Math. Res. Lett.
Volume 22, Number 4, 1243–1253, 2015

A lower bound for the nodal sets of Steklov eigenfunctions

Xing Wang and Jiuyi Zhu

We consider the lower bound of nodal sets of Steklov eigenfunctions on smooth Riemannian manifolds with boundary–the eigenfunctions of the Dirichlet-to-Neumann map. Let N_λ be its nodal set. Assume that zero is a regular value of Steklov eigenfunctions. We show that

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

for some positive constant C depending only on the manifold.

1. Introduction

In this paper, we consider the lower bound estimates for nodal sets

$$N_\lambda = \{x \in \mathcal{M} \mid \phi_\lambda = 0\}$$

of the Steklov eigenfunctions on a smooth Riemannian manifold $(\mathcal{N}, h)$ with boundary $(\mathcal{M}, g)$, where $dim\mathcal{N} = n + 1$ and $h|_{\mathcal{M}} = g$. The Steklov eigenvalue problem is formulated as

$$\begin{cases} \triangle_h \phi_\lambda(x) = 0, & x \in \mathcal{N}, \\ \frac{\partial \phi_\lambda}{\partial \nu}(x) = \lambda\phi_\lambda(x), & x \in \partial\mathcal{N} = \mathcal{M}. \end{cases}$$

Here, ν is an unit outer normal vector on $\mathcal{M}$. The Steklov eigenvalues can also be reduced to the boundary $\mathcal{M}$. Then the ϕ_λ becomes the eigenfunction

Mathematics Subject Classification: 58C40, 28A78, 35P15, 35R01.
Key words and phrases: Nodal sets, Lower bound, Dirichlet-to-Neumann map, Steklov eigenfunctions.

of Dirichlet-to-Neumann operator, i.e.

$$\Lambda\phi_\lambda = \lambda\phi_\lambda.$$

The Dirichlet-to-Neumann operator Λ is defined as

$$\Lambda f = \frac{\partial}{\partial\nu}(Hf)|_{\mathcal{M}}$$

for $f \in H^{\frac{1}{2}}(\mathcal{M})$. Hf is the harmonic extension of f, i.e.

$$\begin{cases} \triangle_h u(x) = 0, & x \in \mathcal{N}, \\ u(x) = f(x), & x \in \partial\mathcal{N} = \mathcal{M} \end{cases}$$

with $u = Hf$. Moreover, the operator Λ is a self-adjoint operator from $H^{\frac{1}{2}}(\mathcal{M})$ to $H^{-\frac{1}{2}}(\mathcal{M})$ and there exists an orthonormal basis $\{\phi_j\}$ of eigenfunctions such that

$$\Lambda\phi_j = \lambda_j\phi_j, \quad \phi_j \in C^\infty(\mathcal{M}), \quad \int_{\mathcal{M}} \phi_j\phi_k\, dV_g = \delta_{jk}.$$

The eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \ldots,$ are ordered in ascending order with counted multiplicity. For simplicity, we choose $n+1$ as the dimension of $\mathcal{N}$, which is a little bit different from the previous work by [1] and [16].

The nodal sets are zero level sets of eigenfunctions. We want to study the asymptotical behavior of the size of nodal sets of Steklov eigenfunctions for large λ. Recently, some remarkable progresses have been made for the upper bound of the size of nodal sets for analytic manifolds. Bellova and Lin [1] proved that if $\mathcal{N}$ is an analytic domain in $\mathbb{R}^n$, then the H^{n-2}-Hausdorff measure of nodal sets of Steklov eigenfunctions has an upper bound of $C\lambda^6$ with C depending only on $\mathcal{N}$. Later on, Zelditch [16] improved their results and showed that the optimal upper bound for the nodal sets is $C\lambda$ for real analytic manifolds. The optimality can be seen from the case that the manifold is a ball.

So far, nothing seems to be known for the lower bound of the nodal sets of Steklov eigenfunctions, even for analytic manifolds. The main goal of our paper is to address the lower bound of nodal sets over general compact smooth manifolds. Quite different from the case for the Laplacian-Beltrami operator, the Dirichlet-to-Neumann operator is a non-local operator, which causes additional difficulty. Fortunately, since we are measuring the whole size of the nodal sets which can be considered as “partial global” quantity, we are able to find a way to overcome the difficulty and carry the argument through.

