

Observations on the Hofer distance between closed subsets

MICHAEL USHER

We prove the elementary but surprising fact that the Hofer distance between two closed subsets of a symplectic manifold can be expressed in terms of the restrictions of Hamiltonians to one of the subsets; this helps explain certain energy-capacity inequalities that appeared recently in [3] and [9]. We also build on [15] to obtain new vanishing results for the Hofer distance between subsets, applicable for instance to singular analytic subvarieties of Kähler manifolds.

This note uses rather elementary arguments to deduce some results about the Hofer distance between closed subsets of a symplectic manifold, defined as the infimal Hofer norm of a Hamiltonian diffeomorphism that maps one subset to the other. In the first section we give an alternative formula (Theorem 1.3) for this distance. This formula helps explain some seemingly-unexpectedly-strong versions of energy-capacity inequalities that appeared recently in [3] and [9], and indeed shows that all energy-capacity inequalities can be expressed in a similar strengthened form. The second section contains new results about the rigid locus defined in [15], in particular connecting it to the Poisson bracket in Corollary 2.3. These results are used to expand the class of subsets on whose orbits the Hofer distance is known to vanish identically. Specifically this vanishing is established for all non-Lagrangian, half-dimensional submanifolds (Corollary 2.7) and all analytic subvarieties (including singular ones) in Kähler manifolds (Theorem 2.15).

Except for the notational conventions that we establish in Section 1.1, the two sections are logically independent of each other. Both sections share the common theme of the relation between the behavior of the restriction of a Hamiltonian to a subset of a symplectic manifold to the dynamics of that subset under the Hamiltonian. For instance one obtains as a special case of either Theorem 1.3 or (independently) of Proposition 2.2 that, if a Hamiltonian H vanishes identically on a closed subset A , then the Hofer

distance between A and its image under the Hamiltonian flow of H will always be zero.

1. Restricting the Hamiltonian

There exists a rich history of results in symplectic topology asserting that, in order for a Hamiltonian diffeomorphism ϕ of a symplectic manifold (M, ω) to behave in a certain way with respect to a subset A of M , the Hofer norm $\|\phi\|_H$ of ϕ must exceed some positive lower bound. Here $\|\phi\|_H$ is by definition the infimal value of $\int_0^1 (\max_M H(t, \cdot) - \min_M H(t, \cdot)) dt$ among smooth compactly supported functions $H: [0, 1] \times M \rightarrow \mathbb{R}$ having time-one map ϕ_H^1 equal to ϕ (more details on the definition will be recalled in Section 1.1). Indeed, the original proofs that $\|\cdot\|_H$ is nondegenerate [8],[10] proceed by proving that, for A equal to a closed Darboux ball in M , there is a number $c_A > 0$ such that any Hamiltonian diffeomorphism ϕ such that $\phi(A) \cap A = \emptyset$ must have $\|\phi\|_H \geq c_A$. If (M, ω) is geometrically bounded, [5] establishes a similar bound with A instead equal to a compact Lagrangian submanifold of M , generalizing an earlier result of [14]. Along similar lines, if A is a compact Lagrangian submanifold of a tame symplectic manifold (M, ω) and if U is either an open set intersecting A or a compact Lagrangian submanifold that intersects A transversely (and nontrivially) then there is a number $c_{A,U} > 0$ such that one has the bound $\|\phi\|_H \geq c_{A,U}$ whenever $\phi(A) \cap \bar{U} = \emptyset$. (This is [15, Theorem 4.9]; see also [2, Corollary 3.7], [7, Theorem J], and [4] for results which cover less general situations but have stronger bounds $c_{A,U}$.)

Recently, similar results to some of those above have appeared in [3, Theorem 1.5(ii)] and in [9, Lemma 9, citing [11]], but with a surprising twist.¹ For certain rather specific classes of symplectic manifolds (M, ω) and Lagrangian submanifolds A , for any open set U intersecting A these authors produce a positive constant $c_{A,U}$ which serves as a lower bound not only for the Hofer norm but also for the apparently-smaller quantity

$$(1) \quad \int_0^1 \left(\max_A H(t, \cdot) - \min_A H(t, \cdot) \right) dt$$

whenever the time-one map of $H: [0, 1] \times M \rightarrow \mathbb{R}$ disjoins A from \bar{U} . This appears counterintuitive: at time, say, 0.5 one would expect the values of $H(0.5, \cdot)$ along $\phi_H^{0.5}(A)$ to be more relevant to the question of whether the

¹We might also mention the results [12, Lemma 2.1, Theorem 2.17(vi)] about Lagrangian spectral invariants, which in retrospect could be seen as anticipating this phenomenon.

Hamiltonian isotopy $\{\phi_H^t\}$ generated by H moves A out of \bar{U} than the values of $H(0.5, \cdot)$ along A , yet it is the latter that contributes to (1). The fact that the maximum and minimum in (1) can be taken over A rather than M is consequential: it plays a key role in the proof of the main result of [9] on the C^0 -rigidity of coisotropic submanifolds.

