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Loop-fusion cohomology and transgression

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'Loop-fusion cohomology' is defined on the continuous loop space of a manifold in terms of Čech cochains satisfying two multiplicative conditions with respect to the fusion and figure-of-eight products on loops. The main result is that these cohomology groups, with coefficients in an abelian group, are isomorphic to those of the manifold and the transgression homomorphism factors through the isomorphism.

In this note we present a refined Čech cohomology of the continuous free loop space $\mathcal{L}M = \mathcal{C}(\mathbb{S}; M)$ of a manifold M. We could instead work throughout with finite energy loops, i.e. Sobolev maps from the circle of order 1. Compared to the standard theory, the cochains are limited by multiplicativity conditions under two products on loops, the fusion product (defined by Stolz and Teichner [8]) and the figure-of-eight product (which appears implicitly in Barrett [1] and explicitly in [6]). The main result of this paper is that the resulting 'loop-fusion' cohomology, $\check{H}^{\bullet}_{\mathrm{lf}}(\mathcal{L}M; A)$, recovers the cohomology of the manifold directly on the loop space.

Figure 1: Fusion (a) and figure-of-eight (b) configurations.

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Theorem. *For each $k \geq 1$ and discrete abelian group A there is an enhanced transgression isomorphism*

$$T_{\mathrm{lf}} : \check{H}^k(M; A) \xrightarrow{\cong} \check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A),$$

forming a commutative diagram with the forgetful map, f, to ordinary cohomology and the standard transgression map T :

$$\begin{array}{ccc} \check{H}^k(M; A) & \xrightarrow{T_{\mathrm{lf}}} & \check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A) \\ & \searrow^{T} & \downarrow f \\ & & \check{H}^{k-1}(\mathcal{L}M; A). \end{array} \tag{1}$$

For $A = \mathbb{Z}$ and $k = 2$ or $k = 3$ this result appears in [6]. There the cohomology classes are represented geometrically by functions and circle bundles over the loop space which satisfy the fusion property and are reparameterization equivariant; the figure-of-eight property follows from these conditions.

The case $k = 2$ with integer coefficients is closely related to the problem of recovering a circle bundle on M up to isomorphism from its holonomy as a function on $\mathcal{L}M$, which has been considered by Teleman [9], Barrett [1] and Caetano-Picken [4]. In [12], Waldorf considers principal bundles for general abelian groups and makes explicit use of the fusion product. The case $k = 3$ corresponds to an association between gerbes on M and circle bundles on $\mathcal{L}M$. Such a construction was first given by Brylinski [2], and in [3], Brylinski and McLaughlin point out that the resulting bundle on the smooth loop space has an action by $\mathrm{Diff}(\mathbb{S})$ and is multiplicative with respect to the composition of loops based at the same point. In [10], [11] Waldorf identifies the fusion property for bundles on $\mathcal{L}M$ given by the transgression of gerbes, and uses this to define an inverse functor.

The extension of such results to $k \geq 3$ to give an explicit transgression of geometric objects, such as higher gerbes, faces the usual obstacles associated with compatibility conditions. Here, the use of Čech cohomology allows for a short and unified treatment of the general case. In particular this shows that the two conditions included in the loop-fusion structure, without equivariance under the rotation action on loops or thin homotopy equivalence, suffice to capture the cohomology of M.

1. Spaces, covers and Čech cohomology

1.1. Base space

Let M be a compact smooth manifold. In the subsequent discussion we fix a Riemann metric on M and $\epsilon > 0$ smaller than the injectivity radius although refinement arguments show that none of the results depend on these choices. For each $m \in M$ let U_m be the open geodesic ball of radius $\epsilon > 0$ centered at m and consider the disjoint union of these balls as a cover of M :

$$\mathcal{U} = \bigsqcup_{m \in M} U_m \longrightarrow M.$$

This is a good cover: for $k \geq 1$, each of the k-fold intersections is empty or contractible. The disjoint union of these intersections is equivalent to the fiber product

$$\mathcal{U}^{(k)} = \mathcal{U} \times_M \cdots \times_M \mathcal{U} \longrightarrow M.$$

Remark 1. It is convenient to work with 'maximal' covers parameterized by the space itself. However it is possible throughout the discussion below to restrict to countable covers as is more conventional in Čech theory. Indeed, one can work here with the cover of M by neighborhoods with centers at a countable dense subset. See the subsequent remark on paths and loops.

Also, though we have assumed M to be smooth and compact for convenience, the result we present applies to a wider variety of spaces. Indeed, we only use that M has a good cover, with respect to which there are compatible good covers of the path and loop spaces as below.

The collection $\{M^n : n \geq 1\}$ forms a simplicial space with the projections $\pi_i : M^n \longrightarrow M^{n-1}$, $1 \leq i \leq n$, as face maps with the convention that π_i omits the ith factor. Similarly $\{\mathcal{U}^n : n \geq 1\}$ is a simplicial space, with face maps also denoted π_i; each $\mathcal{U}^n \longrightarrow M^n$ is also a good cover. Differentials deriving from this simplicial structure will be denoted by ∂.

