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Pullback of regular singular stratified bundles and restriction to curves

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A stratified bundle on a smooth variety X is a vector bundle which is a $\mathscr{D}_X$-module. We show that regular singularity of stratified bundles on smooth varieties in positive characteristic is preserved by pullback and that regular singularity can be checked on curves, if the ground field is large enough.

1. Introduction

If X is a smooth complex variety, then it is proved in [2] that a vector bundle with flat connection (E, ∇) on X is regular singular if and only $\varphi^*(E, \nabla)$ is regular singular for all maps $\varphi : C \to X$ with C a smooth complex curve.

In this short note we analyze an analogous statement for vector bundles with $\mathscr{D}_{X/k}$-action on smooth k-varieties, where k is an algebraically closed field of positive characteristic $p > 0$ and $\mathscr{D}_{X/k}$ the sheaf of differential operators of X relative to k. Vector bundles with such an action are called stratified bundles, see [3]. A notion of regular singularity for stratified bundles was defined and studied in *loc. cit.* under the assumption of the existence of a good compactification, and in [9] in general. We recall this definition in Section 2.

The first result of this article is:

Theorem 1.1. *Let k be an* uncountable *algebraically closed field of characteristic $p > 0$, X a smooth, separated, finite type k-scheme and E a stratified bundle on X. Then E is regular singular if and only if φ^*E is regular singular for every k-morphism $\varphi : C \to X$ with C smooth k-curve.*

In [9, Sec. 8] it is proved that Theorem 1.1 holds without the uncountability condition for stratified bundles with finite monodromy. This relies on work of Kerz, Schmidt and Wiesend, [8]. For stratified bundles with arbitrary

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monodromy, the author does not know at present whether the uncountability condition of k in Theorem 1.1 is necessary or not. For this reason we have to content ourselves with the following general criterion for regular singularity, which easily follows from Theorem 1.1.

Corollary 1.2. *If X is a smooth, separated, finite type k-scheme, and E a stratified bundle on X, then E is regular singular if and only if for every algebraic closure k' of a finitely generated extension of k, and every k'-morphism $\varphi : C \to X_{k'}$ with C a smooth k'-curve, the stratified bundle $\varphi^*(E \otimes k')$ on C is regular singular.*

In the course of the proof we establish a general result on pullbacks, which is of independent interest.

Theorem 1.3. *Let k be an algebraically closed field of characteristic $p > 0$ and $f : Y \to X$ a morphism of smooth, separated, finite type k-schemes. If E is a regular singular stratified bundle on X, then f^*E is a regular singular stratified bundle on Y.*

In other words, if $\mathrm{Strat}(X)$ *denotes the category of stratified bundles on X and* $\mathrm{Strat}^{\mathrm{rs}}(X)$ *its full subcategory with objects the regular singular stratified bundles, then the pullback functor* $f^* : \mathrm{Strat}(X) \to \mathrm{Strat}(Y)$ *restricts to a functor* $f^* : \mathrm{Strat}^{\mathrm{rs}}(X) \to \mathrm{Strat}^{\mathrm{rs}}(Y)$.

The difficulty in proving this theorem is the unavailability of resolution of singularities. Our proof relies on a desingularization result kindly communicated to H. Esnault and the author by O. Gabber. In [10], Theorem 1.3 was only shown in the case that f is dominant.

We conclude the introduction with a brief outline of the article. In Section 2 we recall the definition of regular singularity of a stratified bundle via good partial compactifications. In Section 3 we prove Theorem 1.3, and in Section 4 we establish Theorem 1.1.

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2. Regular singular stratified bundles

Let k denote an algebraically closed field and X a smooth k-variety, i.e. a smooth, separated, finite type k-scheme. If k has characteristic 0, then giving a flat connection ∇ on a vector bundle E on X is equivalent to giving E a left-$\mathscr{D}_{X/k}$-action which is compatible with its $\mathcal{O}_X$-structure. Here $\mathscr{D}_{X/k}$ is the sheaf of rings of differential operators of X relative to k ([5, §16]). If k has positive characteristic, the sheaf of rings $\mathscr{D}_{X/k}$ is still defined, but a flat connection does not necessarily give rise to a $\mathscr{D}_{X/k}$-module structure on E.

Definition 2.1. If X is a smooth, separated, finite type k-scheme, then a *stratified bundle E on X* is a left-$\mathscr{D}_{X/k}$-module E which is coherent with respect to the induced $\mathcal{O}_X$-structure. The category of such objects is denoted by $\mathrm{Strat}(X)$.

The usage of the word "bundle" is justified as a stratified bundle is automatically locally free as an $\mathcal{O}_X$-module. See, e.g., [1, 2.17]. The analogous statement for flat connections is not true in positive characteristic. The notion of a stratification goes back to [6] and a vector bundle on a *smooth* k-scheme equipped with a stratification relative to k in the sense of *loc. cit.* is a stratified bundle in the sense of Definition 2.1, and vice versa.

