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# Best bounds for the Hilbert transform on $L^p(\mathbb{R}^1)$: A corrigendum

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We fix a point left unexplained in [1] by a small modification of the proof.

In the text after (5) in [1] for $R > 100$ define $C_R$ as the circle centered at $iR$ with radius $R' = R - 1/R$, but define a (smaller) lower arc $C_R^L = \left\{iR + R'e^{i\phi} : \frac{5\pi}{4} \leq \phi \leq \frac{7\pi}{4}\right\}$ and a (larger) upper arc $C_R^U = C_R \setminus C_R^L$. Then (6) holds as stated in [1] because of the subharmonicity of $g(u(z), v(z))$. Observe that the following stronger version of (7) holds for some constant $C_f$ (depending only on the Schwartz function $f$):

$$|u(x+iy) + iv(x+iy)| \leq \frac{C_f}{1+|x|+|y|}.$$

This is routine to check for $|x| \geq 2A$, where $[-A, A]$ contains the support of f. Also, for $|x| \leq 2A$ we have

$$\begin{aligned}
|u(x+iy) + iv(x+iy)| &= \frac{1}{\pi}\left|\int_{-\infty}^{\infty} \frac{f(x-t)}{t+iy} dt\right| \\
&= \frac{1}{\pi}\left|\int_{-(A+|x|)}^{A+|x|} \frac{f(x-t) - f(x)}{t+iy} + f(x)\frac{t-iy}{t^2+y^2}\, dt\right| \\
&\leq C'\left(\int_{|t|\leq 3A} \frac{|t|}{|t|+|y|} dt + 2\tan^{-1}\frac{3A}{|y|}\right) \\
&\leq \frac{C''}{1+|y|} \leq \frac{C_f}{1+|x|+|y|}.
\end{aligned}$$

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We conclude from this that

$$(*)\qquad \begin{aligned} |g(u(x+iy),v(x+iy))| &\le |u(x+iy)+iv(x+iy)|^p \\ &\le \left(\frac{C_f}{1+|x|+|y|}\right)^p, \end{aligned}$$

which clearly implies the following versions of (8) and (9) in [1]:

$$\begin{aligned} R'\,|g(u(iR),v(iR))| &\le R'\frac{C}{(1+R)^p} \to 0 \qquad \text{as } R\to\infty, \\ \left|\int_{C_R^U} g(u(z),v(z))ds\right| &\le R'\frac{C}{(1+R)^p} \to 0 \qquad \text{as } R\to\infty. \end{aligned}$$

In view of (6), (8), and (9) in order to obtain (3), it will suffice to show that the integral of $g(u(z),v(z))$ over $C_R^L$ tends to the left hand side of (3). This will be a consequence of the Lebesgue dominated convergence theorem combined with the fact that $C_R^L$ is strictly smaller than half circle. Using parametric equations, the integral $\int_{C_R^L} g(u(z),v(z))ds$ is equal to

$$\int_{\frac{-R'\sqrt{2}}{2}}^{\frac{R'\sqrt{2}}{2}} \frac{g\left(u\left(x+iR-iR'\sqrt{1-\frac{x^2}{R'^2}}\right),v\left(x+iR-iR'\sqrt{1-\frac{x^2}{R'^2}}\right)\right)}{\sqrt{1-\frac{x^2}{R'^2}}}\,dx\,.$$

In view of $(*)$, for all $R>100$, the preceding integrand is bounded by the integrable function $C_f\sqrt{2}(1+|x|)^{-p}$, since $\sqrt{1-\frac{x^2}{R'^2}}$ is bounded below by $\frac{1}{\sqrt{2}}$ in the range of integration. The Lebesgue dominated convergence theorem, therefore, implies that the previously displayed expressions converge to

$$\int_{-\infty}^{\infty} g\left(u\left((x+i0,v\left(x+i0\right)\right)\right)dx = \int_{-\infty}^{\infty} \mathrm{Re}\left[(|f(x)|+iHf(x))^p\right]\,dx,$$

as $R\to\infty$, since for almost all $x$ we have

$$\lim_{z\to x} g(u(z),v(z)) = g(u(x+i0),v(x+i0)) = \mathrm{Re}\left[(|f(x)|+iHf(x))^p\right].$$

This proves (3) and completes the proof.

## References

[1] L. Grafakos, *Best bounds for the Hilbert transform on $L^p(R^1)$*. Math. Res. Let., **4** (1997), no. 4, 469–471.

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