Let's first briefly review the literature concerning the nodal sets of classical eigenfunctions. Let ϕ_λ be an L^2 normalized eigenfunctions of Laplacian-Beltrami on compact manifold $(\mathcal{M}, g)$ without boundary,

$$-\triangle_g \phi_\lambda = \lambda^2 \phi_\lambda.$$

Yau conjectured that for any smooth manifolds, one should control the upper and lower bound of nodal sets of classical eigenfunctions as

$$c\lambda \leq H^{n-1}(N_\lambda) \leq C\lambda$$

where C, c depends only on the manifold $\mathcal{M}$. The conjecture is only verified for real analytic manifold by Donnelly-Fefferman in [5]. For the smooth manifolds, the conjecture is still not settled. Much progresses have been obtained towards the lower bound of nodal sets. Colding and Minicozzi [3], Sogge and Zelditch [12], [13] independently obtained that

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

for smooth manifolds. See also [7] for deriving the same bound by adapting the idea in [12]. For other related works about lower bounds of nodal sets of classical eigenfunctions, see [2], [8], [6], etc, to just mention a few. The methods in [3] and [12] are quite different. Specially, the method in [12] is based on a Dong-type identity in [4] about L^1 norm of $|\nabla \phi_\lambda|$ on the nodal set and the L^1 norm of ϕ_λ on $\mathcal{M}$. Our goal is to adapt their idea to the setting of non-local operator, i.e. Steklov eigenfunctions.

Theorem 1. *Let ϕ_λ be a normalized Steklov eigenfunction and* 0 *be a regular value of ϕ_λ. Then there exists a positive constant C depending only on $\mathcal{N}$ such that*

$$H^{n-1}(N_\lambda) \geq C\lambda^{\frac{3-n}{2}}.$$

With a small modification of the proof, we are also able to get similar lower bounds for general level sets near 0. Denote α-level sets of Steklov eigenfunctions as $L^\alpha_\lambda = \{x \in \mathcal{M} \mid \phi_\lambda = \alpha\}$.

Corollary 1. *Let α be a regular value of ϕ_λ. There exists a positive constant $\epsilon(\mathcal{N})$ such that, for $|\alpha| < \epsilon(\mathcal{N})\lambda^{-\frac{n-1}{4}}$,*

$$H^{n-1}(L_\lambda^\alpha) \geq C\lambda^{\frac{3-n}{2}}$$

with C depending only on $\mathcal{N}$.

2. Preliminaries

In this section, we will review and prepare some general results needed in the proof of Theorem 1. First, we need the following result from [14].

Lemma 1. *The Dirichlet-to-Neumann operator Λ is an elliptic self-adjoint pseudodifferential operator of order 1 over $\mathcal{M}$. Moreover,*

$$\Lambda = \sqrt{-\triangle_g} \ \textit{mod}\ OPS^0(\mathcal{M}).$$

Here, OPS^m denotes the pseudodifferential operator of order m. Since Λ is an elliptic self-adjoint pseudodifferential operator, by the general results in [9] (see also the book of Sogge [11] or [10] for Laplacian-Beltrami operator), we have the following L^p norm estimates.

Lemma 2. *Let ϕ_λ be the Steklov eigenfunction. One has the estimates, for $p \geq 2$,*

$$\|\phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^{\sigma(n,p)} \|\phi_\lambda\|_{L^2(\mathcal{M})}, \tag{2.1}$$

where

$$\sigma(n,p) = \begin{cases} n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty, \\ \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq \frac{2(n+1)}{n-1}. \end{cases}$$

In the whole paper, the notation $A \lesssim B$ or $A \gtrsim B$ denotes $A \leq CB$ or $A \geq CB$ for some generic constant C which does not depend on λ. If we follow exactly the same argument as [12], which makes use of Lemma 2 for $p = \infty$, we can obtain L^p norm estimates for $p = 1$, that is,

$$\|\phi_\lambda\|_{L^1(\mathcal{M})} \gtrsim (1+\lambda)^{-\frac{n-1}{4}} \|\phi_\lambda\|_{L^2(\mathcal{M})}. \tag{2.2}$$

We also need the L^p bounds for the pseudodifferential operators.