In this section we give a simple explanation for these results which give estimates for (1) instead of only for the Hofer norm: they do not, as might first appear, represent some new mysterious action-at-a-distance phenomenon in symplectic topology; rather, by means of elementary considerations about the relationships between Hamiltonians and their time-one maps we will see that the sorts of Hofer norm bounds described above immediately imply identical bounds on the quantity (1).² In particular all of the bounds described in the first paragraph of this section can be combined with Theorem 1.3 below to yield bounds on (1) in the style of [3],[9].

1.1. Conventions

We now establish some basic notational conventions and definitions. Throughout the paper, for a smooth manifold P (possibly with boundary) we denote by $C_0^\infty(P)$ the set of smooth, compactly supported real-valued functions on P . (So if P is already compact, $C_0^\infty(P) = C^\infty(P)$.)

Let (M, ω) be a symplectic manifold without boundary. If $H \in C_0^\infty([0, 1] \times M)$, for each $t \in [0, 1]$ we let $H_t = H(t, \cdot)$ and let X_{H_t} be the vector field obeying $\omega(\cdot, X_{H_t}) = dH_t$. The Hamiltonian isotopy $\{\phi_H^t\}_{t \in [0, 1]}$ is then characterized by the properties that $\phi_H^0 = 1_M$ and $\frac{d\phi_H^t}{dt} = X_{H_t} \circ \phi_H^t$. As usual we denote by $Ham(M, \omega)$ the group consisting of those diffeomorphisms ϕ such that there exists $H \in C_0^\infty([0, 1] \times M)$ with $\phi_H^1 = \phi$.

For any closed subset $B \subset M$ and any $F \in C_0^\infty(M)$ write

$$\text{osc}_B F = \max_B F - \min_B F$$

The Hofer norm on $Ham(M, \omega)$ is then defined by

$$\|\phi\|_H = \inf \left\{ \int_0^1 \text{osc}_M H_t dt \mid \phi_H^1 = \phi \right\}$$

²This is not to say that the aforementioned results in [3] and [9] can be deduced from our argument together with prior results: even just as bounds on the Hofer norm, their lower bounds $c_{A,U}$ are in some cases larger than those given by other methods.

Given a closed subset $A \subset M$, we let

$$\mathcal{L}(A) = \{\phi(A) \mid \phi \in \text{Ham}(M, \omega)\}$$

denote the orbit of A under the Hamiltonian diffeomorphism group. We may then define $\delta: \mathcal{L}(A) \times \mathcal{L}(A) \rightarrow \mathbb{R}$ by setting, for $A_0, A_1 \in \mathcal{L}(A)$, $\delta(A_0, A_1) = \inf\{\|\phi\|_H \mid \phi(A_0) = A_1\}$, or equivalently (and more suggestively for our coming results)

$$\delta(A_0, A_1) = \inf \left\{ \int_0^1 \text{osc}_M H_t dt \mid \phi_H^1(A_0) = A_1 \right\}.$$

1.2. A new formula for δ

For the rest of this section fix a closed subset $A \subset M$. It is easy to see that the function δ defined above is a pseudometric on $\mathcal{L}(A)$ which is invariant under the action of $\text{Ham}(M, \omega)$. In the case that A is a Lagrangian submanifold the study of this pseudometric dates back at least to [13] and [6], and in the latter paper it is shown that if (M, ω) is geometrically bounded and A is a compact Lagrangian submanifold then δ is nondegenerate. See [15] (and also the following section) for results about the behavior of δ when A may not be Lagrangian.

We first prove the following simple lemma:

Lemma 1.1. *Given $H \in C_0^\infty([0, 1] \times M)$ and a closed subset $A \subset M$ there is $K \in C_0^\infty([0, 1] \times M)$ such that:*

- (i) $\phi_H^t(A) = \phi_K^t(A)$ for all t .
- (ii) $\text{osc}_{\phi_H^t(A)} H_t = \text{osc}_{\phi_K^t(A)} K_t$ for all t .
- (iii) For all $t \in [0, 1]$ there is $a \in A$ such that $K(t, \phi_K^t(a)) = 0$.

Proof. The proof splits into three cases depending on whether A and M are compact.

If A is noncompact then the fact that the support of H is compact implies that $H(t, \cdot)|_{\phi_H^t(A)}$ takes the value 0 for all t , so we can simply take $K = H$.