To define regular singularity of a stratified bundle, we introduce some notation.

Definition 2.2.

- If $\overline{X}$ is a smooth, separated, finite type k-scheme and $X \subseteq \overline{X}$ an open subscheme such that $\overline{X} \setminus X$ is the support of a strict normal crossings divisor, then the pair $(X, \overline{X})$ is called *good partial compactification of X*. If in addition $\overline{X}$ is proper over k, then $(X, \overline{X})$ is called *good compactification of X*.
- If $(Y, \overline{Y})$ and $(X, \overline{X})$ are two good partial compactifications, then a *morphism of good partial compactifications* is a morphism $\bar{f} : \overline{Y} \to \overline{X}$ such that $\bar{f}(Y) \subseteq X$.
- If $(Y, \overline{Y})$, $(X, \overline{X})$ are good partial compactifications and if $f : Y \to X$ is a morphism, then we say that f *extends to a morphism of good partial compactifications* if there exits a morphism of good partial compactifications $\bar{f} : (Y, \overline{Y}) \to (X, \overline{X})$, such that $f = \bar{f}|_Y$.

Let X be a smooth, separated, finite type k-scheme. If k has characteristic 0, then there exists a good compactification $(X, \overline{X})$ according to Hironaka's theorem on resolution of singularities. By definition, a vector bundle with flat connection (E, ∇) on X is regular singular, if for some (and equivalently for any) good compactification $(X, \overline{X})$, there exists a torsion free, coherent $\mathcal{O}_{\overline{X}}$-module $\overline{E}$ extending E and a logarithmic connection $\overline{\nabla} : \overline{E} \to \overline{E} \otimes_{\mathcal{O}_{\overline{X}}} \Omega^1_{\overline{X}/k}(\log \overline{X} \setminus X)$ extending ∇.

If k has positive characteristic then it is unknown whether every smooth X admits a good compactification. We work with *all* good *partial* compactifications instead.

Definition 2.3. Let k be an algebraically closed field of positive characteristic and X a smooth, separated, finite type k-scheme.

(a) If E is a stratified bundle on X and $(X, \overline{X})$ a good partial compactification, then E is called $(X, \overline{X})$-*regular singular* if there exists a $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$-module $\overline{E}$, which is coherent and torsion free as an $\mathcal{O}_{\overline{X}}$-module, such that $E \cong \overline{E}|_X$ as stratified bundles. Here $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ is the sheaf of differential operators with logarithmic poles along the underlying normal crossings divisor $\overline{X} \setminus X$, as defined in [3, Sec. 3]; see also Remark 2.4.

(b) A stratified bundle E is called *regular singular* if E is $(X, \overline{X})$-regular singular for all good partial compactifications $(X, \overline{X})$. We write $\mathrm{Strat}^{\mathrm{rs}}(X)$ for the full subcategory of $\mathrm{Strat}(X)$ with objects the regular singular stratified bundles.

Remark 2.4. A good partial compactification $(X, \overline{X})$ gives rise to a logarithmic structure on $\overline{X}$ ([7]), and the associated log-scheme is a log-scheme over $\operatorname{Spec} k$ equipped with its trivial log-structure. Associated to this morphism of log-schemes is a sheaf of logarithmic differential operators $\mathscr{D}_{(X,\overline{X})/k}$, which agrees with the sheaf of rings $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ from [3, Sec. 3]. In local coordinates, if $x \in \overline{X}$ is a closed point and $x_1, \dots, x_n$ étale coordinates around x such that in a neighborhood of x the normal crossings divisor $\overline{X} \setminus X$ is defined by $x_1 \cdot \ldots \cdot x_r = 0$ for some $r \leq n$, then $\mathscr{D}_{\overline{X}/k}(\log \overline{X} \setminus X)$ is spanned by the operators

$$x_1^s \partial_{x_1}^{(s)}, \dots, x_r^s \partial_{x_r}^{(s)}, \partial_{x_{r+1}}^{(s)}, \dots, \partial_{x_n}^{(s)}, s \in \mathbb{Z}_{\geq 0}$$

where $\partial_{x_i}^{(s)}$ is the differential operator such that

$$\partial_{x_i}^{(s)}(x_j^t) = \begin{cases} 0 & i \neq j \\ \binom{t}{s} x_j^{t-s} & i = j. \end{cases}$$

We refer to [9, Sec. 3] for more details.

The notion of regular singularity for stratified bundles is studied in [3, Sec. 3] for smooth varieties X which admit a good compactification and in [9] in general.

We conclude this section by recalling the following fact about regular singularity, which we will use repeatedly in the sequel.

Proposition 2.5. *Let E be a stratified bundle on a smooth, separated, finite type k-scheme X.*

(a) *If $(X, \overline{X})$ is a good partial compactification then E is $(X, \overline{X})$-regular singular if and only if there exists an open subset $\overline{U} \subseteq \overline{X}$ with*

$$\operatorname{codim}_{\overline{X}} \left(\overline{X} \setminus \left(X \cup \overline{U}\right)\right) \geq 2,$$

such that $E|_{\overline{U} \cap X}$ is $(X \cap \overline{U}, \overline{U})$-regular singular.