Lemma 3. *Suppose $P \in OPS^m(\mathcal{M})$. Then*

$$\|P\phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^m \|\phi_\lambda\|_{L^p(\mathcal{M})}, \quad \forall 1 < p < \infty.$$

Specially,

$$\|\nabla_g^m \phi_\lambda\|_{L^p(\mathcal{M})} \lesssim (1+\lambda)^m \|\phi_\lambda\|_{L^p(\mathcal{M})}.$$

Proof. Define the operator $\tilde{P} := P(1+\Lambda)^{-m}$, then $\tilde{P} \in OPS^0(\mathcal{M})$. By the boundedness of zeroth pseudodifferential operator over $L^p(\mathcal{M})$ in [11] or [14], the lemma follows easily. □

3. Lower bounds of nodal sets

In this section, we will obtain the lower bounds of nodal sets of Steklov eigenfunctions. Since the Dirichlet-to-Neumann operator is a non-local operator, we do not need information from the manifold $(\mathcal{N}, h)$. In the following argument, all derivatives and calculations are performed with respect to the manifold $(\mathcal{M}, g)$. We first express the manifold $\mathcal{M}$ as the disjoint union

$$\mathcal{M} = \bigcup_{j=1}^{N_+(\lambda)} D_{j,+} \cup \bigcup_{j=1}^{N_-(\lambda)} D_{j,-} \cup N_\lambda,$$

where $D_{j,+}$ and $D_{j,-}$ are the connected components of the sets $\{x \in \mathcal{M} \mid \phi_\lambda > 0\}$ and $\{x \in \mathcal{M} \mid \phi_\lambda < 0\}$. Using the same idea in [12], we can treat each component separately and then add them up. For simplicity, we just deal with two components. The same argument carries out for many components. Denote

$$D_+ = \{x \in \mathcal{M} \mid \phi_\lambda(x) > 0\}$$

and

$$D_- = \{x \in \mathcal{M} \mid \phi_\lambda(x) < 0\}.$$

For the classical eigenfunctions of Laplacian-Beltrami operator, the singular set is codimension 2. Then zero level sets are smooth submanifolds. It is also shown in [15] that 0 is a regular value for eigenfunctions of second order elliptic differential operators. To the best of the authors' knowledge, it is still unknown whether it is true for Dirichlet-to-Neumann operators. By the Sard's theorem, it is known that almost every level set is regular. Since 0 is assumed to be a regular value of ϕ_λ, then N_λ is a smooth submanifold in $\mathcal{M}$. Note that the boundary $\partial D_{\pm} = N_\lambda$. By the Gauss-Green formula, for any $f \in C^\infty(\mathcal{M})$, we have

$$\int_{D_\pm} div(f\nabla\phi_\lambda)\, dv_g = \int_{N_\lambda} \langle f\nabla\phi_\lambda, \nu\rangle\, ds \tag{3.1}$$

where ds is the surface measure on N_λ induced by the metric g on $\mathcal{M}$, ν is the exterior unit normal vector on N_λ with respect to $D_\pm$, respectively. Note the Gauss-Green formula is taken on $\mathcal{M}$ with metric g. Since $\phi_\lambda = 0$ on N_λ, then $\langle\nabla\phi_\lambda, \nu\rangle = \pm|\nabla\phi_\lambda|$ on N_λ. Thus, (3.1) becomes

$$\int_{D_+} div(f\nabla\phi_\lambda)\, dv_g = -\int_{N_\lambda} f|\nabla\phi_\lambda|\, ds. \tag{3.2}$$

Similarly, we have

$$\int_{D_-} div(f\nabla\phi_\lambda)\, dv_g = \int_{N_\lambda} f|\nabla\phi_\lambda|\, ds. \tag{3.3}$$