Assuming from now on that A is compact, choose an arbitrary $a_0 \in A$ and define $f: [0, 1] \rightarrow \mathbb{R}$ by $f(t) = H(t, \phi_H^t(a_0))$. If M is compact then the lemma will hold with $K(t, m) = H(t, m) - f(t)$. If M is not compact then this latter function might not be compactly supported, but since A

is compact we can find $\beta \in C_0^\infty(M)$ such that $\beta = 1$ on a neighborhood of $\cup_t \{t\} \times \phi_H^t(A)$. Then the lemma will hold with $K(t, m) = \beta(m)(H(t, m) - f(t))$, since the Hamiltonian vector fields of H and K coincide along $\cup_t \{t\} \times \phi_H^t(A)$. \square

The following is well-known:

Proposition 1.2. *For $A_0, A_1 \in \mathcal{L}(A)$ we have*

$$\delta(A_0, A_1) = \inf \left\{ \int_0^1 \text{osc}_{\phi_H^t(A_0)} H_t dt \mid \phi_H^1(A_0) = A_1 \right\}.$$

Proof. Choose any $H \in C_0^\infty([0, 1] \times M)$ such that $\phi_H^1(A_0) = A_1$, and let $\epsilon > 0$. Let $K \in C_0^\infty([0, 1] \times M)$ be as in the previous lemma, applied with $A = A_0$. Define $\Phi: [0, 1] \times M \rightarrow [0, 1] \times M$ by $\Phi(t, m) = (t, \phi_H^t(m))$ and let $\Lambda = \Phi([0, 1] \times A_0)$. Choose a smooth function $\chi: [0, 1] \times M \rightarrow [0, 1]$ such that χ is identically equal to 1 on a neighborhood of the compact set $\Lambda \cap \text{supp}(K)$ and such that $\chi(t, m) = 0$ at all $(t, m) \in [0, 1] \times M$ such that $K(t, m) \notin (\min_{\phi_K^t(A_0)} K_t - \epsilon/2, \max_{\phi_K^t(A_0)} K_t + \epsilon/2)$.

Now let $K' = \chi K$. Notice that the condition on χ implies that, for all t ,

$$\max_{\phi_K^t(A_0)} K_t \leq \max_{\phi_K^t(A_0)} K'_t < \max_{\phi_K^t(A_0)} K_t + \epsilon/2,$$

where in the second inequality we have also used the fact that $\max_{\phi_K^t(A_0)} K_t \geq 0$ (which follows from condition (iii) in Lemma 1.1). Similarly, using that $\min_{\phi_K^t(A_0)} K_t \leq 0$, we have $\min_{\phi_K^t(A_0)} K_t - \epsilon/2 < \min_{\phi_K^t(A_0)} K'_t \leq \min_{\phi_K^t(A_0)} K_t$ and hence, for all t ,

$$\text{osc}_M K'_t < \text{osc}_{\phi_K^t(A_0)} K_t + \epsilon = \text{osc}_{\phi_H^t(A_0)} H_t + \epsilon.$$

The fact that K' coincides with K on a neighborhood of Λ readily implies that $\phi_{K'}^t(A_0) = \phi_K^t(A_0)$ for all t , and in particular that $\phi_{K'}^1(A_0) = A_1$. Thus

$$\delta(A_0, A_1) \leq \inf \left\{ \int_0^1 \text{osc}_{\phi_H^t(A_0)} H_t dt \mid \phi_H^1(A_0) = A_1 \right\} + \epsilon.$$

Since ϵ is arbitrary, we obtain the inequality “ \leq ” in the statement of the proposition, while of course the inequality “ \geq ” is trivial. \square

The main result of this section shows that, instead of taking the oscillation over the time-dependent (and H -dependent) closed set $\phi_H^t(A_0)$ as in Proposition 1.2, we can simply take it over A_0 :

Theorem 1.3. *For $A_0, A_1 \in \mathcal{L}(A)$, we have*

$$\delta(A_0, A_1) = \inf \left\{ \int_0^1 \text{osc}_{A_0} H_t dt \mid \phi_H^1(A_0) = A_1 \right\}$$

Proof. The plan of the proof is to show that, for any $H \in C_0^\infty([0, 1] \times M)$, there exists $K \in C_0^\infty([0, 1] \times M)$ having the properties that

$$(2) \quad \phi_K^1 = \phi_H^1 \quad \text{and} \quad \int_0^1 \text{osc}_{\phi_K^t(A_0)} K_t dt = \int_0^1 \text{osc}_{A_0} H_t dt.$$

In view of Proposition 1.2 this will obviously imply the inequality “ \leq ” in the statement of the theorem, while the inequality “ \geq ” just follows from the fact that $\text{osc}_M H_t \geq \text{osc}_{A_0} H_t$.