For each fixed n the successive fiber products $\left\{(\mathcal{U}^{(k)})^n \equiv (\mathcal{U}^n)^{(k)} : k \geq 1\right\}$ also form a simplicial space with face maps $\iota_j : (\mathcal{U}^n)^{(k+1)} \longrightarrow (\mathcal{U}^n)^{(k)}$ the inclusions of $(k+1)$-fold intersections of the open sets into the k-fold intersections. This second simplicial space underlies the Čech cohomology of M^n. Indeed, for an abelian group A the Čech cochains on M^n with respect to $\mathcal{U}^n$

are the locally constant maps

$$\check{C}^k(M^n; A) \ni \alpha : (\mathcal{U}^n)^{(k+1)} \longrightarrow A, \quad k \in \mathbb{N},$$

with differential

$$\begin{aligned} &\delta : \check{C}^k(M^n; A) \longrightarrow \check{C}^{k+1}(M^n; A), \\ &\delta\alpha = \prod_{j=1}^{k+2} \iota_j^* f^{(-1)^j} : (\mathcal{U}^n)^{(k+2)} \longrightarrow A. \end{aligned} \tag{2}$$

Note that these are *unoriented* Čech cochains, so that α is not required to be odd with respect to permutations acting on the fiberwise factors of $\mathcal{U}^{(k)} \longrightarrow M$.

For a good cover such as $\mathcal{U}^n$, the Čech cohomology is isomorphic to the ordinary cohomology of M^n [5]:

$$\check{H}^\bullet(M^n; A) := H^\bullet(\check{C}^\bullet(M^n; A), \delta) \cong H^\bullet(M^n; A).$$

Lemma 1.1. *For each k, the sequence*

$$\check{H}^k(M; A) \xrightarrow{\partial} \check{H}^k(M^2; A) \xrightarrow{\partial} \check{H}^k(M^3; A) \xrightarrow{\partial} \cdots$$
$$\partial : \check{H}^k(M^n; A) \ni \alpha \longrightarrow \prod_{j=1}^{n+1} \pi_j^* \alpha^{(-1)^j} \in \check{H}^k(M^{n+1}; A)$$

is exact.

Proof. The same computation as for the Čech differential (2) shows that ∂^2 is trivial. Fix a point $\bar{m} \in M$ and consider the inclusions

$$i_n : M^n \hookrightarrow M^{n+1}, \quad (m_1, \ldots, m_n) \longmapsto (\bar{m}, m_1, \ldots, m_n).$$

Then

$$\pi_j \circ i_n = \begin{cases} \mathrm{Id} & j = 1, \\ i_{n-1} \circ \pi_{j-1} & j \geq 2, \end{cases}$$

as maps from M^n to M^n and for $\alpha \in \check{H}^k(M^n; A)$,

$$i_n^* \partial\alpha = \prod_{j=1}^{n+1} i_n^* \pi_j^* \alpha^{(-1)^j} = \alpha^{-1} \left(\partial i_{n-1}^* \alpha^{-1}\right).$$

Thus if $\partial\alpha = 1$ then $\alpha = \partial i_{n-1}^* \alpha^{-1}$. □

1.2. Path space

Let $\mathcal{I}M = \mathcal{C}([0,1];M)$ be the free continuous path space of M; it is a Banach manifold which fibers over M^2 by the endpoint map

$$\varepsilon : \mathcal{I}M \longrightarrow M^2, \quad \varepsilon(\gamma) = \big(\gamma(0), \gamma(1)\big).$$

We make use of the *join product*

$$\begin{gathered} j : \pi_3^*\mathcal{I}M \times_{M^3} \pi_1^*\mathcal{I}M \longrightarrow \pi_2^*\mathcal{I}M, \\ j(\gamma_1,\gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2, \\ \gamma_2\big(2(t-1/2)\big) & 1/2 \le t \le 1, \end{cases} \end{gathered} \tag{3}$$

where $(\gamma_1,\gamma_2) \in \pi_3^*\mathcal{I}M \times_{M^3} \pi_1^*\mathcal{I}M$ if and only if $\gamma_1(1) = \gamma_2(0)$. Note that (3) is a bijection and hence $\pi_3^*\mathcal{I}M \times_{M^3} \pi_1^*\mathcal{I}M$ can be identified with $\mathcal{I}M$ fibering over M^3 by the map $\gamma \longmapsto (\gamma(0), \gamma(1/2), \gamma(1))$.

For $\gamma \in \mathcal{I}M$, let $\Gamma_\gamma = \{\gamma' \in \mathcal{I}M : \sup_t |\gamma(t) - \gamma'(t)| < \epsilon\}$ be the set of paths lying pointwise within the metric tube of radius ϵ around γ. These sets give a good open cover

$$\Gamma = \bigsqcup_{\gamma\in\mathcal{I}M} \Gamma_\gamma \longrightarrow \mathcal{I}M$$

which factors through $\mathcal{U}^2$, and then taking $\Gamma^{(k)} = \Gamma \times_{\mathcal{I}M} \cdots \times_{\mathcal{I}M} \Gamma$ gives commutative diagrams

$$\begin{array}{ccc} \Gamma^{(k)} & \xrightarrow{\ \varepsilon\ } & (\mathcal{U}^2)^{(k)} \\ \downarrow & & \downarrow \\ \mathcal{I}M & \xrightarrow{\ \varepsilon\ } & M^2 \end{array} \tag{4}$$

for each k. Furthermore, join lifts to a well-defined map

$$j : \pi_3^*\Gamma^{(k)} \times_{(\mathcal{U}^3)^{(k)}} \pi_1^*\Gamma^{(k)} \longrightarrow \pi_2^*\Gamma^{(k)}, \tag{5}$$

and there is a natural identification of $\pi_3^*\Gamma^{(k)} \times_{(\mathcal{U}^3)^{(k)}} \pi_1^*\Gamma^{(k)}$ with $\Gamma^{(k)}$.

