

Torsion in the Lichtenbaum Chow group of arithmetic schemes

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We give an example of a smooth arithmetic scheme $\mathfrak{X} \rightarrow B$ over the spectrum of a Dedekind domain and primes p with the property that the p -primary torsion subgroup of the Lichtenbaum Chow group $\mathrm{CH}_L^2(\mathfrak{X})\{p\}$ has positive corank. This also implies that the unramified cohomology group $\mathrm{H}_{\mathrm{nr}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is infinite.

1. Introduction

Let E be an elliptic curve over \mathbb{Q} with conductor N and with complex multiplication by the ring of integers of an imaginary quadratic field K , and let $X = E \times_{\mathbb{Q}} E$. Langer-Raskind [15, Theorem 0.1] have shown that if p is a prime not dividing $6N$, the p -primary torsion subgroup $\mathrm{CH}^2(X)\{p\}$ of the Chow group of codimension 2 cycles is finite. In fact, they show the following: If p is a prime number not dividing $6N$, let B be the spectrum of the ring \mathbb{Z} with the primes dividing $6Np$ inverted, and let \mathfrak{X} be a smooth proper model of X over B . By a result of Mildenhall [18, Theorem 5.8] the kernel of the natural map $\mathrm{CH}^2(\mathfrak{X}) \rightarrow \mathrm{CH}^2(X)$ is finite, which implies that $\mathrm{CH}^2(X)\{p\}$ is finite if and only if $\mathrm{CH}^2(\mathfrak{X})\{p\}$ is finite. Langer-Raskind use this equivalence and prove the quoted finiteness result for the Chow group of the model.

For both X and \mathfrak{X} (or more generally, smooth schemes over a field or the spectrum of a Dedekind domain) Bloch's cycle complex [2] defines a complex $\mathbb{Z}(n)$ of Zariski sheaves (on the small sites X_{Zar} or $\mathfrak{X}_{\mathrm{Zar}}$), and one can consider the groups

$$\mathrm{CH}_M^n(X) = \mathbb{H}_{\mathrm{Zar}}^{2n}(X, \mathbb{Z}(n)) \quad \text{and} \quad \mathrm{CH}_M^n(\mathfrak{X}) = \mathbb{H}_{\mathrm{Zar}}^{2n}(\mathfrak{X}, \mathbb{Z}(n)).$$

It follows from Zariski descent that the group $\mathrm{CH}_M^n(X)$ defined in this way coincides with the classical Chow group $\mathrm{CH}^n(X)$, and an easy argument shows that in our case of interest we also have $\mathrm{CH}^2(\mathfrak{X}) \cong \mathrm{CH}_M^2(\mathfrak{X})$. We may also view $\mathbb{Z}(n)$ as a complex of sheaves in the étale topology (on the small

sites $X_{\text{ét}}$ and $\mathfrak{X}_{\text{ét}}$, and write $\mathbb{Z}(n)_{\text{ét}}$ for this complex. The étale motivic or Lichtenbaum Chow groups we consider are the hypercohomology groups

$$\text{CH}_L^n(X) = \mathbb{H}_{\text{ét}}^{2n}(X, \mathbb{Z}(n)) \quad \text{and} \quad \text{CH}_L^n(\mathfrak{X}) = \mathbb{H}_{\text{ét}}^{2n}(\mathfrak{X}, \mathbb{Z}(n));$$

for details, see Section 2. Since $\text{CH}_L^2(\mathfrak{X})\{p\}$ is a quotient and $\text{CH}^2(\mathfrak{X})\{p\}$ is a subquotient of $\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, the p -primary torsion in both Chow groups is of finite cotyple, and a priori there could be more torsion in the Lichtenbaum Chow group. We show:

Theorem 1.1. *Let E/\mathbb{Q} be an elliptic curve with complex multiplication and conductor N . Let p be a prime such that $p \nmid 6N$ and such that E has ordinary reduction at p , and let S be the set of primes dividing $6Np$. Set $X = E \times_{\mathbb{Q}} E$, and let $\mathfrak{X} \rightarrow \text{Spec}(\mathbb{Z}_S)$ be a smooth proper model of X . Then*

- (a) $\text{CH}_L^2(\mathfrak{X})\{p\}$ has positive corank,
- (b) $\text{CH}_L^2(X)\{p\}$ contains a copy of $\mathbb{Q}_p/\mathbb{Z}_p$.

To prove this, we first establish a sufficient criterion in terms of étale cohomology for $\text{CH}_L^2(\mathfrak{X})\{p\}$ to have positive corank, and then show that this condition is satisfied. The second step involves local computations using p -adic Hodge theory and is already implicit in the proof of Langer-Raskind. For the convenience of the reader we give the full argument.

As suggested by the referee, we state the following corollary. With the notations as in Theorem 1.1, consider the unramified cohomology group

$$\text{H}_{\text{nr}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) = \ker\{\text{H}_{\text{ét}}^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow \bigoplus_{x \in \mathfrak{X}^{(1)}} \text{H}_{\text{ét}}^2(\kappa(x), \mathbb{Q}_p/\mathbb{Z}_p(1))\},$$

where K is the function field of \mathfrak{X} , $\mathfrak{X}^{(a)}$ denotes the set of points $x \in \mathfrak{X}$ such that $\dim \mathcal{O}_{X,x} = a$, and $\kappa(x)$ is the residue field of the point.

Corollary 1.2. *Let $\mathfrak{X} \rightarrow \text{Spec}(\mathbb{Z}_S)$ and p be as in Theorem 1.1. Then the unramified cohomology group $\text{H}_{\text{nr}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is infinite. Furthermore, $\text{H}_{\text{nr}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is infinite.*

2. Preliminaries

We summarize the definitions and properties of motivic and Lichtenbaum cohomology needed in the sequel.