For a general $G \in C_0^\infty([0, 1] \times M)$, consider the two functions

$$\bar{G}(t, m) = -G(t, \phi_G^t(m)) \quad \widehat{G}(t, m) = -G(1 - t, m)$$

A standard calculation shows that \bar{G} generates the Hamiltonian isotopy $\phi_{\bar{G}}^t = (\phi_G^t)^{-1}$. Meanwhile, \widehat{G} is designed to have the property that a map $\gamma: [0, 1] \rightarrow M$ obeys $\gamma'(t) = X_{G_t}(\gamma(t))$ if and only if the time-reversed map $\widehat{\gamma}(t) = \gamma(1 - t)$ obeys $\widehat{\gamma}'(t) = X_{\widehat{G}_t}(\widehat{\gamma}(t))$. In other words, time-one Hamiltonian flowlines for \widehat{G} are precisely time-reversals of time-one flowlines of G ; at the level of isotopies this yields

$$\phi_{\widehat{G}}^t = \phi_G^{1-t} \circ (\phi_G^1)^{-1}$$

In particular we have

$$\phi_{\bar{G}}^1 = \phi_{\widehat{G}}^1 = (\phi_G^1)^{-1}$$

With this said, given $H: [0, 1] \times M \rightarrow \mathbb{R}$ we now produce the function $K: [0, 1] \times M \rightarrow \mathbb{R}$ promised in the first paragraph of the proof:

$$K = \overline{(\widehat{H})} \quad \text{i.e.,} \quad K(t, m) = -\widehat{H}(t, \phi_{\widehat{H}}^t(m)) = H(1 - t, \phi_H^{1-t}((\phi_H^1)^{-1}(m)))$$

We can quickly verify that the two properties in (2) are satisfied: first of all,

$$\phi_K^1 = \phi_{\overline{(\widehat{H})}}^1 = (\phi_{\widehat{H}}^1)^{-1} = ((\phi_H^1)^{-1})^{-1} = \phi_H^1.$$

Meanwhile, we have $\phi_K^t = (\phi_{\widehat{H}}^t)^{-1}$ and so, for $(t, m) \in [0, 1] \times M$,

$$K(t, \phi_K^t(m)) = -\widehat{H}(t, \phi_{\widehat{H}}^t(\phi_K^t(m))) = -\widehat{H}(t, m) = H(1 - t, m).$$

From this we see immediately that, for all t ,

$$\text{osc}_{\phi_K^t(A_0)} K_t = \text{osc}_{A_0} H_{1-t}$$

and hence

$$\int_0^1 \text{osc}_{\phi_K^t(A_0)} K_t dt = \int_0^1 \text{osc}_{A_0} H_{1-t} dt = \int_0^1 \text{osc}_{A_0} H_t dt,$$

proving the second part of (2) and hence the theorem. □

Remark 1.4. In cases where the Hamiltonian H is time-independent simply setting $K = H$ in the above proof will of course lead to a Hamiltonian obeying (2), in view of the conservation of energy property $H \circ \phi_H^t = H$. In this situation one has $\widehat{H} = \overline{H}$, and so the Hamiltonian produced by our proof is indeed just H . However in the time-dependent case \widehat{H} and \overline{H} will generally be distinct and the Hamiltonian K in the proof will generate a different isotopy from the identity to ϕ_H^1 than does H .

Remark 1.5. Since δ is symmetric and since $\text{osc}_{A_1} H_t = \text{osc}_{A_1} \widehat{H}_{1-t}$, it follows from Theorem 1.3 that we also have

$$\delta(A_0, A_1) = \inf \left\{ \int_0^1 \text{osc}_{A_1} H_t dt \mid \phi_H^1(A_0) = A_1 \right\}$$

One can also prove this directly in the style of the above proof, by setting K equal to (\overline{H}) instead of (\widehat{H}) and observing that one then has $K(t, \phi_K^t(m)) = H(1-t, \phi_H^1(m))$ for all $(t, m) \in [0, 1] \times M$.

To connect this to the sorts of estimates described at the beginning of this section, recall that the *displacement energy* of the closed set A is by definition

$$e(A) = \inf \{ \|\phi\|_H \mid \phi(A) \cap A = \emptyset \}.$$

For another subset $U \subset M$ (presumably intersecting A) we likewise define

$$e(A, U) = \inf \{ \|\phi\|_H \mid \phi(A) \cap \bar{U} = \emptyset \}.$$

As originally formulated, the results described in the first paragraph of this section (and many others like them) are lower bounds for $e(A)$ or $e(A, U)$ for various classes of A and U .

Corollary 1.6. *We have*

$$e(A) = \inf \left\{ \int_0^1 \text{osc}_A H_t dt \mid \phi_H^1(A) \cap A = \emptyset \right\} \quad \text{and}$$

$$e(A, U) = \inf \left\{ \int_0^1 \text{osc}_A H_t dt \mid \phi_H^1(A) \cap \bar{U} = \emptyset \right\}$$

Proof. Since A is assumed to be closed we have by definition $e(A) = e(A, A)$, so the first equation is a special case of the second. For the second, simply note that, as an easy consequence of the definitions,

$$e(A, U) = \inf \{ \delta(A, A') \mid A' \in \mathcal{L}(A), A' \cap \bar{U} = \emptyset \}$$

and apply Theorem 1.3. □

The estimates in [3],[9] that motivated this section were lower bounds for the right-hand side in the above corollary; we thus see that any of the numerous methods for estimating $e(A, U)$ in fact yields a similar estimate for this right-hand side.