Remark 2. As noted in Remark 1 above, it is possible to work throughout with countable covers. For instance, one can restrict to neighborhoods centered on paths which are finite combinations of affine geodesic segments over rational intervals with end-points in a chosen countable dense set in the manifold. The resulting cover has the crucial property of being closed under joins, and the induced countable cover of loop space, considered below, is closed with respect to the two loop-fusion operations. Moreover the conditions we need for the covers of M and $\mathcal{I}M$, namely (4) and (5) are satisfied on more general spaces, including non-compact and singular cases, but then the independence of choices needs to be established.

The definition of the Čech cochain complex above carries over to $\mathcal{I}M$ (since finite dimensionality of M is not used) giving

$$\check{C}^k(\mathcal{I}M;A) \ni f : \Gamma^{(k+1)} \longrightarrow A, \quad \delta f = \prod_{j=1}^{k+2} \iota_j^* f^{(-1)^j} \in \check{C}^{k+1}(\mathcal{I}M;A),$$

where we reuse the notation $\iota_j : \Gamma^{(k+1)} \longrightarrow \Gamma^{(k)}$ for the face maps of the simplicial space $\left\{\Gamma^{(k)}; k \geq 1\right\}$, and observe that again $\check{H}^k(\mathcal{I}M;A) \cong H^k(\mathcal{I}M;A)$ since Γ is a good cover of a paracompact space.

The identification of $\pi_3^*\Gamma \times_{\mathcal{U}^3} \pi_1^*\Gamma$ with Γ and (5) gives a second chain map on $\check{C}^\bullet(\mathcal{I}M;A)$ associated to the simplicial structure on $\{M^n : n \geq 1\}$:

$$\bar{\partial} : \check{C}^k(\mathcal{I}M;A) \longrightarrow \check{C}^k(\mathcal{I}M;A), \quad \bar{\partial} f = \pi_3^* f^{-1}\, \pi_1^* f^{-1}\, j^*(\pi_2^* f) : \Gamma^{(k)} \longrightarrow A.$$

This does not lead to a complex, i.e. $\bar{\partial}^2$ is not trivial, since $\mathcal{I}M$ is not itself a simplicial space over $\{M^n : n \geq 1\}$; reparameterization is required to compare pullbacks of paths.

The constant paths may be identified as an inclusion $M \subset \mathcal{I}M$. Let

$$\check{C}_0^k(\mathcal{I}M;A) = \left\{ f \in \check{C}^k(\mathcal{I}M;A) : f|_M = 1 \right\}$$

denote the subcomplex of cochains which are trivial on them. Since the join map restricts to the trivial map on constant paths, $\bar{\partial} : \check{C}_0^\bullet(\mathcal{I}M;A) \longrightarrow \check{C}_0^\bullet(\mathcal{I}M;A)$.

Lemma 1.2. *The subcomplex* $(\check{C}_0^\bullet(\mathcal{I}M;A), \delta)$ *is acyclic.*

Proof. The short exact sequence of chain complexes

$$0 \longrightarrow \check{C}_0^\bullet(\mathcal{I}M;A) \longrightarrow \check{C}^\bullet(\mathcal{I}M;A) \longrightarrow \check{C}^\bullet(M;A) \longrightarrow 0$$

induces a long exact sequence in cohomology, and since there is a deformation retraction of $\mathcal{I}M$ onto M it follows that $\check{H}^\bullet(\mathcal{I}M;A) \cong \check{H}^\bullet(M;A)$ and in particular $\check{H}_0^\bullet(\mathcal{I}M;A) = 0$. □

1.3. Loop space

For $l \geq 1$ we denote by $\mathcal{I}^{[l]}M$ the fiber product

$$\mathcal{I}^{[l]}M = \mathcal{I}M \times_{M^2} \cdots \times_{M^2} \mathcal{I}M,$$

and observe that $\mathcal{I}^{[2]}M = \{(\gamma_1, \gamma_2) : \gamma_1(t) = \gamma_2(t),\ t = 0, 1\}$ may be identified with the Banach manifold of free continuous loops by *fusion* of paths:

$$\psi : \mathcal{I}^{[2]}M \xrightarrow{\cong} \mathcal{L}M = \mathcal{C}(\mathbb{S}; M), \quad \ell(t) = \psi(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(t) & 0 \leq t \leq 1, \\ \gamma_2(-t) & -1 \leq t \leq 0, \end{cases}$$

where $\mathbb{S}$ is parameterized as $[-1, 1]/(\{-1\} \sim \{1\})$ for later convenience.

The set $\left\{\mathcal{I}^{[l]}M : l \geq 1\right\}$ forms another simplicial space, with face maps given by the fiber projections $\varrho_j : \mathcal{I}^{[l]}M \longrightarrow \mathcal{I}^{[l-1]}M$, $1 \leq j \leq l$, and the set $\left\{\Gamma^{[l]} : l \geq 1\right\}$ forms a good cover, where

$$\Gamma^{[l]} = \Gamma \times_{\mathcal{U}^2} \cdots \times_{\mathcal{U}^2} \Gamma \longrightarrow \mathcal{I}^{[l]}M$$

is lifted from the path space with k-fold overlaps

$$(\Gamma^{(k)})^{[l]} \equiv (\Gamma^{[l]})^{(k)} = \Gamma^{[l]} \times_{\mathcal{I}^{[l]}M} \cdots \times_{\mathcal{I}^{[l]}M} \Gamma^{[l]}.$$

For brevity of notation, we denote the corresponding cover of loop space by

$$\Lambda = \Gamma^{[2]} \longrightarrow \mathcal{L}M.$$

We will denote differentials derived from this simplicial space or its cover by d.

