E_* \, dd_{t+i\tau}^c \chi_\epsilon(t + i\tau)d\mu_L(x, v) = \left(\int_{\gamma_{x,v}} \Delta_{t+i\tau} \chi_\epsilon ds \right) dV_\omega,$$

where dV_ω is the Kähler volume form $\frac{\omega^m}{m!}$ (see §1.1.)

In fact, $d\mu_L$ is equivalent under E to the contact volume form $\alpha \wedge \omega_\rho^{m-1}$ where $\alpha = d^c\sqrt{\rho}$. Hence the claim amounts to saying that the Kähler volume form is $d\tau$ times the contact volume form. In particular it is a smooth (and of course signed) multiple J of the Kähler volume form dV_ω, and we do not need to know the coefficient function J beyond that it is bounded above and below by constants independent of λ. We then have

$$(39) \qquad \int_{S^* M} \int_{\mathbb{C}} \log |\psi_j^{\mathbb{C}}|^2 (\exp_x(t + i\tau)v) dd_{t+i\tau}^c \chi_\epsilon(t + i\tau)d\mu_L(x, v)$$
$$= \int_{M_\tau} \log |\psi_j^{\mathbb{C}}|^2 J dV.$$

To complete the proof of (37) it suffices to prove that the right side is $\geq -C\lambda$ for some $C > 0$.

We use the well-known

Lemma 11. *(Hartog's Lemma; (see [HoI-IV, Theorem 4.1.9]): Let $\{u_j\}$ be a sequence of subharmonic functions in an open set $X \subset \mathbb{R}^m$ which have a*

uniform upper bound on any compact set. Then either $u_j \to -\infty$ uniformly on every compact set, or else there exists a subsequence u_{j_k} which is convergent to some $u \in L^1_{loc}(X)$. Further, $\limsup_n u_n(x) \le u(x)$ with equality almost everywhere. For every compact subset $K \subset X$ and every continuous function f,

$$\limsup_{n \to \infty} \sup_K (u_n - f) \le \sup_K (u - f).$$

In particular, if $f \ge u$ and $\epsilon > 0$, then $u_n \le f + \epsilon$ on K for n large enough.

This Lemma implies the desired lower bound on (39): there exists $C > 0$ so that

$$(40) \qquad \frac{1}{\lambda} \int_{M_\tau} \log |\psi_\lambda^{\mathbb{C}}| J dV \ge -C.$$

For if not, there exists a subsequence of eigenvalues λ_{j_k} so that

$$\frac{1}{\lambda_{j_k}} \int_{M_\tau} \log |\psi_{\lambda_{j_k}}^{\mathbb{C}}| J dV \to -\infty.$$

By Proposition 2, $\{\frac{1}{\lambda_{j_k}} \log |\psi_{\lambda_{j_k}}^{\mathbb{C}}|\}$ has a uniform upper bound. Moreover the sequence does not tend uniformly to $-\infty$ since $||\psi_\lambda^{\mathbb{C}}||_{L^2(M)} = 1$. It follows that a further subsequence tends in L^1 to a limit u and by the dominated convergence theorem the limit of (40) along the sequence equals $\int_{M_\tau} u J dV \ne -\infty$. This contradiction concludes the proof of (40), hence (37), and thus the theorem. $\qquad\square$

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