2. New properties of the rigid locus

An immediate consequence of Theorem 1.3 is that, for any closed subset $A \subset M$, if a function $H \in C_0^\infty([0, 1] \times M)$ obeys $H|_{[0,1] \times A} = 0$, then $\delta(A, \phi_H^1(A)) = 0$.³ We will obtain below in Proposition 2.2 a strengthening of this result, in preparation for which we now recall some terminology from [15].

Again fixing a closed subset $A \subset M$, we write

$$\bar{\Sigma}_A = \{ \phi \in Ham(M, \omega) \mid \delta(A, \phi(A)) = 0 \}.$$

(The notation refers to the fact that this is the closure of the stabilizer Σ_A of A with respect to the Hofer topology on $Ham(M, \omega)$; in particular $\bar{\Sigma}_A$ is a subgroup of $Ham(M, \omega)$, see [15, Proposition 2.2].) The *rigid locus* of A is then defined to be the set

$$R_A = \bigcap_{\phi \in \bar{\Sigma}_A} \phi^{-1}(A)$$

So obviously $R_A \subset A$ (take $\phi = 1_M$). It is easy to see that if $R_A = A$ then δ is nondegenerate on $\mathcal{L}(A)$. A less obvious fact (originally proven as [15,

³Of course this is not surprising in the special case that A is a coisotropic submanifold, since then the hypothesis implies that $\phi_H^1(A) = A$.

Lemma 4.2(iii)]; this is also a special case of Proposition 2.1 below) is that if $R_A = \emptyset$ then δ vanishes identically on $\mathcal{L}(A)$.

Our main results in this section are strong new restrictions on the structure of the rigid locus R_A (Corollaries 2.3 and 2.6) which are then applied in Corollary 2.7 and Theorem 2.15 to obtain new classes of subsets A for which it always holds that $R_A = \emptyset$ (implying that in these cases the pseudometric δ vanishes identically).

For any open subset $U \subset M$ let Ham_U denote the subgroup of $Ham(M, \omega)$ consisting of Hamiltonian diffeomorphisms generated by (extensions by zero of) Hamiltonians $H \in C_0^\infty([0, 1] \times U)$.

Proposition 2.1. *For any closed set $A \subset M$ we have*

$$Ham_{M \setminus R_A} \subset \bar{\Sigma}_A.$$

Proof. The proof is very similar to that of [15, Lemma 4.2(iii)], which concerns the case that $R_A = \emptyset$. Given $x \in M \setminus R_A$ we may find $\psi_x \in \bar{\Sigma}_A$ such that $\psi_x(x) \notin A$; since A is closed we can then find a neighborhood U_x of x such that $\psi_x(U_x) \cap A = \emptyset$. Then

$$\psi_x Ham_{U_x} \psi_x^{-1} = Ham_{\psi_x(U_x)} \subset \bar{\Sigma}_A$$

(indeed every element of $Ham_{\psi_x(U_x)}$ preserves A). So since $\bar{\Sigma}_A$ is a subgroup of $Ham(M, \omega)$ and $\psi_x \in \bar{\Sigma}_A$ it follows that $Ham_{U_x} \leq \bar{\Sigma}_A$.

We have thus found an open cover $\{U_x\}_{x \in M \setminus R_A}$ of $M \setminus R_A$ with the property that each Ham_{U_x} is contained in $\bar{\Sigma}_A$. But the fragmentation lemma of [1, III.3.2] (applied to the symplectic manifold $M \setminus R_A$, which is an open subset of M) asserts that $Ham_{M \setminus R_A}$ is generated by $\cup_x Ham_{U_x}$, so that $Ham_{M \setminus R_A} \subset \bar{\Sigma}_A$. □

The following shows that a Hamiltonian which only vanishes on R_A , not necessarily on all of A , continues to have the property that its flow sends A to sets which lie a distance zero away from A .

Proposition 2.2. *Suppose that $H \in C_0^\infty([0, 1] \times M)$ has $H|_{[0,1] \times R_A} = 0$. Then $\phi_H^s \in \bar{\Sigma}_A$ for all $s \in [0, 1]$.*

Proof. Define $H^s(t, m) = sH(st, m)$ for $s \in [0, 1]$. Then $\phi_{H^s}^1 = \phi_H^s$, so since $H^s|_{[0,1] \times R_A} = 0$ whenever $H|_{[0,1] \times R_A} = 0$ it suffices to prove the result for $s = 1$.