(b) *If there exists a dense open subset $U \subseteq X$ such that $E|_U$ is regular singular, then E is regular singular.*

Proof. (a) This is [9, Prop. 4.3].

(b) Assume that $E|_U$ is regular singular. It follows from the first part of this proposition that all we have to show is that for any good partial compactification $(X, \overline{X})$ there exists an open subset $\overline{U} \subseteq \overline{X}$ containing all generic points of $\overline{X} \setminus X$, such that $(U, \overline{U})$ is a good partial compactification. Write $\eta_1, \ldots, \eta_d \in \overline{X}$ for the codimension 1 points not contained in X. Let $\overline{U}_i'$ be an open neighborhood of η_i and Z_i the closure of $(\overline{U}_i' \cap X) \setminus U$ in $\overline{X}$. Defining $\overline{U}_i := (U \cup \overline{U}_i') \setminus Z_i$, the open subset $\overline{U} := \bigcup_{i=1}^d \overline{U}_i \subseteq \overline{X}$ does the job (note that $Z_i \cap U = \emptyset$).

□

Remark 2.6. Once we have proved Theorem 1.3, we will also know that if E is a regular singular stratified bundle on X then E is regular singular when restricted to any open subset.

3. Pullback of regular singular stratified bundles

In this section we construct the pullback functor for regular singular stratified bundles. We first recall a basic fact, which is obvious from the perspective of log-schemes.

Proposition 3.1 ([9, Prop. 4.4]). *Let k be an algebraically closed field of positive characteristic and $\bar{f} : (Y, \overline{Y}) \to (X, \overline{X})$ a morphism of good partial compactifications over k (Definition 2.3). Write $f := \bar{f}|_Y : Y \to X$. If E is an $(X, \overline{X})$-regular singular stratified bundle on X, then f^*E is $(Y, \overline{Y})$-regular singular.*

Next, we show that in order to prove Theorem 1.3, it suffices to study dominant morphisms and closed immersions separately.

Proposition 3.2. *Theorem 1.3 is true, if and only if it is true for all closed immersions and all dominant morphisms.*

Proof. Without loss of generality we may assume that X and Y are connected. Let $f : Y \to X$ be as in Theorem 1.3, and let $i : Z \hookrightarrow X$ be the closed immersion given by the scheme theoretic image of f. Since Y is reduced, so is Z. We factor f as $Y \xrightarrow{g} Z \xrightarrow{i} X$. Note that g is dominant. Since Z is reduced, there exists an open subscheme $U \subseteq X$, such that $U \cap Z$ is regular. Define $V := f^{-1}(U)$, and consider the sequence of maps

$$V \xrightarrow{g|_V} Z \cap U \xrightarrow{i|_{Z\cap U}} U \xrightarrow{j} X$$

where $j : U \hookrightarrow X$ is the open immersion. The two outer maps are dominant, the middle map is a closed immersion and all four varieties are regular. We apply the assumption of this proposition from right to left.

Let E be a regular singular stratified bundle on X. By assumption $E|_U$ is regular singular on U, then $(i|_{Z\cap U})^*E|_U$ is regular singular on $Z \cap U$ and finally $g|_V^*(i_{Z\cap U})^*E|_U = f|_V^*E|_U$ is regular singular on V. According to Proposition 2.5 this means that f^*E is regular singular. $\square$

From this proposition together with Proposition 3.1 we see directly that to prove Theorem 1.3, it suffices to prove the following statement.

Proposition 3.3. *Let k be an algebraically closed field and $f : Y \to X$ a morphism of smooth, separated, finite type k-schemes.* Assume that f is either dominant or a closed immersion.

If $(Y, \overline{Y})$ is a good partial compactification, then there exist

- *an open subset $\overline{V} \subseteq \overline{Y}$ containing all generic points of $\overline{Y} \setminus Y$, and*
- *a good partial compactification $(X, \overline{X})$, such that f induces a morphism of good partial compactifications*

$$\bar{f} : (\overline{V} \cap Y, \overline{V}) \to (X, \overline{X}).$$

The remainder of this section is devoted to the proof of Proposition 3.3. We first treat the dominant case (Lemma 3.4), then the case of a closed immersion (Proposition 3.9).