Let X be a smooth quasi-projective variety over a field k . The motivic cohomology groups of X with coefficients in an abelian group A are defined as

$$H_M^n(X, A(n)) = \mathbb{H}_{\text{Zar}}^m(X, (z^n(-, \bullet) \otimes A)[-2n]),$$

where $z^n(-, \bullet)$ is the complex of Zariski sheaves given by Bloch’s cycle complex [2]. In particular, $\text{CH}^n(X) = H_M^{2n}(X, \mathbb{Z}(n))$ is the Chow group of codimension n cycles. The complex $z^n(-, \bullet)$ is also a complex of étale sheaves [2, §11] whose hypercohomology groups define the étale motivic or Lichtenbaum cohomology

$$H_L^m(X, A(n)) = \mathbb{H}_{\text{ét}}^m(X, (z^n(-, \bullet)_{\text{ét}} \otimes A)[-2n]);$$

in particular, $\text{CH}_L^n(X) = H_L^{2n}(X, \mathbb{Z}(n))$. It is known that the comparison map

$$H_M^m(X, A(n)) \rightarrow H_L^m(X, A(n))$$

is an isomorphism with rational coefficients. Furthermore, if $\ell \nmid \text{char}(k)$ is a prime, there is a quasi-isomorphism $(\mathbb{Z}/\ell^r\mathbb{Z})(n)_{\text{ét}} \xrightarrow{\sim} \mu_{\ell^r}^{\otimes n}$ [10, Theorem 1.5], thus with finite coefficients Lichtenbaum and étale cohomology coincides

$$H_L^m(X, \mathbb{Z}/\ell^r(n)) \cong H_{\text{ét}}^m(X, \mu_{\ell^r}^{\otimes n}).$$

Let R be a Dedekind domain and let $\mathfrak{X} \rightarrow B = \text{Spec}(R)$ be an essentially smooth B -scheme. The complex $z^n(-, \bullet)$ on \mathfrak{X} also defines a complex of sheaves for the Zariski and for the étale cohomology. Following Geisser [11], we define the motivic and étale motivic or Lichtenbaum cohomology of \mathfrak{X} as

$$H_M^m(\mathfrak{X}, A(n)) = \mathbb{H}_{\text{Zar}}^m(\mathfrak{X}, A(n)) \quad \text{and} \quad H_L^m(\mathfrak{X}, A(n)) = \mathbb{H}_{\text{ét}}^m(\mathfrak{X}, A(n));$$

we also set $\text{CH}_M^n(\mathfrak{X}) = H_M^{2n}(\mathfrak{X}, \mathbb{Z}(n))$ and $\text{CH}_L^n(\mathfrak{X}) = H_L^{2n}(\mathfrak{X}, \mathbb{Z}(n))$. We remark that it not clear that the groups $H_M^m(\mathfrak{X}, \mathbb{Z}(n))$ coincides with the cohomology of Bloch’s cycle complex of abelian groups in general (this is known, for example, in case X is essentially of finite type over $B = \text{Spec}(A)$, where A is a discrete valuation ring [11, Theorem 3.2]). However, in our case of interest we can compare the exact localization sequences along the evident maps

$$\begin{array}{ccccccc} \text{CH}^2(X, 1) & \rightarrow & \bigoplus_{v \notin S} \text{Pic}(Y_v) & \rightarrow & \text{CH}^2(\mathfrak{X}) & \rightarrow & \text{CH}^2(X) \rightarrow 0 \\ \cong \downarrow & & \cong \downarrow & & \downarrow & & \downarrow \cong \\ H_M^3(X, \mathbb{Z}(2)) & \rightarrow & \bigoplus_{v \notin S} H_M^2(Y_v, \mathbb{Z}(1)) & \rightarrow & H_M^4(\mathfrak{X}, \mathbb{Z}(2)) & \rightarrow & H_M^4(X, \mathbb{Z}(2)) \rightarrow 0 \end{array}$$

where the isomorphisms result from the facts that for X one has Zariski descent, and that for a smooth variety Y one has $\mathbb{Z}(1)_{\text{Zar}} \sim \mathcal{O}_Y^\times[-1]$ [2, Corollary 6.4]. Hence $\text{CH}^2(\mathfrak{X}) \cong \text{CH}_M^2(\mathfrak{X})$, i.e. the motivic Chow group of the model we consider does indeed coincide with the classical Chow group.

We remark that if $\epsilon : \mathfrak{X}_{\text{ét}} \rightarrow \mathfrak{X}_{\text{Zar}}$ is the canonical morphism of sites and \mathfrak{X} is essentially of finite type over B , the map $\mathbb{Q}(n)_{\text{Zar}} \rightarrow R\epsilon_*\mathbb{Q}(n)_{\text{ét}}$ is a quasi-isomorphism [11, Proposition 3.6], hence we have with rational coefficients

$$H_M^m(\mathfrak{X}, \mathbb{Q}(n)) \cong H_L^m(\mathfrak{X}, \mathbb{Q}(n)).$$

Let $\mathfrak{X} \rightarrow B$ be an essentially smooth scheme over the spectrum of a Dedekind ring. In what follows we will use two properties of the Lichtenbaum cohomology groups of \mathfrak{X} whose proofs require the use of the proof of the Bloch-Kato conjecture, i.e. the assertion that for a field F and a prime number $p \nmid \text{char}(F)$ the norm-residue map between Milnor K -theory and étale cohomology

$$K_m^M(F)/p^r \rightarrow H_{\text{ét}}^m(F, \mu_{p^r}^{\otimes m})$$

is an isomorphism. This has been shown for the prime $p = 2$ by Voevodsky [23], and in general by Rost-Voevodsky [24], see also [25]. Making use of the norm-residue isomorphism, Geisser has shown the following [11, Theorem 1.2]:

(A) The morphism $\mathbb{Z}(n)_{\text{Zar}} \rightarrow \tau_{\leq n+1}R\epsilon_*\mathbb{Z}(n)_{\text{ét}}$ is a quasi-isomorphism, thus

$$H_M^m(\mathfrak{X}, \mathbb{Z}(n)) \cong H_L^m(\mathfrak{X}, \mathbb{Z}(n)) \text{ for } m \leq n + 1.$$

(B) If the prime p is invertible in B , there is a quasi-isomorphism of complexes of étale sheaves $\mathbb{Z}/p^r(n)_{\text{ét}} \xrightarrow{\sim} \mu_{p^r}^{\otimes n}$ (where the sheaf $\mu_{p^r}^{\otimes n}$ is in degree 0). Hence

$$H_L^m(\mathfrak{X}, \mathbb{Z}/p^r(n)) \cong H_{\text{ét}}^m(\mathfrak{X}, \mu_{p^r}^{\otimes n}) \text{ for } m \in \mathbb{Z}.$$

3. Proof of the Theorem

Proof. (of Theorem 1.1) Let $p > 3$ be a prime such that E has good ordinary reduction at p , and let $X = E \times_{\mathbb{Q}} E$ and $\mathfrak{X} \rightarrow \text{Spec}(\mathbb{Z}_S)$ be as in Theorem 1.1; in particular, the prime p is invertible in \mathbb{Z}_S . Note that by standard arguments involving the Leray spectral sequence associated with the morphism $\mathfrak{X} \rightarrow \text{Spec}(\mathbb{Z}_S)$, the étale cohomology groups $H_{\text{ét}}^m(\mathfrak{X}, \mu_{p^r}^{\otimes n})$ are finite, the groups $H_{\text{ét}}^m(\mathfrak{X}, \mathbb{Q}_p(n))$ are finite-dimensional \mathbb{Q}_p -vector spaces, and the

groups $H_{\text{ét}}^m(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ are of finite cotype, see [15, proof of Lemma 3.1], for instance.

In both topologies $\tau = \text{Zar}$ and $\tau = \text{ét}$, the map ‘multiplication by p^r ’ induces the short exact sequence $0 \rightarrow \mathbb{Z}(2)_\tau \rightarrow \mathbb{Z}(2)_\tau \rightarrow \mathbb{Z}/p^r(2)_\tau \rightarrow 0$ of complexes of sheaves, from which we obtain short exact universal coefficient sequences in motivic and Lichtenbaum cohomology. Comparing these sequences along the morphism $\mathfrak{X}_{\text{ét}} \rightarrow \mathfrak{X}_{\text{Zar}}$, we get the following commutative diagram with exact rows

$$(1) \quad \begin{array}{ccccccc} 0 & \rightarrow & H_M^3(\mathfrak{X}, \mathbb{Z}(2))/p^r & \rightarrow & H_M^3(\mathfrak{X}, \mathbb{Z}/p^r(2)) & \rightarrow & CH^2(\mathfrak{X})[p^r] \rightarrow 0 \\ & & \cong \downarrow & & \kappa_{p^r} \downarrow & & \downarrow \\ 0 & \rightarrow & H_L^3(\mathfrak{X}, \mathbb{Z}(2))/p^r & \rightarrow & H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) & \rightarrow & CH_L^2(\mathfrak{X})[p^r] \rightarrow 0 \end{array}$$

Here we have used the quasi-isomorphism $\mathbb{Z}(n) \xrightarrow{\sim} \tau_{\leq n+1} R\epsilon_* \mathbb{Z}(n)_{\text{ét}}$ from (A) to obtain the left vertical isomorphism and the quasi-isomorphism $\mathbb{Z}/p^r(2)_{\text{ét}} \xrightarrow{\sim} \mu_{p^r}^{\otimes 2}$ from (B), which allows us to identify in the bottom row the groups

$$H_L^3(\mathfrak{X}, \mathbb{Z}/p^r(2)) \cong H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}).$$

Note that, as remarked above, $H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2})$ is finite. Furthermore, the kernel of the map $CH^2(\mathfrak{X}) \rightarrow CH^2(X)$ is finite [18, Theorem 5.8] and $CH^2(X)[p^r]$ is finite [15, Theorem 0.1], thus $CH^2(\mathfrak{X})[p^r]$ is finite, and hence all groups in the above diagram are finite as well. Taking the direct limit over all r of the bottom row shows that $CH_L^2(\mathfrak{X})\{p\}$ is a quotient of $H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$, thus is itself of finite cotype. Hence we have an isomorphism

$$CH_L^2(\mathfrak{X})\{p\} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus k} \oplus F, \quad F \text{ a finite group.}$$

If A is an abelian group, let $T_p(A) = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A)$ be the p -adic Tate module, and let $V_p(A) = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ be the corresponding Tate \mathbb{Q}_p -vector space. From the above description of $CH_L^2(\mathfrak{X})\{p\}$ we see that $T_p(CH_L^2(\mathfrak{X})\{p\}) \cong \mathbb{Z}_p^{\oplus k}$, thus $V_p(CH_L^2(\mathfrak{X})\{p\}) \cong \mathbb{Q}_p^{\oplus k}$, and we have

$$CH_L^2(\mathfrak{X})\{p\} \text{ has positive corank} \Leftrightarrow V_p(CH_L^2(\mathfrak{X})\{p\}) \neq 0.$$