So assume that $H|_{[0,1] \times R_A} = 0$ and for any natural number n let $f_n: \mathbb{R} \rightarrow \mathbb{R}$ be a smooth, nondecreasing function such that $f_n(s) = s$ for $|s| \geq \frac{1}{n}$ and

$f_n(s) = 0$ for $|s| < \frac{1}{2n}$. Then $\|f_n \circ H - H\|_{C^0} \leq \frac{1}{n}$, and so $\phi_{f_n \circ H}^1 \rightarrow \phi_H^1$ as $n \rightarrow \infty$ with respect to the Hofer topology on $Ham(M, \omega)$. But $f_n \circ H$ vanishes on the neighborhood $\{|H| < \frac{1}{2n}\}$ of $[0, 1] \times R_A$ and has support contained in the (compact) support of H , so $\phi_{f_n \circ H}^1 \in Ham_{M \setminus R_A}$. Thus by Proposition 2.1, $\phi_{f_n \circ H}^1 \in \bar{\Sigma}_A$ for all n . But $\bar{\Sigma}_A$ is closed in the Hofer topology, so it follows that $\phi_H^1 \in \bar{\Sigma}_A$ also. \square

For the rest of the paper we will focus on autonomous Hamiltonians $H \in C_0^\infty(M)$. We continue to denote by ϕ_H^t the Hamiltonian flow of the function on $[0, 1] \times M$ defined by $(t, m) \mapsto H(m)$.

For a general closed subset $B \subset M$ we denote

$$I_B = \{H \in C_0^\infty(M) \mid H|_B = 0\}.$$

Corollary 2.3. *Letting $\{F, G\} = \omega(X_F, X_G)$ denote the Poisson bracket, the subset $I_{R_A} \subset C_0^\infty(M)$ is closed under $\{\cdot, \cdot\}$.*

Proof. Let $F, G \in I_{R_A}$. It follows immediately from the definition of R_A (and the fact that $\bar{\Sigma}_A$ is a subgroup of $Ham(M, \omega)$) that R_A is preserved by all elements of $\bar{\Sigma}_A$, so since Proposition 2.2 asserts that $\phi_F^t \in \bar{\Sigma}_A$ for all t , we have $\phi_F^t(R_A) = R_A$ for all t . So the fact that $G \in I_{R_A}$ implies that $G \circ \phi_F^t$ vanishes identically on R_A for all t . Thus for $x \in R_A$ we have

$$\{F, G\}(x) = (dG)_x(X_F) = \left. \frac{d}{dt} G(\phi_F^t(x)) \right|_{t=0} = 0,$$

i.e. $\{F, G\} \in I_{R_A}$. \square

Remark 2.4. Corollary 2.3 imposes rather strong restrictions on the possible geometry of the rigid locus R_A of any closed subset. It is a standard (and easily checked) fact that if $B \subset M$ is a submanifold then I_B is closed under $\{\cdot, \cdot\}$ if and only if B is coisotropic. Thus if the rigid locus is a submanifold then it is coisotropic.

Also we recover the fact ([15, Corollary 4.5]) that, if $A \subset M$ is a submanifold, δ can be nondegenerate on $\mathcal{L}(A)$ only if A is coisotropic: indeed if δ were nondegenerate we would have $R_A = A$, and as just noted, given that $R_A = A$ is a submanifold R_A is coisotropic.

Corollary 2.5. *Let $x \in R_A$, and suppose that $F_1, \dots, F_k \in I_{R_A}$ have the property that $(dF_1)_x, \dots, (dF_k)_x \in T_x^*M$ are linearly independent. Then the*

map

$$\begin{aligned} \psi: \mathbb{R}^k &\rightarrow M \\ (a_1, \dots, a_k) &\mapsto \phi_{\sum a_i F_i}^1(x) \end{aligned}$$

has image contained in R_A , and restricts to a sufficiently small ball around the origin as an embedding.

Proof. The linearization of ψ at $\vec{0} \in \mathbb{R}^k$ sends the standard basis vectors e_1, \dots, e_k to $(X_{F_1})_x, \dots, (X_{F_k})_x$, and these are linearly independent by the assumption that $(dF_1)_x, \dots, (dF_k)_x \in T_x^*M$ are linearly independent. Thus the restriction of ψ to a suitably small neighborhood of $\vec{0}$ is an immersion, and its restriction to a smaller neighborhood is an embedding.

Because each function $\sum a_i F_i$ belongs to I_{R_A} , by Proposition 2.2 we have $\phi_{\sum a_i F_i}^1 \in \bar{\Sigma}_A$ for each \vec{a} . Since R_A is preserved by the action of any element of $\bar{\Sigma}_A$, and since $x \in R_A$, for each $\vec{a} \in \mathbb{R}^k$ it follows that $\psi(\vec{a}) = \phi_{\sum a_i F_i}^1(x) \in R_A$. \square

The following resolves a question that was raised in [15, Section 4.2].

Corollary 2.6. *Let $A \subset M$ be any closed subset such that $R_A \neq \emptyset$ and suppose that $N \subset M$ is any submanifold which is closed as a subset. If $\dim N < \frac{1}{2} \dim M$ then N does not contain R_A , while if N is connected and $\dim N = \frac{1}{2} \dim M$ then N does not properly contain R_A .*

Proof. Suppose to the contrary that we have $R_A \subset N$ where N is as in the statement of the corollary. Let $k = \dim M - \dim N$, and choose any $x \in R_A$. We can then obtain functions F_1, \dots, F_k as in Corollary 2.5 by taking a coordinate chart around x in which N appears as $\{\vec{0}\} \times \mathbb{R}^{\dim M - k}$ and then multiplying the first k coordinate functions by a cutoff function which is equal to 1 on a small neighborhood of x . Hence Corollary 2.5 gives an embedding $B^k(\delta) \hookrightarrow R_A \subset N$ of a small k -dimensional ball $B^k(\delta)$, with image containing x .