Lemma 3.4. *Proposition 3.3 is true for dominant morphisms $f : Y \to X$.*

Proof. This is essentially [11, Ex. 8.3.16]. Without loss of generality, we may assume X and Y to be irreducible. If $(Y, \overline{Y})$ is a good partial compactification, we may assume that $\overline{Y} \setminus Y$ is a smooth divisor, say with generic point η. If X' is a normal compactification of X, then after removing a closed subset of codimension ≥ 2 from $\overline{Y}$, f extends to a morphism $f' : \overline{Y} \to X'$. If $f'(\eta) \in X$ there is nothing to do; we may take $\overline{X} = X$. Otherwise, [11, Ex. 8.3.16] tells us that there is a blow-up $X'' \to X'$ of X' in $\overline{\{f'(\eta)\}}$ such that f' extends to a map $f'' : \overline{Y} \to X''$ such that $f''(\eta)$ is a normal codimension 1 point of X''. We define $\overline{X}$ to be a suitable neighborhood of $f''(\eta)$ to finish the proof. $\square$

Let k be an algebraically closed field and X a normal, irreducible, separated, finite type k-scheme. We write $k(X)$ for the function field of X. We recall a few basic definitions:

Definition 3.5. Let v be a discrete valuation on $k(X)$.

- We write $\mathcal{O}_v \subseteq k(X)$ for its valuation ring, $\mathfrak{m}_v$ for the maximal ideal of $\mathcal{O}_v$ and $k(v)$ for its residue field.
- If X' is a model of $k(X)$, then a point $x \in X'$ is called *center of* v, if $\mathcal{O}_{X',x} \subseteq \mathcal{O}_v \subseteq k(X)$, and $\mathfrak{m}_v \cap \mathcal{O}_{X',x} = \mathfrak{m}_x$.

- v is called *geometric* if there exists a model X' of $k(X)$ such that v has a center $\xi \in X'$ which is a normal codimension 1 point. In this case $\mathcal{O}_v = \mathcal{O}_{X',\xi}$.

Remark 3.6. Recall that if X' is separated over k, then v has at most one center on X' and if X' is proper, then v has precisely one center on X'.

Proposition 3.7 ([11, Ch. 8, Thm. 3.26], [11, Ch.8 , Ex. 3.14]). *Let X be a normal, irreducible, separated, finite type k-scheme and v a discrete valuation on $k(X)$.*

(a) *We have the inequality*

$$\operatorname{trdeg}_k k(v) \leq \dim X - 1 \tag{1}$$

where $k(v)$ is the residue field of $\mathcal{O}_v$.

(b) *The discrete valuation v is geometric if and only if equality holds in* (1).

(c) *Let X' be a normal, proper compactification of X and x_0 the center of v on X'. If $k(v)/k(x_0)$ is finitely generated, we can make* (b) *more precise:*

Define $X'_0 := X'$. Inductively define

$$\varphi_n : X'_n \to X'_{n-1}$$

as the blow-up of X'_{n-1} in the reduced closed subscheme defined by $\overline{\{x_{n-1}\}}$, where x_{n-1} is the center of v on X'_{n-1}. If (1) *is an equality for v, then for large n, φ_n is an isomorphism, i.e. for large n, x_n is a codimension* 1 *point of X'_n.*

Now we prove Proposition 3.3 in the case where f is a closed immersion.

Lemma 3.8. *Let $i : Y \hookrightarrow X$ be a closed immersion of normal, irreducible, separated, finite type k-schemes, and let v be a discrete geometric valuation of $k(Y)$ with center y on Y such that $k(v)/k(y)$ is finitely generated. Write $X_0 := X$, $Y_0 := Y$, $y_0 := y$, and inductively define*

$$X_n \to X_{n-1}$$

as the blow-up of X_{n-1} in $\overline{\{y_{n-1}\}}$, Y_n as the proper transform of Y_{n-1}, and y_n as the center of v on Y_n. Then for large n, the center y_n of v has codimension 1 *in Y_n.*

Proof. This follows directly from Proposition 3.7, using that the induced map $Y_n \to Y_{n-1}$ between the proper transforms is naturally isomorphic to the blow-up of Y_{n-1} in $\overline{\{y_{n-1}\}}$. □

Proposition 3.9. *Proposition 3.3 is true for closed immersions.*

Proof. Let $i : Y \hookrightarrow X$ be a closed immersion of smooth, connected, separated, finite type k-schemes. Let $(Y, \overline{Y})$ be a good partial compactification. Without loss of generatlity we may assume that $\overline{Y} \setminus Y$ is irreducible, and hence (the support of) a smooth divisor. To prove the lemma, we may replace $\overline{Y}$ by open neighborhoods $\overline{V}$ of the generic point η of $\overline{Y} \setminus Y$ and Y by $\overline{V} \cap Y$.

Let X' be a normal, proper k-scheme containing X as a dense open subscheme. After possibly removing a closed subset of codimension ≥ 2 from $\overline{Y}$ we may assume that i extends to a morphism $i' : \overline{Y} \to X'$. Note that $i'(\eta) \in X' \setminus X$: otherwise we would have $i'(\eta) \in i'(Y)$, as $i'(\overline{Y})$ is irreducible, and then the valuation on $k(Y)$ associated with η would have two centers, which is impossible as Y is separated over k.