Consider the \mathbb{Q}_p -vector spaces

$$\begin{aligned} H_M(\mathfrak{X}) &= [\varprojlim_r H_M^3(\mathfrak{X}, \mathbb{Z}(2))/p^r] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \\ H_L(\mathfrak{X}) &= [\varprojlim_r H_L^3(\mathfrak{X}, \mathbb{Z}(2))/p^r] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \end{aligned}$$

Since $\text{CH}^2(\mathfrak{X})\{p\}$ is finite, $T_p(\text{CH}^2(\mathfrak{X})\{p\}) = 0$ and $V_p(\text{CH}^2(\mathfrak{X})\{p\}) = 0$. Taking in (1) the inverse limit over all r (which is exact in this case) and tensoring with \mathbb{Q}_p , we obtain the following commutative diagram whose bottom row is exact

$$\begin{array}{ccccccc}
 H_M(\mathfrak{X}) & \xrightarrow{\cong} & [\varprojlim_r H_M^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p & & & & \\
 (2) \quad \cong \downarrow & & \kappa \downarrow & & & & \\
 0 \rightarrow H_L(\mathfrak{X}) & \rightarrow & H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) & \rightarrow & V_p(\text{CH}_L^2(\mathfrak{X})\{p\}) & \rightarrow & 0
 \end{array}$$

If $\mathbb{K} = \text{im}(\kappa)$, this immediately implies that

$$(3) \quad V_p(\text{CH}_L^2(\mathfrak{X})\{p\}) \cong H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))/\mathbb{K}.$$

If $\mathbb{Q}(X)$ is the function field of X , define

$$N H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) = \ker\{H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) \rightarrow H^3(\mathbb{Q}(X), \mu_{p^r}^{\otimes 2})\},$$

and

$$N H^3(\mathfrak{X}, \mathbb{Q}_p(2)) = [\varprojlim_r N H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Note that the \mathbb{Q}_p -vector space $N H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$ defined above does not necessarily coincide with the first level of the usual coniveau filtration, i.e. if

$$N' H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) = \varinjlim \ker\{H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow H_{\text{ét}}^3(\mathfrak{X} \setminus \mathfrak{Z}, \mathbb{Q}_p(2))\},$$

where the limit is taken over all closed subschemes $\mathfrak{Z} \subset \mathfrak{X}$ of codimension one, there is an injective map $N' H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow N H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$, which is not an isomorphism in general, see [15, Remark 2.1].

We claim that $\mathbb{K} = \text{im}(\kappa) \subseteq N H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$. To see this, note first that

$$(4) \quad \mathbb{K} = [\varprojlim_r \text{im}(\kappa_{p^r})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The composition $H_M^3(\mathfrak{X}, \mathbb{Z}/p^r(2)) \xrightarrow{\kappa_{p^r}} H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) \rightarrow H_{\text{ét}}^3(\mathbb{Q}(X), \mu_{p^r}^{\otimes 2})$ coincides with $H_M^3(\mathfrak{X}, \mathbb{Z}/p^r(2)) \rightarrow H_M^3(\mathbb{Q}(X), \mathbb{Z}/p^r(2)) \rightarrow H_{\text{ét}}^3(\mathbb{Q}(X), \mathbb{Z}/p^r(2))$, where the first map is the zero map, because the motivic Zariski sheaf $\mathcal{H}_M^3(\mathbb{Z}/p^r(2))$ on X is trivial (this follows from the existence of a Gersten resolution [2, §10] and a local argument; see also [11, §4]). Thus $\text{im}(\kappa_{p^r}) \subseteq N H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2})$, and our claim follows from (4).

Given $\mathbb{K} \subseteq N\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$, the composition of (3) with the evident quotient map $\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))/\mathbb{K} \rightarrow \mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))/N\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$ yields a surjective map

$$(5) \quad V_p(\text{CH}_L^2(\mathfrak{X})\{p\}) \rightarrow \mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))/N\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)).$$

In particular, to show $\text{CH}_L^2(\mathfrak{X})\{p\}$ has positive corank it suffices to show

$$(6) \quad \dim_{\mathbb{Q}_p} N\mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \leq \dim_{\mathbb{Q}_p} \mathbb{H}_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) - 1.$$

Let G_S be the Galois group of the maximal extension of \mathbb{Q} in $\overline{\mathbb{Q}}$ which is unramified outside of S . For $v \in S$ let $X_v = \mathfrak{X} \times_B \mathbb{Q}_v$ and write $G_v = \text{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ for the absolute Galois group and $H^n(G_v, M) = H^n(\mathbb{Q}_v, M)$ for the continuous Galois cohomology groups with values in a G_v -module M .

Let $V = \mathbb{H}_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2))$ and consider the localization map

$$H^1(G_S, V) \rightarrow \bigoplus_{v \in S} H^1(G_v, V);$$

we recall that by results of Flach [7] this map becomes surjective after passing to a suitable quotient on the right hand side.