By (3.2) and (3.3), we obtain

$$2\int_{N_\lambda} f|\nabla\phi_\lambda|\, ds = \int_{D_-} div(f\nabla\phi_\lambda)\, dv_g - \int_{D_+} div(f\nabla\phi_\lambda)\, dv_g. \tag{3.4}$$

To obtain a lower bound of nodal sets of Steklov eigenfunctions, we need to choose some appropriate test functions. It turns out that $f \equiv 1$ and $f = \sqrt{1 + |\nabla\phi_\lambda|^2}$ are good choices. Let $f \equiv 1$. We are able to establish the following proposition.

Proposition 1. *There exists a positive constant $K(\mathcal{N})$ such that, for $\lambda > K(\mathcal{N})$,*

$$\int_{N_\lambda} |\nabla\phi_\lambda|\, ds \geq \frac{\lambda^2}{4}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Proof. Set $f = 1$ in (3.4), we have

$$2\int_{N_\lambda} |\nabla\phi_\lambda|\, ds = \int_{D_-} \triangle\phi_\lambda\, dv_g - \int_{D_+} \triangle\phi_\lambda\, dv_g. \tag{3.5}$$

From Lemma 1, we know that

$$\sqrt{-\triangle_g} = \Lambda + P_0,$$

where $P_0 \in OPS^0(\mathcal{M})$. It follows that

$$-\triangle = \Lambda^2 + P_1 + P_0^2,$$

where $P_1 = \Lambda P_0 + P_0 \Lambda \in OPS^1(\mathcal{M})$. Therefore,

$$\triangle \phi_\lambda = -(\Lambda^2 + P_1 + P_0^2)\phi_\lambda = -\lambda^2 \phi_\lambda - P_1 \phi_\lambda - P_0^2 \phi_\lambda.$$

Substituting the latter identity into (3.5) implies that

$$\begin{aligned} 2\int_{N_\lambda} |\nabla \phi_\lambda|\, ds &= -\int_{D_-} \lambda^2 \phi_\lambda - \int_{D_-} P_1 \phi_\lambda - \int_{D_-} P_0^2 \phi_\lambda \\ &\quad + \int_{D_+} \lambda^2 \phi_\lambda + \int_{D_+} P_1 \phi_\lambda + \int_{D_+} P_0^2 \phi_\lambda \\ &\geq -\lambda^2 \int_{D_-} \phi_\lambda + \lambda^2 \int_{D_+} \phi_\lambda - \int_{\mathcal{M}} |P_1 \phi_\lambda| - \int_{\mathcal{M}} |P_0^2 \phi_\lambda| \\ &= \lambda^2 \|\phi_\lambda\|_{L^1(\mathcal{M})} - (1+\lambda)\|\tilde{P}_0 \phi_\lambda\|_{L^1(\mathcal{M})} - \|P_0^2 \phi_\lambda\|_{L^1(\mathcal{M})}, \end{aligned} \tag{3.6}$$

where $\tilde{P}_0 = P_1(1+\Lambda)^{-1} \in OPS^0(\mathcal{M})$. Now there are two "bad" terms in (3.6):

$$\|\tilde{P}_0 \phi_\lambda\|_{L^1(\mathcal{M})}, \quad \|P_0^2 \phi_\lambda\|_{L^1(\mathcal{M})}.$$

We are able to control them by the L^1 norm of ϕ_λ multiplied by an ϵ power of λ. We can establish the following lemma.

Lemma 4. *Let $P \in OPS^0(\mathcal{M})$. Then for any positive constant ϵ, there exists a positive constant $C = C(\mathcal{N}, \epsilon)$ such that*

$$\|P\phi_\lambda\|_{L^1(\mathcal{M})} \leq C\lambda^\epsilon \|\phi_\lambda\|_{L^1(\mathcal{M})}. \tag{3.7}$$