By Lemma 3.8 there exists a modification $X'' \to X'$, which is an isomorphism over X, such that (perhaps after again removing a closed subset of codimension ≥ 2 from $\overline{Y}$) i extends to a map $i'' : \overline{Y} \to X''$ such that $i''(\eta)$ is a codimension 1 point of the closure of $i''(\overline{Y})$ in X''. Thus, replacing X' by X'' we may assume that $i : Y \hookrightarrow X$ extends to a closed immersion $i' : \overline{Y} \hookrightarrow X'$.

It remains to show that we can replace X' by a chain of blow-ups with centers over $X' \setminus X$, and $\overline{Y}$ with its proper transform, such that $i'(\eta)$ is a regular point of X' and a regular point of $X' \setminus X$.

For this we use a desingularization result kindly communicated to us by Ofer Gabber.

Lemma 3.10 (Gabber). (a) *Let $\mathcal{O}$ be a noetherian local integral domain and $\mathfrak{p} \subseteq \mathcal{O}$ a prime ideal such that $\mathcal{O}/\mathfrak{p}$ is of dimension* 1 *and such that the normalization of $\mathcal{O}/\mathfrak{p}$ is finite over $\mathcal{O}/\mathfrak{p}$. Write $X := X_0 := \operatorname{Spec} \mathcal{O}$, $C := C_0 := \operatorname{Spec} \mathcal{O}/\mathfrak{p}$. For $n \geq 0$ let X_{n+1} be the blow-up of X_n at the closed points of C_n, and C_{n+1} the proper transform of C_n in X_{n+1}. For large n, C_n is regular, and at every closed point ξ of*

C_n we have that if

$$\mathfrak{p}_n := \ker\left(\mathcal{O}_{X_n,\xi} \to \mathcal{O}_{C_n,\xi}\right),$$

then for every $m \in \mathbb{N}$ the $\mathcal{O}_{C_n,\xi}$-module $\mathfrak{p}_n^m/\mathfrak{p}_n^{m+1}$ is torsion-free.

(b) *If $\mathcal{O}_\mathfrak{p}$ is regular, then for large n so is $\mathcal{O}_{X_n,\xi}$.*

Proof. The normalization of $\mathcal{O}/\mathfrak{p}$ is finite, and it is obtained by blowing up singular points repeatedly. Thus we may assume that $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring.

After this reduction, the map $C_{n+1} \to C_n$ induced by the blow-up $X_{n+1} \to X_n$ is an isomorphism. In particular, $C_n = C_{n-1} = \ldots = C_0 = \operatorname{Spec} \mathcal{O}/\mathfrak{p}$.

Write $\mathcal{O}'$ for the local ring of X_1 in the closed point of C_1, and $\pi \in \mathcal{O}$ for a lift of a uniformizer of the discrete valuation ring $\mathcal{O}/\mathfrak{p}$. To ease notation we will also write π for its image in $\mathcal{O}/\mathfrak{p}$. Then $\mathcal{O}'$ is the localization of the ring $\sum_{i\geq 0} \pi^{-i}\mathfrak{p} \subseteq \operatorname{Frac}(\mathcal{O})$ at a suitable maximal ideal. Moreover, $\mathfrak{p}_1$ is the localization of the ideal generated by $\pi^{-1}\mathfrak{p}$. From this it is not difficult to see that we get a surjective $\mathcal{O}/\mathfrak{p}$-linear morphism

$$\varphi : \pi^{-m}\mathcal{O}/\mathfrak{p} \otimes_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}^m/\mathfrak{p}^{m+1} \twoheadrightarrow \mathfrak{p}_1^m/\mathfrak{p}_1^{m+1},$$

defined by $\pi^{-m} \otimes x \mapsto \pi^{-m}x$.

The $\mathcal{O}/\mathfrak{p}$-module $\ker(\varphi)$ is torsion. Indeed, if $x \in \mathfrak{p}^m$ is an element such that $\pi^{-m}x \in \mathfrak{p}_1^{m+1}$, then $\pi^{-m}x \in \pi^{-(m+1)}\mathfrak{p}^{m+1}$, so $\pi x \in \mathfrak{p}^{m+1}$. This implies that φ induces a surjective map on torsion submodules.

Now let $e_m \geq 0$ be the smallest integer such that multiplication with π^{e_m} kills the torsion submodule of $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ or equivalently of $\pi^{-m}\mathcal{O}/\mathfrak{p} \otimes_{\mathcal{O}/\mathfrak{p}} \mathfrak{p}^m/\mathfrak{p}^{m+1}$. Similarly, let $e'_m \geq 0$ be the integer such that multiplication with $\pi^{e'_m}$ kills the torsion submodule of $\mathfrak{p}_1^m/\mathfrak{p}_1^{m+1}$. Since φ induces a surjective map on torsion submodules, it follows that $e_m \geq e'_m$. If $e_m > 0$, we claim that $e_m > e'_m$. For this it is sufficient to show that for every element $x \in \mathfrak{p}^m$ such that $\pi x \in \mathfrak{p}^{m+1}$, we have $\pi^{-m} \otimes x \in \ker(\varphi)$. But this is clear: $\varphi(\pi^{-m} \otimes x) = \pi^{-(m+1)}\pi x \in \mathfrak{p}_1^{m+1}$.