In general, given a \mathbb{Q}_p -linear representation V of $G_{\mathbb{Q}}$, we have for a non-archimedean completion \mathbb{Q}_v of \mathbb{Q} the following \mathbb{Q}_p -subspaces of $H^1(\mathbb{Q}_v, V)$

$$H_f^1(G_v, V) \subseteq H_g^1(G_v, V) \subseteq H^1(G_v, V),$$

defined by Bloch-Kato in [3, §3] as follows: If $v \neq p$, let \mathbb{Q}_v^{nr} be the maximal unramified extension of \mathbb{Q}_v , set $H_g^1(\mathbb{Q}_v, V) = H^1(\mathbb{Q}_v, V)$, and

$$H_f^1(\mathbb{Q}_v, V) = \ker\{H^1(\mathbb{Q}_v, V) \rightarrow H^1(\mathbb{Q}_v^{nr}, V)\}.$$

If $v = p$, let B_{cris} and B_{DR} be the rings defined by Fontaine in [8], and set

$$\begin{aligned} H_f^1(\mathbb{Q}_v, V) &= \ker\{H^1(\mathbb{Q}_v, V) \rightarrow H^1(\mathbb{Q}_v, V \otimes_{\mathbb{Q}_p} B_{\text{cris}})\}, \\ H_g^1(\mathbb{Q}_v, V) &= \ker\{H^1(\mathbb{Q}_v, V) \rightarrow H^1(\mathbb{Q}_v, V \otimes_{\mathbb{Q}_p} B_{DR})\}. \end{aligned}$$

If T is a G_S -stable \mathbb{Z}_p -lattice, define A by the exactness of the sequence

$$0 \rightarrow T \xrightarrow{\iota} V \xrightarrow{\text{Pr}} A \rightarrow 0.$$

Let $W_v = H_f^1(\mathbb{Q}_v, V) \subseteq H^1(\mathbb{Q}_v, V)$ (where we set $H_f^1(\mathbb{Q}_v, V) = 0$ for the archimedean place $v = \infty$). Let $M_{\mathbb{Q}}$ be the set of all valuations of \mathbb{Q} and

define

$$H_{f,\mathbb{Z}}^1(\mathbb{Q}, V) = \{x \in H^1(\mathbb{Q}, V) \mid x_v \in H_f^1(\mathbb{Q}_v, V) \text{ for all } v \in M_{\mathbb{Q}}\}.$$

Using Tate’s global duality theorem, Flach has shown [7, Proposition 1.4] (cp. [3, Lemma 5.16]) that the localization maps induce an exact sequence

$$H^1(G_S, A) \rightarrow \bigoplus_{v \in S} \frac{H^1(\mathbb{Q}_v, A)}{\text{pr}_*(W_v)} \rightarrow \iota_*^{-1}(W')^*,$$

where

$$W' = H_{f,\mathbb{Z}}^1(\mathbb{Q}, V^*(1)) \text{ and } V^*(1) \cong H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1)).$$

Flach also showed that given our assumptions on E the Selmer groups associated with V and $V^*(1)$ are finite over \mathbb{Q} [7]. This implies that the group W' appearing in the above exact sequence is finite, and further that for the \mathbb{Q}_p -vector space $V = H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2))$ the localization maps induce a surjective map [15, Corollary 5.4]

$$(7) \quad H^1(G_S, V) \rightarrow \bigoplus_{v \in S} \frac{H^1(\mathbb{Q}_v, V)}{H_f^1(\mathbb{Q}_v, V)}.$$

Consider now a smooth proper model $\mathfrak{X} \rightarrow \text{Spec}(\mathbb{Z}_S)$ of X as in Theorem 1.1. The surjective localization map from (7) fits into the commutative diagram

$$\begin{array}{ccc} H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) & \rightarrow & \bigoplus_{v \in S} H_{\text{ét}}^3(X_v, \mathbb{Q}_p(2)) \\ \downarrow & & \downarrow \\ H^1(G_S, V) & \xrightarrow{\text{onto}} & \bigoplus_{v \in S} H^1(\mathbb{Q}_v, V) / H_f^1(\mathbb{Q}_v, V) \end{array}$$

where the left vertical map is induced by the Hochschild-Serre spectral sequence, and the right vertical map comes from the Hochschild-Serre spectral sequence, followed by the evident quotient map. We claim the left vertical map $H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow H^1(G_S, V)$ is surjective. Indeed, $E_2^{r,s} = H^r(G_S, H^s(\overline{X}, \mathbb{Q}_p(2))) = 0$ for $r > 2$ by cohomological dimension [4, 2.11], and $E_2^{0,3} = 0$ by a standard specialization argument, see [5, Theorem 1.5]. Thus $\text{gr}_F^0 H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) = 0$ and $E_2^{1,2} = \text{gr}_F^1 H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$, i.e. the claimed surjection is the evident map $F^1 H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow \text{gr}_F^1 H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$. Since $p \in S$, we may pass first to the quotient $H^1(\mathbb{Q}_p, V) / H_f^1(\mathbb{Q}_p, V)$, and then further

to $H^1(\mathbb{Q}_p, V)/H^1_g(\mathbb{Q}_p, V)$ to obtain a surjective map of \mathbb{Q}_p -vector spaces

$$(8) \quad H^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow H^1(\mathbb{Q}_p, V)/H^1_g(\mathbb{Q}_p, V).$$

In our next step we show that the surjective map (8) factors through the quotient $H^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/NH^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$. In the Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r(\mathbb{Q}_p, H^s_{\text{ét}}(\overline{X}, \mathbb{Q}_p(2))) \Rightarrow H^{r+s}_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p(2))$$

$E_2^{0,3} = 0$ by a weight argument, and $E_2^{2,1} = 0$ by [1, Proposition 2.4]. Since furthermore $E_2^{r,s} = 0$ for $r > 2$ by cohomological dimension, this implies that

$$(9) \quad H^3_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p(2)) \cong H^1(\mathbb{Q}_p, V).$$

Let $\mathfrak{X}_{\mathbb{Z}_p} = \mathfrak{X} \times_B \mathbb{Z}_p$, and let $j : X_{\mathbb{Q}_p} \rightarrow \mathfrak{X}_{\mathbb{Z}_p}$ be the inclusion. It is immediate that the image of $NH^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$ in $H^1(\mathbb{Q}_p, V)$ is contained in the image of $NH^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p}, \mathbb{Q}_p(2))$ in $H^1(\mathbb{Q}_p, V)$. In particular, to show the map (8) factors through $H^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))/NH^3_{\text{ét}}(\mathfrak{X}, \mathbb{Q}_p(2))$ it suffices to show that the image of $NH^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p}, \mathbb{Q}_p(2))$ in $H^1(\mathbb{Q}_p, V)$ is contained in the subspace $H^1_g(\mathbb{Q}_p, V)$.