Proof. Let $\delta > 0$. By Hölder's inequality,

$$\|P\phi_\lambda\|_{L^1(\mathcal{M})} \lesssim \|P\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} \lesssim \|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})},$$

where we have used Lemma 3. As we know,

$$\begin{aligned} \|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} &\leq \|\phi_\lambda\|_{L^\infty(\mathcal{M})}^{\frac{\delta}{1+\delta}} \|\phi_\lambda\|_{L^1(\mathcal{M})}^{\frac{1}{1+\delta}} \\ &= \left(\frac{\|\phi_\lambda\|_{L^\infty(\mathcal{M})}}{\|\phi_\lambda\|_{L^1(\mathcal{M})}} \right)^{\frac{\delta}{1+\delta}} \|\phi_\lambda\|_{L^1(\mathcal{M})}. \end{aligned} \tag{3.8}$$

Thanks to Lemma 2, we know

$$\|\phi_\lambda\|_{L^\infty(\mathcal{M})} \lesssim \lambda^{\frac{n-1}{2}}.$$

By (2.2),

$$\|\phi_\lambda\|_{L^1(\mathcal{M})} \gtrsim \lambda^{-\frac{n-1}{4}}.$$

Thus, from (3.8), we have

$$\|\phi_\lambda\|_{L^{1+\delta}(\mathcal{M})} \lesssim \lambda^{\frac{3(n-1)\delta}{4(1+\delta)}} \|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Selecting δ so small that $\frac{3(n-1)\delta}{4(1+\delta)} \leq \epsilon$, then the proposition is shown. □

With aid of Lemma 4, we continue the proof of Proposition 1. Let's go back to (3.6). We want to control the other two "bad" terms in the right hand side of (3.6) by $\lambda^2\|\phi_\lambda\|_{L^1(\mathcal{M})}$. In order to achieve it, one applies (3.7) and chooses $0 < \epsilon < 1$. Since (3.7) holds for any positive constant ϵ, for instance, we may choose $\epsilon = 1/2$. Then we obtain

$$2\int_{N_\lambda} |\nabla\phi_\lambda|\, ds \geq \lambda^2\|\phi_\lambda\|_{L^1(\mathcal{M})} - C(1+\lambda)^{\frac{3}{2}}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

Choosing λ appropriately large which depends only on $\mathcal{N}$, we finally arrive at

$$2\int_{N_\lambda} |\nabla\phi_\lambda|\, ds \geq \frac{\lambda^2}{2}\|\phi_\lambda\|_{L^1(\mathcal{M})}.$$

We are done with the proof of Proposition 1. □

Next we select the test function as $f = \sqrt{1+|\nabla\phi_\lambda|^2}$. We are able to prove the following proposition.

Proposition 2. *There exists a positive constant $C = C(\mathcal{N})$ such that*

$$\int_{N_\lambda} |\nabla\phi_\lambda|^2\, ds \leq C(1+\lambda)^3. \tag{3.9}$$

Proof. Let $f = \sqrt{1+|\nabla\phi_\lambda|^2}$ in (3.4). We derive that

$$\begin{aligned} 2\int_{N_\lambda} \sqrt{1+|\nabla\phi_\lambda|^2}|\nabla\phi_\lambda|\,ds &= \int_{D_-} div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)\,dv_g \\ &\quad - \int_{D_+} div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)\,dv_g \\ &\le \int_{\mathcal{M}} |div(\sqrt{1+|\nabla\phi_\lambda|^2}\nabla\phi_\lambda)|\,dv_g \\ &\lesssim \int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{-1/2}|\nabla^2\phi_\lambda||\nabla\phi_\lambda|^2\,dv_g \\ &\quad + \int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{1/2}|\triangle\phi_\lambda|\,dv_g. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} \int_{N_\lambda} |\nabla\phi_\lambda|^2\,ds &\lesssim \int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2)^{1/2}|\nabla^2\phi_\lambda|\,dv_g \\ &\lesssim \left(\int_{\mathcal{M}} (1+|\nabla\phi_\lambda|^2\,dv_g\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} |\nabla^2\phi_\lambda|^2\,dv_g\right)^{\frac{1}{2}} \\ &\lesssim (1+\lambda)^3, \end{aligned} \tag{3.10}$$

where Lemma 3 has been used in last inequality. □

We are ready to give the proof of Theorem 1. We use an idea in [7] by Hezari and Sogge.