Repeating the argument for the blow-ups $X_2 \to X_1$, $X_3 \to X_2$, and so on, it follows that for fixed m, there exists a minimal integer $N(m) \geq 0$ such that $\mathfrak{p}_n^m/\mathfrak{p}_n^{m+1}$ is torsion free for all $n \geq N(m)$. It remains to see that the sequence $N(m)$ is bounded. Consider the associated graded ring $\bigoplus_{m\geq 0} \mathfrak{p}^m/\mathfrak{p}^{m+1}$. The subset of elements which are killed by a power of π is an ideal of this noetherian ring, hence finitely generated. Thus the sequence of numbers e_m is bounded, which implies that the sequence $N(m)$ is bounded. This completes the proof (a).

Finally, lets prove (b). Assume that $\mathcal{O}/\mathfrak{p}$ is a discrete valuation ring, that $\mathcal{O}_\mathfrak{p}$ is regular and that $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ is a free $\mathcal{O}/\mathfrak{p}$-module for every $m \geq 0$. To prove that $\mathcal{O}$ is regular, it suffices to show that $\operatorname{Spec}\mathcal{O}/\mathfrak{p} \to \operatorname{Spec}\mathcal{O}$ is a regular immersion, i.e. that for every $m \geq 0$ the natural (surjective) map

$$\operatorname{Sym}^m \mathfrak{p}/\mathfrak{p}^2 \twoheadrightarrow \mathfrak{p}^m/\mathfrak{p}^{m+1} \tag{2}$$

is an isomorphism of $\mathcal{O}/\mathfrak{p}$-modules. Write K for the fraction field of $\mathcal{O}/\mathfrak{p}$. Looking at the commutative diagram

$$\begin{array}{ccc} \mathcal{O} & \twoheadrightarrow & \mathcal{O}/\mathfrak{p} \\ \downarrow & & \downarrow \\ \mathcal{O}_\mathfrak{p} & \twoheadrightarrow & K = \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} \end{array}$$

we see that (2) is an isomorphism after tensoring with K. In particular, $\operatorname{Sym}^m \mathfrak{p}/\mathfrak{p}^2$ and $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ have the same rank r. As $\mathfrak{p}^m/\mathfrak{p}^{m+1}$ is a free $\mathcal{O}/\mathfrak{p}$-module by assumption, it follows that (2) can be identified with is a surjective endomorphism of a free $\mathcal{O}/\mathfrak{p}$-module of rank r and hence is an isomorphism. □

Using Lemma 3.10 we finish the proof of Proposition 3.9. Let $\varphi : \mathcal{O}_{X',i'(\eta)} \to \mathcal{O}_{\overline{Y},\eta}$ be the morphism induced by i'. Then φ is surjective, as i' is a closed immersion by construction, and if $\mathfrak{p} = \ker(\varphi)$, then $\mathfrak{p}$ is prime. As $\mathcal{O}_{\overline{Y},\eta} = \mathcal{O}_{X',i'(\eta)}/\mathfrak{p}$ is 1-dimensional, we can apply Gabber's Lemma 3.10 to $\mathcal{O}_{X',i'(\eta)}$ and $\mathfrak{p}$: It shows that after replacing X' by a chain of blow-ups with centers over $X' \setminus X$, and $\overline{Y}$ with its proper transform, $i'(\eta)$ lies in X''_{reg}. Thus, after removing a closed subset of codimension ≥ 2 from $\overline{Y}$ we have $i'(\overline{Y}) \subseteq X'_{\mathrm{reg}}$.

Moreover, as $i'(\eta)$ is a regular point of $i'(\overline{Y})$, there is a regular system of parameters $h_0, \ldots, h_n$ of $\mathcal{O}_{X',i'(\eta)}$ such that $(h_1, \ldots, h_n) = \mathfrak{p} = \ker(\mathcal{O}_{X',i'(\eta)} \to \mathcal{O}_{\overline{Y},\eta})$, and such that h_0 is the uniformizer of the discrete valuation ring $\mathcal{O}_{\overline{Y},\eta} = \mathcal{O}_{X',i'(\eta)}/\mathfrak{p}$.

Without loss of generality we may assume that $X' \setminus X$ is the support of a Cartier divisor with local equation g around $i'(\eta)$. Then $g \in \mathfrak{m}_{i'(\eta)}$, where $\mathfrak{m}_{i'(\eta)}$ is the maximal ideal of $\mathcal{O}_{X',i'(\eta)}$, and we claim that g can be written

$$g = uh_0^m + \sum_{i=1}^{n} a_i h_i$$

with $u \in \mathcal{O}^\times_{X',i'(\eta)}$, $a_i \in \mathcal{O}_{X',i'(\eta)}$. Indeed, since $i'(\overline{Y}) \cap X \neq \emptyset$, we see that g has nonzero image in $\mathcal{O}_{X',i'(\eta)}/(h_1,\dots,h_n) = \mathcal{O}_{\overline{Y},\eta}$ which is a discrete valuation ring with uniformizer h_0. So $g = \bar{u}h_0^m \mod (h_1,\dots,h_n)$ with $\bar{u} \in \mathcal{O}^\times_{\overline{Y},\eta}$. Any lift u of $\bar{u}$ to $\mathcal{O}_{X',i'(\eta)}$ is a unit, so the claim follows. Moreover, $m > 0$, since $i'(\overline{Y}) \not\subseteq X$.