If C^\bullet is a complex of sheaves on $\mathfrak{X}_{\mathbb{Z}_p}$, we write $\tau_{\leq n}C^\bullet$ for its truncation consisting of terms in degree $\leq n$. In particular, if $C^\bullet = Rj_*\mu_{p^r}^{\otimes 2}$, then

$$H^3(\mathfrak{X}_{\mathbb{Z}_p}, \tau_{\leq 2}Rj_*\mathbb{Q}_p(2)) = [\varprojlim_r H^3(\mathfrak{X}_{\mathbb{Z}_p}, \tau_{\leq 2}Rj_*\mu_{p^r}^{\otimes 2})] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

and it follows from [16, Lemma 5-4] that there is an inclusion

$$(10) \quad NH^3_{\text{ét}}(\mathfrak{X}_{\mathbb{Z}_p}, \mathbb{Q}_p(2)) \subseteq H^3(\mathfrak{X}_{\mathbb{Z}_p}, \tau_{\leq 2}Rj_*\mathbb{Q}_p(2)).$$

We need a result from p -adic Hodge theory, for an overview in a more general setting, see [20, §2], for instance. There is a natural (injective) pull-back map

$$(11) \quad \alpha : H^3(\mathfrak{X}_{\mathbb{Z}_p}, \tau_{\leq 2}Rj_*\mathbb{Q}_p(2)) \rightarrow H^3_{\text{ét}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p(2)) \cong H^1(\mathbb{Q}_p, V).$$

Let $\iota : Y_p \rightarrow \mathfrak{X}_{\mathbb{Z}_p}$ be the inclusion of the closed fiber, and let $s_r^{\log}(n)$ be the log syntomic complex defined by Kato in [14]. By a result of Tsuji [22, Theorem 5.1], we have in $D^b_{\text{ét}}(\mathfrak{X}, \mathbb{Z}/p^r\mathbb{Z})$ for $0 \leq n \leq p - 2$ a quasi-isomorphism

$$\eta : s_r^{\log}(n) \xrightarrow{\sim} \iota_*\iota^*(\tau_{\leq n}Rj_*\mu_{p^r}^{\otimes n});$$

in particular, this holds for $n = 2$, provided $p > 3$. Consider now the group

$$H^n(\mathfrak{X}_{\mathbb{Z}_p}, s_{\mathbb{Q}_p}^{\log}(n)) = [\varprojlim_r H^n(\mathfrak{X}_{\mathbb{Z}_p}, s_r^{\log}(n))] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Since we assume $p > 3$, the quasi-isomorphism η induces an isomorphism

$$(12) \quad H^3(\mathfrak{X}_{\mathbb{Z}_p}, s_{\mathbb{Q}_p}^{\log}(2)) \xrightarrow{\cong} H^3(\mathfrak{X}_{\mathbb{Z}_p}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)).$$

Let η' be the composition of this isomorphism with the map α from (11), i.e.

$$\eta' : H^3(\mathfrak{X}_{\mathbb{Z}_p}, s_{\mathbb{Q}_p}^{\log}(2)) \rightarrow H^1(\mathbb{Q}_p, V).$$

The result we need is the following inclusion [17, Lemma 2.12] (which by [17], and more generally [19, Theorem 3.1], is in fact known to be an equality)

$$(13) \quad \text{im}(\eta') \subseteq H_g^1(\mathbb{Q}_p, V).$$

In particular, by (10), (12) and (13) the image of $N H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2))$ in $H^1(\mathbb{Q}_p, V)$ lies in the subspace $H_g^1(\mathbb{Q}_p, V)$, thus (8) induces a surjective map

$$(14) \quad H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) / N H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p(2)) \rightarrow H^1(\mathbb{Q}_p, V) / H_g^1(\mathbb{Q}_p, V) =: \Theta_p.$$

It remains to show $\Theta_p \neq 0$. The dimension of this space can be computed as follows [15, pg. 16] (see also [16, proof of Lemma 4-5]): Let $W = V(1)^* \cong H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1))$, viewed as a $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -module. Then

$$\dim_{\mathbb{Q}_p} \Theta_p = \dim_{\mathbb{Q}_p} D_{dR}(W) / D_{dR}^0(W) - \dim_{\mathbb{Q}_p} D_{\text{cris}}(W)^{f=1} + \dim_{\mathbb{Q}_p} W^{G_p}.$$

For the first term we have isomorphisms

$$D_{dR}(W) = H_{dR}^2(X/\mathbb{Q}) \otimes \mathbb{Q}_p \text{ and } D_{dR}^0(W) = F_{\text{Hdg}}^1 H_{dR}^2(X/\mathbb{Q}) \otimes \mathbb{Q}_p,$$

hence the dimension of this term is equal to $\dim H^2(X, \mathcal{O}_X) = 1$.