Proof of Theorem 1. On one hand, by Proposition 2,

$$\int_{N_\lambda} |\nabla\phi_\lambda|\,ds \le \left(\int_{N_\lambda} |\nabla\phi_\lambda|^2\,ds\right)^{\frac{1}{2}} |N_\lambda|^{\frac{1}{2}} \lesssim \lambda^{\frac{3}{2}}|N_\lambda|^{\frac{1}{2}}. \tag{3.11}$$

On the other hand, from Proposition 1, we have

$$\int_{N_\lambda} |\nabla\phi_\lambda|\,ds \ge \frac{\lambda^2}{4}\|\phi_\lambda\|_{L^1(\mathcal{M})} \gtrsim \lambda^{2-\frac{n-1}{4}}, \tag{3.12}$$

where we have used (2.2) in the last inequality. Combining the estimates (3.11) and (3.12), we arrive at

$$|N_\lambda| \gtrsim \lambda^{\frac{3-n}{2}}.$$

□

Acknowledgements

It is our pleasure to thank Professor Christopher D. Sogge for many fruitful discussions throughout the preparation of this work. We appreciate his insightful and useful comments, which helped to improve this paper much. We also thank referees for constructive comments. The research of J. Zhu is partially supported by National Science Foundation grant DMS-1500468.

References

[1] K. Bellova and F. H. Lin, *Nodal sets of Steklov eigenfunctions.* `arXiv:1402.4323`.

[2] J. Brüning, *Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators.* Math. Z., **158**(1978), 15–21.

[3] T. H. Colding and W. P. Minicozzi II, *Lower bounds for nodal sets of eigenfunctions.* Comm. Math. Phys., **306**(2011), 777–784.

[4] R.-T. Dong, *Nodal sets of eigenfunctions on Riemann surfaces.* J. Differential Geom., **36**(1992), 493–506.

[5] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds.* Invent. Math., **93**(1988), 161–183.

[6] Q. Han and F. H. Lin, *Nodal sets of solutions of Elliptic Differential Equations.* Book in preparation (online at `http://www.nd.edu/qhan/nodal.pdf`).

[7] H. Hezari and C. D. Sogge, *A natural lower bound for the size of nodal sets.* Anal. PDE., **5**(2012), no. 5, 1133–1137.

[8] D. Mangoubi, *A remark on recent lower bounds for nodal sets.* Comm. Partial Differential Equations, **36**(2011), no. 12, 2208–2212.

[9] A. Seeger and C. D. Sogge, *Bounds for eigenfunctions of differential operators.* Indiana Math. J., **38**(1989), 669–682.

[10] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds.* J. Funct. Anal., **77**(1988), 123–138.

[11] C. D. Sogge, *Fourier integrals in classical analysis.* Cambridge Tracts in Mathematics, **105**, Cambridge University Press, Cambridge, 1993.

[12] C. D. Sogge and S. Zelditch, *Lower bounds on the Hausdorff measure of nodal sets*. Math. Res. Lett., **18**(2011), 25–37.

[13] C. D. Sogge and S. Zelditch, *Lower bounds on the Hausdorff measure of nodal sets II*. Math. Res. Lett., **19**(2012), no. 6, 1361–1364.

[14] M. Taylor, *Partial differential equations II: Qualitative studies of linear equation*. Applied mathematicsal Sciences, **116**, Springer-Verlag, New York, 1996.

[15] K. Uhlenbeck, *Genetric properties of eigenfunctions*. Amer. J. Math., **98**(1976), no. 4, 1059–1078.

[16] S. Zelditch, *Measure of nodal sets of analytic steklov eigenfunctions*. `arXiv:1403.0647`.

Department of Mathematics, Johns Hopkins University
Baltimore, MD 21218, USA
E-mail address: `xwang@math.jhu.edu`

Department of Mathematics, Johns Hopkins University
Baltimore, MD 21218, USA
E-mail address: `jzhu43@math.jhu.edu`

Received November 14, 2014