If for every $i > 0$ the term $a_i h_i$ is divisible by h_0^m we are done, because in this case we can write $g = h_0^m \cdot \text{unit}$, so around $i'(\eta)$ the reduced induced structure on $X' \setminus X$ is $V(h_0)$, hence regular, and $i'(\overline{Y})$ intersects $X' \setminus X$ in $i'(\eta)$ transversally.

If $a_i h_i$ is not divisible by h_0^m for all $i > 0$, then we blow up X' in $\overline{\{i'(\eta)\}}$ and replace X' by this blow-up and $\overline{Y}$ by its proper transform (note that this does not change $\overline{Y}$, as $i'(\eta)$ is of codimension 1 in $i'(\overline{Y})$). Then the local ring $\mathcal{O}_{X',i'(\eta)}$ has a regular system of parameters $(h_0, h_1/h_0, \dots, h_n/h_0)$. Hence, repeating this process m-times, we can write

$$g = h_0^m\Big(u + \sum_{i=1}^n a_i h_i / h_0^m\Big)$$

in $\mathcal{O}_{X',i'(\eta)}$, and we conclude as in the previous paragraph. $\square$

4. Checking for regular singularities on curves

We continue to denote by k an algebraically closed field of positive characteristic p, and by X a smooth, connected, separated, finite type k-scheme.

To prove Theorem 1.1, we first establish the following easy lemma:

Lemma 4.1. *Let S be a noetherian scheme, $X \to S$ a smooth morphism of finite type k-schemes with $X = \operatorname{Spec} A$ affine, $U = \operatorname{Spec} A[t^{-1}]$, and $t \in A$ a regular element. Assume that the closed subscheme $D := V(t) \subseteq X$ is irreducible and smooth over S. If $g \in A[t^{-1}]$, then the set*

$$\operatorname{Pol}_{\leq n}(g) := \{s \in S \mid g|_{U_s} \in \Gamma(U_s, \mathcal{O}_{U_s}) \text{ has pole order } \leq n \text{ along } D_s\}$$

is a constructible subset of S.

Proof. Note that since $D \to S$ is smooth, $D_s \subseteq X_s$ is a smooth divisor for every $s \in S$, so it makes sense to talk about the pole order of $g|_{U_s}$ along D_s.

Since $\operatorname{Pol}_{\leq n}(g) = \operatorname{Pol}_{\leq 0}(t^n g)$, it suffices to show that $\operatorname{Pol}_{\leq 0}(g)$ is constructible.

The element g defines a commutative diagram of S-schemes

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "g"] & X \arrow[d, "g"] \\ \mathbb{A}^1_S \arrow[r, hook] & \mathbb{P}^1_S. \end{tikzcd}$$

The image $g(X) \subseteq \mathbb{P}^1_S$ is a constructible set, so $g(X) \cap (\{\infty\} \times S)$ is a constructible subset of $\mathbb{P}^1_S$. If $\mathrm{pr} : \mathbb{P}^1_S \to S$ is the structure morphism of $\mathbb{P}^1_S$, then $\mathrm{pr}(g(X) \cap (\{\infty\} \times S))$ is a constructible subset of S. Finally note that $S \setminus \mathrm{Pol}_{\leq 0}(g) = \mathrm{pr}((g(X) \cap (\{\infty\} \times S))$. □

We are now ready to prove Theorem 1.1 with respect to a fixed good partial compactification.

Proposition 4.2. *Let $(X, \overline{X})$ be a good partial compactification and E a stratified bundle on X. Assume that for every k-morphism $\bar{\varphi} : \overline{C} \to \overline{X}$ with $\overline{C}$ a smooth k-curve, the stratified bundle $\varphi^* E$ is $(C, \overline{C})$-regular singular, where $C := \bar{\varphi}^{-1}(X)$, $\varphi := \bar{\varphi}|_C$.*

If the base field k is uncountable, then E is $(X, \overline{X})$-regular singular.

Proof. Let k be an uncountable algebraically closed field of characteristic $p > 0$. We immediately reduce to the case where X is connected and $\dim X \geq 2$. By removing a closed set of codimension ≥ 2 from $\overline{X}$ we may assume that $D := (\overline{X} \setminus X)_{\mathrm{red}}$ is a smooth divisor; by treating its components separatedly, we may assume that D is irreducible with generic point η. Shrinking $\overline{X}$ further, we may assume that

- $\overline{X} = \operatorname{Spec} A$ is affine,
- there exist étale coordinates $x_1, \ldots, x_n \in A$ such that $D = V(x_1)$.
- E corresponds to a free $A[x_1^{-1}]$-module, say with basis $e_1, \ldots, e_r$.