Passing to $\mathcal{I}^{[l]}M$ in (3) gives rise to a map

$$j^{[l]} : \pi_3^*\mathcal{I}^{[l]}M \times_{M^3} \pi_1^*\mathcal{I}^{[l]}M \longrightarrow \pi_2^*\mathcal{I}^{[l]}M, \tag{6}$$

and its local version

$$j^{[l]} : \pi_3^*(\Gamma^{[l]})^{(k)} \times_{(\mathcal{U}^3)^{(k)}} \pi_1^*(\Gamma^{[l]})^{(k)} \longrightarrow \pi_2^*(\Gamma^{[l]})^{(k)}. \tag{7}$$

In the case $l = 2$, we call this the *figure-of-eight product* on loops as in [6]. The product of two loops $\ell_1 = \psi(\gamma_{11}, \gamma_{12})$ and $\ell_2 = \psi(\gamma_{21}, \gamma_{22})$ such that $\ell_1(1) = \ell_2(0)$ is the loop $\ell_3 = \psi\big(j(\gamma_{11}, \gamma_{21}), j(\gamma_{12}, \gamma_{22})\big)$. See Figure 1(b). The domain in (6) with $l = 2$ may be identified with the subspace of *figure-of-eight loops* in M:

$$\mathcal{L}_8 M = \{\ell \in \mathcal{L}M : \ell(1/2) = \ell(-1/2)\} \longrightarrow M^3.$$

This Banach manifold fibers over M^3 and has a good cover given by the domain in (7) with $l = 2$ and $k = 1$. Unlike the case $l = 1$, $\mathcal{L}_8 M$ cannot be identified with the full loop space nor is $j^{[2]}$ invertible.

There is another product on loop space, considered already in [8], associated to $\mathcal{I}^{[3]}M$. If $(\gamma_1, \gamma_2, \gamma_3) \in \mathcal{I}^{[3]}M$, then $\ell_3 = \psi(\gamma_1, \gamma_3)$ is the *fusion product* (on loops) of $\ell_1 = \psi(\gamma_1, \gamma_2)$ and $\ell_2 = \psi(\gamma_2, \gamma_3)$. See Figure 1(a).

Within the Čech cochain complex $\big(\check{C}^\bullet(\mathcal{L}M; A), \delta\big)$ for loop space, where

$$\check{C}^k(\mathcal{L}M; A) \ni f : \Lambda^{(k+1)} \longrightarrow A, \quad \delta f = \prod_{j=1}^{k+2} \iota_j^* f^{(-1)^j} \in \check{C}^{k+1}(\mathcal{L}M; A),$$

consider the subcomplex of *fusion cochains*

$$\check{C}^k_{\mathrm{fus}}(\mathcal{L}M; A) = \Big\{ f \in \check{C}^k(\mathcal{L}M; A) : df = 1 \Big\}, \\ df = \varrho_1^* f^{-1}\, \varrho_2^* f\, \varrho_3^* f^{-1} \in \check{C}^k(\mathcal{I}^{[3]}M; A).$$

Note that $d^2 : \check{C}^k(\mathcal{I}^{[l]}; A) \longrightarrow \check{C}^k(\mathcal{I}^{[l+2]}M; A)$ is trivial and $\delta d = d\delta$ so this is indeed a subcomplex.

The subspace $\mathcal{L}_8 M \subset \mathcal{L}M$ is closed under fusion so $\check{C}^\bullet_{\mathrm{fus}}(\mathcal{L}_8 M; A)$ is well-defined, and imposing a condition over the figure-of-eight product leads to the *loop-fusion* subcomplex

$$\check{C}^k_{\mathrm{lf}}(\mathcal{L}M; A) = \Big\{ f \in \check{C}^k_{\mathrm{fus}}(\mathcal{L}M; A) : \bar{\partial} f = \delta g \text{ for } g \in \check{C}^{k-1}_{\mathrm{fus}}(\mathcal{L}_8 M; A) \Big\}, \\ \bar{\partial} f = \pi_3^* f^{-1}\, \pi_1^* f^{-1}\, (j^{[2]})^*(\pi_2^* f) \in \check{C}^k_{\mathrm{fus}}(\mathcal{L}_8 M; A). \tag{8}$$

Thus, this complex consists of those fusion cochains which are multiplicative with respect to the figure-of-eight product up to a fusion boundary. The image of $\bar{\partial}$ on these chains lies in the space of fusion Čech cochains on the space of figure-of-eight loops; though we do not need to consider it here, $\bar{\partial}^2$ may be sensibly defined (it is not automatically trivial). That (8) is a subcomplex follows from the fact that $\delta\bar{\partial} = \bar{\partial}\delta$. It is also the case that $d\bar{\partial} = \bar{\partial} d$ on suitably defined spaces, in particular as maps from $\check{C}^k(\mathcal{I}M; A)$ to $\check{C}^{k+1}(\mathcal{I}M; A)$ and from $\check{C}^k(\mathcal{L}M; A)$ to $\check{C}^{k+1}(\mathcal{L}_8 M; A)$.