For the second term, let $E_{\mathbb{F}_p}$ be the reduction of E at the prime p , and let $X_{\mathbb{F}_p} = E_{\mathbb{F}_p} \times E_{\mathbb{F}_p}$. By the crystalline conjecture (cp. [9] and [6]) and the

crystalline Tate conjecture (proved for abelian varieties in [21] and [13])

$$D_{cris}(W)^{f=1} \cong (H_{cris}^2(X_{\mathbb{F}_p}/W(\mathbb{F}_p)) \otimes \mathbb{Q}_p)^{\phi_p=p} \cong \text{Pic}(X_{\mathbb{F}_p}) \otimes \mathbb{Q}_p,$$

and therefore

$$\dim_{\mathbb{Q}_p} D_{cris}(W)^{f=1} = 2 + \dim_{\mathbb{Q}}(\text{End}_{\mathbb{F}_p}(E_{\mathbb{F}_p}) \otimes \mathbb{Q}) = 4.$$

For the remaining term, let $V_p(E) = H^1(\bar{E}, \mathbb{Q}_p(1))$, so that

$$\dim_{\mathbb{Q}_p} W^{G_p} = 2 + \dim_{\mathbb{Q}_p} \text{End}_{G_p}(V_p(E)).$$

Here we use that we have ordinary reduction at p . It follows that the prime p splits in the CM field K , \mathbb{Q}_p contains K , and the complex multiplication is defined over \mathbb{Q}_p . Thus $\dim_{\mathbb{Q}_p} \text{End}_{G_p}(V_p(E)) = 2$, and therefore

$$(15) \quad \dim_{\mathbb{Q}_p} \Theta_p = 1 - 4 + 4 = 1.$$

Now the combination of (6), (14) and (15) implies that $\text{CH}_{\mathbb{L}}^2(\mathfrak{X})\{p\}$ has positive corank, and completes the proof of part (a) of Theorem 1.1.

To prove $\text{CH}_{\mathbb{L}}^2(X)\{p\}$ contains a copy of $\mathbb{Q}_p/\mathbb{Z}_p$ we show that the kernel of the map

$$\text{CH}_{\mathbb{L}}^2(\mathfrak{X})\{p\} \rightarrow \text{CH}_{\mathbb{L}}^2(X)\{p\}$$

is finite. Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & \bigoplus_{v \notin S} H_{\text{ét}}^1(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) & & & \\ & & & \downarrow & & & \\ 0 & \rightarrow & H_{\mathbb{L}}^3(\mathfrak{X}, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & \text{CH}_{\mathbb{L}}^2(\mathfrak{X})\{p\} \rightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \downarrow \gamma \\ 0 & \rightarrow & H_{\mathbb{L}}^3(X, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H_{\text{ét}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & \text{CH}_{\mathbb{L}}^2(X)\{p\} \rightarrow 0 \\ & & & & \downarrow & & \\ & & & & \bigoplus_{v \notin S} H_{\text{ét}}^2(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) & & \end{array}$$

whose middle column is the localization sequence in étale cohomology. We will make again use of Mildenhall’s result [18, Theorem 5.8] that $\Sigma = \ker\{\text{CH}^2(\mathfrak{X}) \rightarrow \text{CH}^2(X)\}$ is a finite group. Since in bidegree $(3, 2)$ we can identify motivic and Lichtenbaum cohomology for both \mathfrak{X} and X , we may

rewrite the localization sequence in motivic cohomology with integral coefficients in the form

$$H_{\mathbb{L}}^3(\mathfrak{X}, \mathbb{Z}(2)) \rightarrow H_{\mathbb{L}}^3(X, \mathbb{Z}(2)) \xrightarrow{\partial} \bigoplus_{v \notin S} CH^1(Y_v) \rightarrow \Sigma \rightarrow 0.$$

From the map ‘multiplication by p^r ’ we obtain a long exact sequence

$$\Sigma[p^r] \rightarrow \text{im}(\partial)/p^r \rightarrow \bigoplus_{v \notin S} CH^1(Y_v)/p^r \rightarrow \Sigma/p^r \rightarrow 0,$$

and taking the direct limit over all r it is immediate that the map

$$\text{coker}(\alpha) = \text{im}(\partial) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \bigoplus_{v \notin S} CH^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

has finite kernel. If Y_v is a smooth closed fiber, we have from the Kummer sequence an injective map $CH^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow H_{\text{ét}}^2(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1))$ such that the diagram

$$\begin{array}{ccc} \text{coker}(\alpha) & \rightarrow & \text{coker}(\beta) \\ \downarrow & & \downarrow \text{into} \\ \bigoplus_{v \notin S} CH^1(Y_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{\text{into}} & \bigoplus_{v \notin S} H_{\text{ét}}^2(Y_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \end{array}$$

commutes. Thus the kernel of the top horizontal map is finite. Given this, it suffices to show the image of $\ker(\beta) \rightarrow \ker(\gamma)$, or equivalently, the image of

$$\bigoplus_{v \notin S} CH_{\mathbb{L}}^1(Y_v)\{p\} \rightarrow CH_{\mathbb{L}}^2(\mathfrak{X})\{p\}$$

is finite, which also follows easily from Mildenhall’s Theorem. The diagram

$$\begin{array}{ccc} \bigoplus_{v \notin S} CH^1(Y_v)\{p\} & \rightarrow & CH^2(\mathfrak{X})\{p\} \\ \cong \downarrow & & \downarrow \\ \bigoplus_{v \notin S} CH_{\mathbb{L}}^1(Y_v)\{p\} & \rightarrow & CH_{\mathbb{L}}^2(\mathfrak{X})\{p\} \end{array}$$

commutes, and the top horizontal map factors through the finite group $\Sigma\{p\}$. This proves our second claim (b), and completes the proof of Theorem 1.1. \square

Remark 3.1. If E/\mathbb{Q} does not have complex multiplication and $p > 3$ is a prime such that E has good ordinary reduction at p , it follows from the proof of [16, Lemma 4-5] that $\dim_{\mathbb{Q}_p} \Theta_p = 0$.