If $\dim N < \frac{1}{2} \dim M$ this immediately gives a contradiction since in this case $k > \dim N$ but we have just embedded a k -dimensional ball into N . In the remaining case that $\dim N = \frac{1}{2} \dim M$ (so $\dim N = k$) the k -dimensional ball that we have embedded into $R_A \subset N$ necessarily contains a neighborhood of x in N . Since $x \in R_A$ was chosen arbitrarily this proves that R_A is open in N . But as an immediate consequence of its definition, R_A is also closed. So since N is assumed connected and by hypothesis $R_A \neq \emptyset$, it must be that $R_A = N$. \square

Corollary 2.7. *Let $A \subset M$ be a submanifold of dimension $\frac{1}{2} \dim M$ which is connected and closed as a subset. Then the pseudometric δ on $\mathcal{L}(A)$ either vanishes identically or is nondegenerate. In particular if A is not Lagrangian then δ vanishes identically.*

Proof. Taking $N = A$ in Corollary 2.6, since one always has $R_A \subset A$ we see that the hypothesis implies that either $R_A = A$ or $R_A = \emptyset$, *i.e.* (by [15, Lemma 4.2]) either δ is nondegenerate or δ vanishes identically. If A is not Lagrangian (equivalently, not coisotropic) then the first alternative cannot hold (by [15, Corollary 4.5], or Corollary 2.3 above). \square

Remark 2.8. As mentioned earlier, Chekanov showed in [6] that if (M, ω) is geometrically bounded and A is a compact Lagrangian submanifold then δ is nondegenerate. The same paper contains an example (attributed to Sikorav) of a compact Lagrangian submanifold of a non-geometrically-bounded symplectic manifold for which δ vanishes identically.

Remark 2.9. For submanifolds of codimension strictly between 1 and $\frac{1}{2} \dim M$ it is possible for δ to neither be nondegenerate nor vanish identically, as explained in [15, Remark 1.5].

Remark 2.10. Corollary 2.6 also evidently implies (again taking $N = A$) that if $A \subset M$ is a connected closed submanifold of dimension at most $\frac{1}{2} \dim M$ and if $B \subset A$ is any proper closed subset then $R_B = \emptyset$ and so δ vanishes identically on $\mathcal{L}(B)$.

2.1. Subvarieties

In this subsection we prove Theorem 2.15, asserting that δ vanishes identically on $\mathcal{L}(A)$ whenever A is a (possibly singular) complex analytic subvariety of a Kähler manifold. Accordingly let (M, ω, J) be a Kähler manifold (so ω is a symplectic form and J is an ω -compatible integrable almost complex structure). We then obtain a Riemannian metric $g: TM \times_M TM \rightarrow \mathbb{R}$ defined by $g(v, w) = \omega(v, Jw) = \omega(-Jv, w)$. Define maps $\theta_\omega, \theta_g: TM \rightarrow T^*M$ by $\theta_\omega(v) = \omega(v, \cdot)$ and likewise $\theta_g(v) = g(v, \cdot)$. Thus $\theta_g = -\theta_\omega \circ J$. Since ω is nondegenerate, θ_ω and θ_g are invertible, and we see that $\theta_g^{-1} = J \circ \theta_\omega^{-1}$. Define the dual metric $g^*: T^*M \times_M T^*M \rightarrow \mathbb{R}$ by $g^*(\alpha, \beta) = g(\theta_g^{-1}(\alpha), \theta_g^{-1}(\beta))$. Of course by the definition of θ_g we have $g^*(\alpha, \beta) = \alpha(\theta_g^{-1}(\beta))$.

Proposition 2.11. *Let $U \subset M$ be an open subset and let $f: U \rightarrow \mathbb{C}$ be a holomorphic function, written as $f = u + iv$ where $u, v: U \rightarrow \mathbb{R}$. Then the Poisson bracket of u and v is given everywhere on U by*

$$\{u, v\} = g^*(du, du) = g^*(dv, dv)$$

Proof. In our present notation the Hamiltonian vector field of u is given by $X_u = -\theta_\omega^{-1}(du)$. So

$$\{u, v\} = \omega(X_u, X_v) = dv(X_u) = -dv(\theta_\omega^{-1}(du)) = dv(J\theta_g^{-1}(du))$$

But the Cauchy–Riemann equation for the holomorphic function f amounts to the statement that $dv \circ J = du$, so the above gives $\{u, v\} = du(\theta_g^{-1}(du)) = g^*(du, du)$.