The *loop-fusion* cohomology of $\mathcal{L}M$ is then defined to be

$$\check{H}^{\bullet}_{\mathrm{lf}}(\mathcal{L}M;A) = H^{\bullet}\big(\check{C}^{\bullet}_{\mathrm{lf}}(\mathcal{L}M;A),\delta\big) \longrightarrow \check{H}^{\bullet}(\mathcal{L}M;A), \tag{9}$$

with its homomorphism, f, to ordinary Čech cohomology induced by the inclusion of $\check{C}^{\bullet}_{\mathrm{lf}}(\mathcal{L}M;A)$ in $\check{C}^{\bullet}(\mathcal{L}M;A)$. The identification of this cohomology with that of the manifold, in the main result, shows that it does not depend on the choice of metric or parameter ϵ.

2. Transgression and regression

We proceed to the proof of the Theorem above.

2.1. Transgression

We first construct the map T_{lf}. Let $\alpha \in \check{C}^k(M;A)$ be a cocycle for $k \geq 1$, and consider

$$\varepsilon^*\partial\alpha \in \check{C}^k_0(\mathcal{I}M;A), \quad \partial\alpha = \pi_1^*\alpha^{-1}\pi_2^*\alpha \in \check{C}^k(M^2;A).$$

Since $\delta\varepsilon^*\partial\alpha = \varepsilon^*\partial\delta\alpha = 1$ and $\check{C}^{\bullet}_0(\mathcal{I}M;A)$ is exact by Lemma 1.2, it follows that $\varepsilon^*\partial\alpha = \delta\beta$ for some $\beta \in \check{C}^{k-1}_0(\mathcal{I}M;A)$; set

$$\omega = d\beta = \varrho_1^*\beta^{-1}\,\varrho_2^*\beta \in \check{C}^{k-1}(\mathcal{L}M;A). \tag{10}$$

Then $\varepsilon\circ\varrho_1 = \varepsilon\circ\varrho_2$ implies

$$\delta\omega = d\delta\beta = \varrho_1^*(\varepsilon^*\partial\alpha)^{-1}\,\varrho_2^*(\varepsilon^*\partial\alpha) = 1.$$

Moreover $d^2 = 1$ so

$$d\omega = d^2\beta = 1 \implies \omega \in \check{C}^{k-1}_{\mathrm{fus}}(\mathcal{L}M;A).$$

Finally, ω is fusion-figure-of-eight since $\bar{\partial}\omega = d\bar{\partial}\beta$ and $\bar{\partial}\beta$, which lies in $\check{C}^k_0(\mathcal{I}M;A)$ by Lemma 1.2, is a boundary. Indeed, for any path $\gamma = j(\gamma_1,\gamma_2)$,

$$\begin{aligned}\delta\bar{\partial}\beta(\gamma) &= \bar{\partial}\varepsilon^*\partial\alpha(\gamma) = \varepsilon^*\partial\alpha^{-1}(\gamma_1)\,\varepsilon^*\partial\alpha^{-1}(\gamma_2)\,\varepsilon^*\partial\alpha(\gamma)\\ &= \alpha\big(\gamma_1(0)\big)\alpha^{-1}\big(\gamma_1(1)\big)\alpha\big(\gamma_2(0)\big)\alpha^{-1}(\gamma_2(1))\alpha^{-1}\big(\gamma(0)\big)\alpha\big(\gamma(1)\big) = 1.\end{aligned}$$

Thus $\bar{\partial}\beta$ is a cocycle and as $\check{C}^{\bullet}_0(\mathcal{I}M;A)$ is acyclic there exists an element $\eta \in \check{C}^{k-2}_0(\mathcal{I}M;A)$ such that $\bar{\partial}\beta = \delta\eta$. It follows that

$$\bar{\partial}\omega = d\bar{\partial}\beta = d\delta\eta = \delta d\eta, \quad d(d\eta) = 1 \implies \omega \in \check{C}^{k-1}_{\mathrm{lf}}(\mathcal{L}M;A).$$

Consider next the effect of the choices made. If $\beta' \in \check{C}_0^{k-1}(\mathcal{I}M; A)$ is another cochain such that $\delta\beta' = \varepsilon^*\partial\alpha$, then $\delta(\beta'\beta^{-1}) = 1$ implies that $\beta' = \beta\delta\nu$ for some $\nu \in \check{C}_0^{k-2}(\mathcal{I}M; A)$, which alters ω by the boundary term $\delta d\nu$. Similarly if $\alpha' = \alpha\delta\mu$ is another representative for $[\alpha] \in \check{H}^k(M; A)$, it follows that $\omega' = \omega\delta\sigma$, where σ is the result of the same construction applied to μ. Thus the *transgression map*

$$T_{\mathrm{lf}} : \check{H}^k(M; A) \longrightarrow \check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A), \quad T_{\mathrm{lf}}[\alpha] = [\omega]^{-1} \tag{11}$$

is well-defined.