If we write $\delta^{(m)}_{x_1} := x_1^m \partial^{(m)}_{x_1} \in \mathscr{D}_{X/k}$ (see Remark 2.4), then for $f \in A$ we can also write

$$\delta^{(m)}_{x_1}(fe_i) = \sum_{j=1}^{r} b^{(m)}_{ij}(f) e_j, \text{ with } b^{(m)}_{ij}(f) \in A[x_1^{-1}].$$

To show that E is regular singular, it suffices to show that the pole order of the elements $b^{(m)}_{ij}(f)$ along x_1 is bounded by some $N \in \mathbb{N}$, because then the

$\mathscr{D}_{\overline{X}/k}(\log D)$-module generated by $\sum_{i=1}^r e_i A$ is contained in $\left(\bigoplus_{i=1}^r x_1^{-N} e_i A\right)$, and thus finitely generated over A.

Write A as the quotient of a polynomial ring $k[y_1, \dots, y_d]$ and $\bar{y}_i$ for the image of y_i in A. It then suffices to show that the pole order of $b_{ij}^{(m)}(\bar{y}_c^h)$ has a common upper bound, for $1 \le i, j \le r$, $1 \le c \le d$, $m, h \ge 0$.

Define $S := \mathbb{A}_k^{n-1} = \operatorname{Spec} k[x_2, \dots, x_n]$. We get a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] & \overline{X} \arrow[r, "\text{étale}"] \arrow[dr, "\text{smooth}"'] & \mathbb{A}_S^1 \arrow[d] \\ & & S \end{tikzcd} \tag{3}$$

We are now in the situation of Lemma 4.1. For $N \in \mathbb{N}$ consider the constructible sets $\mathrm{Pol}_{\le N}(b_{ij}^{(m)}(\bar{y}_c^h)) \subseteq S$, and define

$$\mathbf{P}_{\le N} := \bigcap_{i,j,m,c,h} \overline{\mathrm{Pol}_{\le N}(b_{ij}^{(m)}(\bar{y}_c^h))}.$$

This is a closed subset of S. Now since for every closed point $s \in S$ the fiber X_s is a regular curve over k meeting D transversally, we see that by assumption $E|_{X_s}$ is $(X_s, \overline{X}_s)$-regular singular. But this means that there is some $N_s \ge 0$, such that $s \in \mathbf{P}_{\le N_s}$. In other words, the union $\bigcup_{N \ge 0} \mathbf{P}_{\le N}$ contains all closed points of S. Since k is uncountable and since the $\mathbf{P}_{\le N}$ are closed subsets of S, this means there exists some $N_0 \ge 0$, such that $\mathbf{P}_{\le N_0} = S$. The definition of $\mathbf{P}_{\le N_0}$ and Lemma 4.1 imply that the sets $\mathrm{Pol}_{\le N_0}(b_{ij}^{(m)}(\bar{y}_c^h))$ are *dense constructible* subsets of S. But a dense constructible subset of an irreducible noetherian space contains an open dense subset by [4, Prop. 10.14]. This shows that the pole order of $b_{ij}^{(m)}(\bar{y}_c^h)$ along x_1 is bounded by N_0, and thus that E is $(X, \overline{X})$-regular singular. $\square$

Now we can easily finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We have proved Theorem 1.3 which shows that if E is a regular singular stratified bundle on X, then $\varphi^* E$ is regular singular for every map $\varphi : C \to X$ with C a smooth k-variety.

For the converse, assume that $\varphi^* E$ is regular singular for every morphism $\varphi : C \to X$ with C a smooth k-curve. We have to show that E is $(X, \overline{X})$-regular singular with respect to every good partial compactification $(X, \overline{X})$. But the assumptions of Proposition 4.2 are satisfied, so E is $(X, \overline{X})$-regular singular. $\square$

Finally, we give a proof of Corollary 1.2.

Proof of Corollary 1.2. We need to show that E is $(X, \overline{X})$-regular singular for every good partial compactification $(X, \overline{X})$, whenever the condition of this corollary is satisfied. We may assume that $\overline{X} \setminus X$ is irreducible. As in the proof of Theorem 1.1, we reduce to $\overline{X}$ affine and E free, so that showing that E is $(X, \overline{X})$-regular singular boils down to showing that the pole order of a certain set of functions in $k(X)$ has a common bound. This is independent of the coefficients, so we may base change to a field $K \supseteq k$, K algebraically closed and uncountable (e.g. $K := \overline{k((t))}$). Then we can apply the theorem, to see that E_K is regular singular if it is regular singular along all smooth K-curves. But every such curve is defined over a subextension k' of K/k, finitely generated over k. The corollary follows. □

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