Meanwhile since J is an isometry with respect to g , the adjoint of J is an isometry with respect to g^* , and so the fact that $dv \circ J = du$ implies that $g^*(dv, dv) = g^*(du, du)$. □

Definition 2.12. Let A be a closed subset of the Kähler manifold (M, ω, J) and let $x \in X$. A **holomorphic reducing chart** (U, V, ψ, f) for A around x consists of the following data:

- A connected open neighborhood $U \subset M$ of x having compact closure.
- An open set $V \subset \mathbb{C}^n$, and a holomorphic chart $\psi: V \rightarrow M$ such that $\bar{U} \subset \psi(V)$.
- A holomorphic function $f: V \rightarrow \mathbb{C}$ such that $f|_{\psi^{-1}(R_A)} = 0$ where R_A is the rigid locus of A

Proposition 2.13. *If (U, V, ψ, f) is a holomorphic reducing chart for A around x then there is an open subset $V' \subset V$ which contains $\psi^{-1}(\bar{U})$ such that, for each $j = 1, \dots, n$, $(U, V', \psi|_{V'}, \frac{\partial f}{\partial z_j})$ is also a holomorphic reducing chart for A around x .*

Proof. Let $\beta: M \rightarrow [0, 1]$ be a smooth function which is identically equal to 1 on some open set U' containing \bar{U} but whose support is compact and contained in $\psi(V)$. Define $F: M \rightarrow \mathbb{C}$ to be equal to $\beta \cdot (f \circ \psi^{-1})$ on $\psi(V)$ and to 0 on $M \setminus \psi(V)$, and let u and v be, respectively, the real and imaginary parts of F . Since $f|_{\psi^{-1}(R_A)} = 0$, we have $u|_{R_A} = v|_{R_A} = 0$. So by Corollary 2.3, $\{u, v\}|_{R_A} = 0$.

Now the restriction of $F = u + iv$ to U' is holomorphic, so by Proposition 2.11 we have $\{u, v\}|_{U'} = g^*(du, du) = g^*(dv, dv)$. So since $\{u, v\}|_{R_A} = 0$

we obtain $dF = 0$ at each point of $R_A \cap U'$. Letting $V' = \psi^{-1}(U')$, we have $F|_{V'} = f \circ (\psi|_{V'})^{-1}$, so we deduce that $df = 0$ at each point of $(\psi|_{V'})^{-1}(R_A)$. The fact that each $(U, V', \psi|_{V'}, \frac{\partial f}{\partial z_j})$ is a holomorphic reducing chart then follows immediately from the definition. \square

Corollary 2.14. *If (U, V, ψ, f) is a holomorphic reducing chart for A around x such that f is not identically zero on $\psi^{-1}(U)$, then $x \notin R_A$.*

Proof. By Proposition 2.13 and induction, for each multi-index α we obtain a holomorphic reducing chart for A around x of the form $(U, V_\alpha, \psi|_{V_\alpha}, \frac{\partial^{|\alpha|} f}{\partial z^\alpha})$ where V_α is a neighborhood of \bar{U} . If it were the case that $x \in R_A$ we would then obtain that f vanishes to infinite order at $\psi^{-1}(x)$. But since $\psi^{-1}(U)$ is connected and f is holomorphic this implies that $f|_{\psi^{-1}(U)}$ is identically zero. \square

Theorem 2.15. *Let A be a complex analytic subvariety of positive codimension in a Kähler manifold (M, ω, J) , or more generally any closed subset of a complex analytic subvariety of positive codimension. Then $R_A = \emptyset$, and so δ vanishes identically on $\mathcal{L}(A)$.*

Proof. By definition, M is covered by the images of holomorphic charts $\psi_\alpha: V_\alpha \rightarrow M$ each having the property that $\psi_\alpha^{-1}(A)$ is contained in the zero locus of some holomorphic function $f_\alpha: V_\alpha \rightarrow \mathbb{C}$ that is not identically zero on any nonempty open subset. Since $R_A \subset A$, then, if U is any connected open subset whose closure is compact and contained in $\psi_\alpha(V_\alpha)$ the tuple $(U, V_\alpha, \psi_\alpha, f_\alpha)$ is a holomorphic reducing chart for A around any point of U . Such a U can be found for any $x \in \psi_\alpha(V_\alpha)$, so Corollary 2.14 shows that $\psi_\alpha(V_\alpha) \cap R_A = \emptyset$. So since the various $\psi_\alpha(V_\alpha)$ cover M , $R_A = \emptyset$. \square

Remark 2.16. As Remark 2.10 and the proof of Theorem 2.15 illustrate, arguments that show that a point x does not lie in the rigid locus of some subset A often also show that $x \notin R_B$ whenever B is a closed subset of A . It seems natural to expect that one always has the inclusion $R_B \subset R_A$ whenever $B \subset A$, but I do not know a proof of this statement.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA
200 D. W. BROOKS DR., ATHENS, GA 30602, USA
E-mail address: `usher@math.uga.edu`

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