2.2. Regression

Next we define a map which is shown below to be the inverse of T_{lf}. Suppose $\omega \in \check{C}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A)$ is a cocycle, so

$$\delta\omega = 1, \quad d\omega = 1, \quad \bar{\partial}\omega = \delta\nu, \quad d\nu = 1.$$

Then ω gives *descent data* for the trivial principal A-bundle

$$\Gamma^{(k)} \times A \longrightarrow \Gamma^{(k)} \tag{12}$$

over $(\mathcal{U}^2)^{(k)}$. That is, multiplication by ω determines a relation on the fibers, with the content of $d\omega = 1$ being that this is an equivalence relation so inducing a well-defined principal A-bundle $P_k \longrightarrow (\mathcal{U}^2)^{(k)}$:

$$(P_k)_{(m,m')} = \big\{(\gamma, a) \in \Gamma^{(k)} \times A : \varepsilon(\gamma) = (m, m')\big\}/\sim_\omega,$$
$$(\gamma, a) \sim_\omega (\gamma', a') \iff a = \omega(\gamma, \gamma')a'.$$

The condition $\delta\omega = 1$ implies that P_k is a simplicial bundle (see [3], [7]), i.e. the bundle over $(\mathcal{U}^2)^{(k+1)}$ consisting of the alternating tensor products of the pullbacks of P_k by the maps $\iota_j : (\mathcal{U}^2)^{(k+1)} \longrightarrow (\mathcal{U}^2)^{(k)}$ is canonically trivial:

$$\delta P_k = \bigotimes_j \iota_j^* P_k^{(-1)^j} \cong (\mathcal{U}^2)^{(k+1)} \times A \longrightarrow (\mathcal{U}^2)^{(k+1)}.$$

Similarly, ν determines a principal A-bundle

$$R_{k-1} = \Gamma^{(k-1)} \times A/\sim_\nu \longrightarrow (\mathcal{U}^3)^{(k-1)},$$

and by functoriality of descent there is a canonical isomorphism

$$\partial P_k \cong \delta R_{k-1} \longrightarrow (\mathcal{U}^3)^{(k)}, \quad \partial P_k = \pi_1^* P_k^{-1} \otimes \pi_2^* P_k \otimes \pi_3^* P_k^{-1}. \tag{13}$$

The components of $(\mathcal{U}^2)^{(k)}$ and $(\mathcal{U}^3)^{(k-1)}$ are contractible so there exist sections

$$s : (\mathcal{U}^2)^{(k)} \longrightarrow P_k, \quad \text{and} \quad r : (\mathcal{U}^3)^{(k-1)} \longrightarrow R_{k-1}.$$

These pull back to give sections δs of δP_k and δr of δR_{k-1} and as δP_k is canonically trivial δs gives rise to a cocycle

$$\kappa = \delta s \in \check{C}^k(M^2; A), \quad \delta\kappa = \delta\delta s = 1,$$

where $\delta^2 s$ coincides with the canonical trivialization of $\delta^2 P$ for any section s. Another choice of section s' alters κ by a term $\delta\gamma$, where $\gamma \in \check{C}^{k-1}(M^2; A)$ is fixed by $s' = s\,\gamma$. Thus $[\kappa] \in \check{H}^k(M^2; A)$ is determined by ω. Similarly, another choice ω' such that $\omega' = \omega\delta\mu$, $d\mu = 1$ leads to a bundle P'_k and a canonical isomorphism $P'_k \cong P_k \otimes \delta Q_{k-1}$, where Q_{k-1} is formed by descent using μ. If $\kappa = \delta s$ and $\kappa' = \delta s'$ for respective sections s and s' of P_k and P'_k and q is any section of Q_{k-1}, then $s' = (s \otimes \delta q)\,\nu$ for some $\nu \in \check{C}^{k-1}(M^2; A)$ and $\kappa' = \kappa\delta^2 q\,\delta\nu = \kappa\delta\nu$. Thus the map from $\check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A)$ to $\check{H}^k(M^2; A)$ is well-defined.

Finally, we may compare ∂s and δr as sections of (13); then $\partial s = \delta r\,\tau$ for some $\tau \in \check{C}^{k-1}(M^3; A)$, from which it follows that

$$\partial\kappa = \delta(\partial s) = \delta^2 r\,\delta\tau = \delta\tau \in \check{C}^k(M^3; A).$$

(A different choice of r leads to $\partial\kappa = \delta\tau'$ for some other $\tau' \in \check{C}^{k-1}(M^3; A)$.) Thus $\partial[\kappa] = 1 \in \check{H}^k(M^3; A)$ and so by Lemma 1.1, $[\kappa] = \partial[\alpha]$ for a unique class $[\alpha] \in \check{H}^k(M; A)$. It follows that the *regression map* is well-defined by

$$R : \check{H}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A) \longrightarrow \check{H}^k(M; A), \quad R[\omega] = [\alpha]^{-1}. \tag{14}$$

Proposition 2.1. *The maps* (11) *and* (14) *are inverses.*

Proof. To see that $T_{\mathrm{lf}}R = \mathrm{Id}$ fix a cocycle $\omega \in \check{C}^{k-1}_{\mathrm{lf}}(\mathcal{L}M; A)$ and let $\alpha \in \check{C}^k(M; A)$ represent $R[\omega]^{-1}$, so that $\partial\alpha = \kappa\delta\nu$ for some $\nu \in \check{C}^{k-1}(M^2; A)$, where $\kappa = \delta s \in \check{C}^k(M^2; A)$ for a choice of section s of the bundle P_k. Replacing s by $s\nu^{-1}$ if necessary, we may assume that $\partial\alpha = \kappa = \delta s$.