We prove Corollary 1.2.

Proof. (of Corollary 1.2) It suffices to show that there is an exact sequence

$$(16) \quad H_M^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{\kappa} H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow H_{\text{nr}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

Given this, we have using (A) a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & H_M^3(\mathfrak{X}, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H_M^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & \text{CH}^2(\mathfrak{X})\{p\} \rightarrow 0 \\ & & \cong \downarrow & & \kappa \downarrow & & \downarrow \\ 0 & \rightarrow & H_L^3(\mathfrak{X}, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \rightarrow & H_{\text{ét}}^3(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \rightarrow & \text{CH}_L^2(\mathfrak{X})\{p\} \rightarrow 0 \end{array}$$

and Corollary 1.2 follows from Theorem 1.1. To establish (16) we first work with finite coefficients. Consider the spectral sequence (cp. [11, §4])

$$E_{1,M}^{a,b} = \bigoplus_{x \in \mathfrak{X}^{(a)}} H_M^{b-a}(\kappa(x), \mathbb{Z}/p^r\mathbb{Z}(2-a)) \Rightarrow H_M^{a+b}(\mathfrak{X}, \mathbb{Z}/p^r\mathbb{Z}(2)),$$

where $E_{1,M}^{a,b} = 0$ for $a > 2$, since $\mathbb{Z}/p^r\mathbb{Z}(n) = 0$ for $n < 0$. Moreover, $E_{2,M}^{0,3} \subseteq H^0(X, \mathcal{H}_M^3(\mathbb{Z}/p^r\mathbb{Z}(2))) = 0$, which implies that there is a surjective map

$$(17) \quad H_M^3(\mathfrak{X}, \mathbb{Z}/p^r\mathbb{Z}(2)) \rightarrow E_{2,M}^{1,2}.$$

By construction, the group $E_{2,M}^{1,2}$ coincides with the homology of the complex

$$(18) \quad K_2(K)/p^r \rightarrow \bigoplus_{x \in \mathfrak{X}^{(1)}} \kappa(x)^\times/p^r \rightarrow \bigoplus_{x \in \mathfrak{X}^{(2)}} \mathbb{Z}/p^r\mathbb{Z};$$

here we have used that $K_i(F)/p^r \cong H_M^i(F, \mathbb{Z}/p^r\mathbb{Z}(i))$ for a field F and $i = 0, 1, 2$. There is an analogous spectral sequence for étale cohomology

$$E_{1,\text{ét}}^{a,b} = \bigoplus_{x \in \mathfrak{X}^{(a)}} H_{\text{ét}}^{b-a}(\kappa(x), \mu_{p^r}^{\otimes(2-a)}) \Rightarrow H_{\text{ét}}^{a+b}(\mathfrak{X}, \mu_{p^r}^{\otimes 2}).$$

In this spectral sequence $E_{1,\text{ét}}^{a,b} = 0$ if $a > b$, since étale cohomology vanishes in negative degrees. Note $E_{2,\text{ét}}^{0,3} = H_{\text{nr}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2})$, thus there is an exact sequence

$$(19) \quad 0 \rightarrow E_{2,\text{ét}}^{1,2} \rightarrow H_{\text{ét}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}) \rightarrow H_{\text{nr}}^3(\mathfrak{X}, \mu_{p^r}^{\otimes 2}),$$

where $E_{2,\text{ét}}^{1,2}$ is the homology of the complex

$$(20) \quad H_{\text{ét}}^2(K, \mu_{p^r}^{\otimes 2}) \rightarrow \bigoplus_{x \in \mathfrak{X}^{(1)}} H_{\text{ét}}^1(\kappa(x), \mu_{p^r}^{\otimes 1}) \rightarrow \bigoplus_{x \in \mathfrak{X}^{(2)}} H^0(\kappa(x), \mu_{p^r}^{\otimes 0})$$

Now the complexes (18) and (20) are isomorphic, since $K_i(F)/p^r \cong H_{\text{ét}}^i(F, \mu_{p^r}^{\otimes i})$ for a field F and $i = 0, 1, 2$. Thus $E_{2,M}^{1,2} \cong E_{2,\text{ét}}^{1,2}$, and we obtain from (17) and (19) the exact sequence $H_M^3(\mathcal{X}, \mathbb{Z}/p^r\mathbb{Z}(2)) \rightarrow H_{\text{ét}}^3(\mathcal{X}, \mu_{p^r}^{\otimes 2}) \rightarrow H_{\text{nr}}^3(\mathcal{X}, \mu_{p^r}^{\otimes 2})$; given this, the exact sequence (16) follows by taking the evident colimit.

For the second statement of the Corollary note that there is an inclusion

$$H_{\text{nr}}^3(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \subseteq H_{\text{nr}}^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

□

Remark 3.2. Let X be a smooth quasi-projective variety over a field k . From [12, Proposition 2.9] we have the exact sequence

$$0 \rightarrow \text{CH}^2(X) \rightarrow \text{CH}_L^2(X) \rightarrow H_{\text{nr}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow 0,$$

and Theorem 1.1 also implies that for the surface X/\mathbb{Q} the groups $\text{CH}^2(X)$ and $\text{CH}_L^2(X)$ differ by an infinite group; however, if $\bar{X} = X \times_{\mathbb{Q}} \bar{\mathbb{Q}}$, then $\text{CH}^2(\bar{X}) = \text{CH}_L^2(\bar{X})$.

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