Consider the transgression of α. This involves a choice of element $\beta \in \check{C}^{k-1}_0(\mathcal{I}M; A)$ such that $\delta\beta = \varepsilon^*\partial\alpha = \varepsilon^*\kappa$ but there is a natural choice available. Namely, the section s of P_k lifts canonically to a section of the trivial

A-bundle over $\Gamma^{(k)}$, from which P_k is descended, and so defines a cochain

$$\widetilde{s} \in \check{C}_0^{k-1}(\mathcal{I}M; A), \quad \widetilde{s}(\gamma) = a \iff s\big(\varepsilon(\gamma)\big) = [(\gamma, a)] \in P_k.$$

That $\widetilde{s}$ is trivial on constant paths is a consequence of the fact that the fusion condition implies that the descent data ω for P_k is trivial on constant loops. Since δP_k is trivially descended from the trivial bundle over $\Gamma^{(k+1)}$,

$$\delta\widetilde{s} = \widetilde{(\delta s)} = \varepsilon^*\delta s = \varepsilon^*\kappa,$$

and hence $\beta = \widetilde{s} \in \check{C}_0^{k-1}(\mathcal{I}M; A)$ is an element such that $\delta\beta = \varepsilon^*\kappa$. It then follows that $d\beta = \varrho_1^*\widetilde{s}^{-1}\,\varrho_2^*\widetilde{s} \equiv \omega \in \check{C}_{\mathrm{lf}}^{k-1}(\mathcal{L}M; A)$ since

$$\begin{gathered}\widetilde{s}(\gamma)\,\widetilde{s}(\gamma')^{-1} = a\,a'^{-1}, \quad \text{such that} \\ s\big(\varepsilon(\gamma)\big) = s\big(\varepsilon(\gamma')\big) = [(\gamma, a)] = [(\gamma', a')] \iff a = \omega(\gamma, \gamma')\,a'.\end{gathered}$$

In the other direction, fix a cocycle $\alpha \in \check{C}^k(M^2; A)$ and suppose $T_{\mathrm{lf}}[\alpha]^{-1}$ is represented by $\omega \in \check{C}_{\mathrm{lf}}^{k-1}(\mathcal{L}M; A)$, given by $\omega = d\beta$ where $\delta\beta = \varepsilon^*\partial\alpha \in \check{C}_0^k(\mathcal{I}M; A)$. The regression of ω involves a choice of section of the bundle P_k, but here too there is a natural one which recovers $\partial\alpha \in \check{C}^k(M^2; A)$. Indeed, since $\omega = \varrho_1^*\beta^{-1}\,\varrho_2^*\beta$, the equivalence relation defining P_k takes the particular form

$$P_k \ni [(\gamma, a)] = [(\gamma', a')] \iff a = \beta(\gamma)\beta(\gamma')^{-1}a',$$

and an appropriate section of P_k is defined by

$$s(m, m') = [(\gamma, \beta(\gamma))] = [(\gamma', \beta(\gamma'))],$$

since this equivalence class is independent of the particular $\gamma \in \varepsilon^{-1}(m, m')$. With s so defined, it follows that $\delta s \in \check{C}^k(M; A)$ is given by

$$\delta s(m, m') = [(\gamma, \delta\beta(\gamma)] = [(\gamma, \varepsilon^*\partial\alpha(\gamma))] = \partial\alpha(m, m'). \qquad \square$$

2.3. Compatibility

The commutativity of the diagram (1) asserts that the 'enhanced transgression' map constructed above is compatible with transgression in the usual sense. The latter corresponds to pullback of cohomology under the evaluation map followed by projection onto the second factor under the decomposition for the product:

$$\begin{aligned}(15)\quad \mathrm{ev}^* : H^k(M;A) &\longrightarrow H^k(\mathbb{S}\times\mathcal{L}M;A)\\ &= H^k(\mathcal{L}M;A)\oplus H^{k-1}(\mathcal{L}M;A)\longrightarrow H^{k-1}(\mathcal{L}M;A).\end{aligned}$$

To realize this in Čech cohomology, fix a small parameter $\delta>0$ and consider the open cover $\mathcal{S}=\bigsqcup_{(t,l)\in\mathbb{S}\times\mathcal{L}M} S_{t,l}$ of $\mathbb{S}\times\mathcal{L}M$, where

$$(16)\quad \begin{aligned}&S_{t,l}=\left\{(t',l')\in\mathbb{S}\times\mathcal{L}M : l'\in\Lambda_l,\ t'\in(t-\delta,t+\delta),\ l'(t')\in U_{l(t)}\right\},\\ &S_{t,l}\longrightarrow\Lambda_l,\quad S_{t,l}\longrightarrow I_t\subset\mathbb{S},\quad \mathrm{ev}:S_{t,l}\ni(t',l')\longmapsto l'(t')\in U_{l(t)}.\end{aligned}$$

The interval $I_t=(t-\delta,t+\delta)\subset\mathbb{S}$ is to be interpreted as the 'short' signed interval on $\mathbb{S}$. This is a good cover, with respect to which we consider the Čech complex on $\mathbb{S}\times\mathcal{L}M$. The evaluation map $\mathrm{ev}:\mathbb{S}\times\mathcal{L}M\longrightarrow M$ and projections $\mathbb{S}\times\mathcal{L}M\longrightarrow\mathcal{L}M$ and $\mathbb{S}\times\mathcal{L}M\longrightarrow\mathbb{S}$ lift to maps of the covers $\mathcal{S}\longrightarrow\mathcal{U}$, $\mathcal{S}\longrightarrow\Lambda$ and $\mathcal{S}\longrightarrow\mathcal{V}$, respectively, where $\mathcal{V}$ is the cover of $\mathbb{S}$ by intervals of length 2δ around each point.

The first factor in the product (15) corresponds to pullback to $\mathcal{L}M$ under the evaluation map at any fixed point on the circle. Consequently, to consider the projection to the second factor of (15) we modify the pullback $\mathrm{ev}^*\alpha\in\check{C}^k(\mathbb{S}\times\mathcal{L}M;A)$ to

$$(17)\qquad \alpha'=(\mathrm{ev}_0^*\alpha)^{-1}\,\mathrm{ev}^*\alpha\in\check{C}^k(\mathbb{S}\times\mathcal{L}M;A)$$

instead, where $\mathrm{ev}_0:\mathbb{S}\times\mathcal{L}M\ni(t,\ell)\longmapsto\ell(0)\in M$ factors through the projection to $\mathcal{L}M$. Then the class of (17) projects to zero in $\check{H}^k(\mathcal{L}M;A)$ and has the same projection as $\mathrm{ev}^*\alpha$ to $\check{H}^{k-1}(\mathcal{L}M;A)$.

To compute the latter, consider the space $[-1,1]\times\mathcal{L}M$ which maps to $\mathbb{S}\times\mathcal{L}M$ by the identification of the endpoints. This has a good cover $\mathcal{T}=\bigsqcup_{t,l}T_{t,l}$ where $T_{t,l}$ is defined as in (16) except that the interval is restricted to $[-1,1]$. The map to $\mathbb{S}\times\mathcal{L}M$ then lifts to a continuous map of the covers. The image of (17) lies in the subcomplex $\check{C}^k_0([-1,1]\times\mathcal{L}M;A)$ of chains which are trivial at $\{0\}\times\mathcal{L}M$. This subcomplex is acyclic as in the proof of Lemma 1.2 since $[-1,1]\times\mathcal{L}M$ retracts onto $\{0\}\times\mathcal{L}M$. Thus

$$\alpha'=\delta\sigma,\quad \sigma\in\check{C}^{k-1}([-1,1]\times\mathcal{L}M;A),$$

and the transgression class is represented by the difference

$$(18)\qquad \left(\sigma|_{\{1\}\times\mathcal{L}M}\right)\left(\sigma^{-1}|_{\{-1\}\times\mathcal{L}M}\right)\in\check{C}^{k-1}(\mathcal{L}M;A).$$

That this is a cocycle follows from the fact that its Čech differential is the difference of α' at 1 and -1 which is trivial since α' is pulled back from the circle.

On the other hand, the initial portion of the enhanced transgression construction in §2.1 may be modified as follows. Consider the pullback

$$\widetilde{\varepsilon}^*\partial\alpha \in \check{C}^k([0,1]\times\mathcal{I}M;A),$$
$$\widetilde{\varepsilon}:[0,1]\times\mathcal{I}M\longrightarrow M^2,\quad \widetilde{\varepsilon}(t,\gamma)=\big(\gamma(0),\gamma(t)\big).$$

As before, this lies in an exact subcomplex and as a result $\widetilde{\varepsilon}^*\partial\alpha=\delta\widetilde{\beta}$, where $\widetilde{\beta}\in\check{C}^{k-1}([0,1]\times\mathcal{I}M;A)$. The restriction $\beta=\widetilde{\beta}|_{\{1\}\times\mathcal{I}M}$ to a cochain on $\mathcal{I}M$ reduces to the earlier construction and $\widetilde{\beta}|_{\{0\}\times\mathcal{I}M}$ is trivial. Then the product

$$\sigma=\varsigma_1^*\widetilde{\beta}\,\varsigma_2^*\widetilde{\beta}\in\check{C}^{k-1}([-1,1]\times\mathcal{I}^{[2]}M;A),$$
$$\varsigma_i:[-1,1]\times\mathcal{I}^{[2]}M\longrightarrow[0,1]\times\mathcal{I}M,$$
$$\varsigma_1\big(t,(\gamma_1,\gamma_2)\big)=(\max(0,t),\gamma_1),\quad \varsigma_2\big(t,(\gamma_1,\gamma_2)\big)=(-\min(0,t),\gamma_2)$$

is a cochain on $[-1,1]\times\mathcal{L}M$ with differential equal to α'. Indeed,

$$\delta\sigma\big(t,\ell\big)=(\varsigma_1^*\delta\widetilde{\beta}\varsigma_2^*\delta\widetilde{\beta})\big(t,(\gamma_1,\gamma_2)\big)=\begin{cases}\alpha(\gamma_1(t))\alpha^{-1}(\gamma_1(0)) & 0\le t\le 1,\\ \alpha(\gamma_2(-t))\alpha^{-1}(\gamma_2(0)) & -1\le t\le 0,\end{cases}$$
$$=\alpha\big(\ell(t)\big)\alpha^{-1}\big(\ell(0)\big),$$

where $\ell=\psi(\gamma_1,\gamma_2)$. Finally, observe that the transgression class (18) is represented by the 'enhanced transgression' class $d\beta^{-1}$:

$$\big(\sigma|_{\{1\}\times\mathcal{L}M}\big)\,\big(\sigma^{-1}|_{\{-1\}\times\mathcal{L}M}\big)\,(\gamma_1,\gamma_2)=\widetilde{\beta}(1,\gamma_1)\,\widetilde{\beta}^{-1}(1,\gamma_2)=d\beta^{-1}(\gamma_1,\gamma_2).$$

This completes the proof of the Theorem.

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