

# Quantitative uniqueness estimates for second order elliptic equations with unbounded drift

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In this paper we derive quantitative uniqueness estimates at infinity for solutions to an elliptic equation with unbounded drift in the plane. More precisely, let  $u$  be a real solution to  $\Delta u + W \cdot \nabla u = 0$  in  $\mathbf{R}^2$ , where  $W$  is real vector and  $\|W\|_{L^p(\mathbf{R}^2)} \leq K$  for  $2 \leq p < \infty$ . Assume that  $u$  satisfies certain a priori assumption at 0. For  $2 < p < \infty$ , if  $\|u\|_{L^\infty(\mathbf{R}^2)} \leq C_0$ , then  $u$  satisfies the following asymptotic estimates at  $R \gg 1$

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-C_1 R^{1-2/p} \log R),$$

where  $C_1 > 0$  depends on  $p, K, C_0$ . For  $p = 2$ , if  $|u(z)| \leq |z|^m$  for  $|z| > 1$  with some  $m > 0$ , then

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq C_2 \exp(-C_3 (\log R)^2),$$

where  $C_2 > 0$  depends on  $m$  and  $C_3$  depends on  $m, K$ . Using the scaling argument in [BK05], these quantitative estimates are easy consequences of estimates of the maximal vanishing order for solutions of the local problem. The estimate of the maximal vanishing order is a quantitative form of the strong unique continuation property.

## 1. Introduction

In this work we consider the second order elliptic equation with an unbounded drift term

$$(1.1) \quad \Delta u + W \cdot \nabla u = 0 \quad \text{in } \mathbf{R}^2,$$

where  $W = (W_1, W_2)$  is a real vector-valued function with  $L^p$  bound for  $2 \leq p < \infty$ . Here we are interested in the lower bound of the decay rate for

any nontrivial solution  $u$ . When  $p = \infty$ , the problem is related to Landis' conjecture [KL88]. That is, let  $u$  be a solution of (1.1) with  $\|W\|_{L^\infty(\mathbf{R}^2)} \leq 1$  and  $\|u\|_{L^\infty(\mathbf{R}^2)} \leq C_0$  and  $|u(z)| \leq \exp(-C|x|^{1+})$  for some  $C > 0$ , then  $u$  is trivial. If one applies a suitable Carleman estimate to (1.1) and a scaling device in [BK05], the best exponent one can get is 2, namely, under the same conditions stated above except  $|u(z)| \leq \exp(-C|x|^{2+})$ , then  $u$  is trivial (see [Da12], [LW13] for quantitative forms of this result). Moreover, in [Da12], the author constructed a Meshkov type example showing that the exponent 2 is in fact optimal for *complex-valued*  $W$  and  $u$ .

In a recent paper [KLW14], the authors studied Landis' conjecture for second order elliptic equations in the plane in the real setting, including (1.1) with real-valued  $W$  and  $u$ . It was proved in [KLW14] that if  $u$  is a real-valued solution of (1.1) satisfying  $|u(z)| \leq \exp(C_0|z|)$ ,  $|\nabla u(0)| = 1$ , and  $\|W\|_{L^\infty(\mathbf{R}^2)} \leq 1$ , then

$$(1.2) \quad \inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-CR \log R) \quad \text{for } R \gg 1$$

where  $C$  depends on  $C_0$ .

In this paper, we would like to study estimates like (1.2) for  $2 \leq p < \infty$ . For complex-valued  $W$  satisfying

$$(1.3) \quad |W(z)| \leq C\langle z \rangle^{-s}, \quad s \geq 0,$$

where  $\langle z \rangle = \sqrt{1 + |z|^2}$ , the lower bound of the decay rate for  $u$  is  $\exp(-R^{2-2s} f(\log R))$  for  $s < 1/2$  and is  $\exp(-R\tilde{f}(\log R))$  for  $s \geq 1/2$ , where  $f(\log R)$  and  $\tilde{f}(\log R)$  are functions of  $\log R$  which grow slower than any positive power of  $R$  (see [Da12], [LW13]). Here our assumption on  $W$  will be an integral bound rather than a pointwise bound as in (1.3). Precisely, we prove that

**Theorem 1.1.** *Let  $u \in W_{loc}^{2,p}(\mathbf{R}^2)$  be a real solution of (1.1) with  $2 \leq p < \infty$ .*

(i) *Assume  $2 < p < \infty$ . Let  $|u(z)| \leq C_0$  for some  $C_0 > 0$ ,*

$$(1.4) \quad \|W\|_{L^p(\mathbf{R}^2)} \leq \tilde{K},$$

*and  $|\nabla u(0)| = 1$ , then*

$$\inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-C_1 R^{1-2/p} \log R)$$

*for  $R \gg 1$ , where  $C_1$  depends on  $p, \tilde{K}$ , and  $C_0$ .*

(ii) For  $p = 2$ , assume that  $|u(z)| \leq |z|^m$  for  $|z| > 1$  with some  $m > 0$ . If

$$(1.5) \quad \|W\|_{L^2(\mathbf{R}^2)} \leq K$$

and

$$1 \leq \int_{B_1} |\nabla u|^2,$$

then

$$(1.6) \quad \inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq C_2 \exp(-C_3(\log R)^2)$$

for  $R \gg 1$ , where  $C_2$  depends  $m$  and  $C_3$  depends on  $m, K$ .

Hereafter, we denote  $B_r(a)$  the ball of radius  $r$  centered at  $a$ . When  $a = 0$ , we simply denote  $B_r(a) = B_r$ .

Using the scaling argument in [BK05], Theorem 1.1 is an easy consequence of the estimate of the maximal vanishing order of the solution  $v$  to

$$(1.7) \quad \Delta v + A \cdot \nabla v = 0 \quad \text{in } B_8$$

with

$$(1.8) \quad \|A\|_{L^p(B_8)} \leq K.$$

It suffices to take  $K \geq 1$ . The proof of the maximal vanishing order of  $v$  relies on a nice reduction of (1.7) to a  $\bar{\partial}$  equation. Having the  $\bar{\partial}$  equation, we then derive the vanishing order by using Hadamard's three circle theorem. The case  $p = 2$  needs special attention due to the fact that the Cauchy transform fails to be a bounded map from  $L^2(B_8)$  to  $L^\infty(B_8)$ .

One can also study the same problem in an exterior domain. However, the arguments used here can not be applied to the case of exterior domain due to the non-simplicity of the scaled domain for the local problem. Precisely, there will be a "hole" in  $B_8$  for (1.7). Nonetheless, it may be possible to apply the method, based on the Carleman estimate, used in our early paper [KLW14] to treat the exterior domain problem. We leave such extension to the interested reader.

The estimate of the maximal vanishing order of  $v$  provides us a quantitative form of the strong unique continuation property (SUCP) for (1.7). Note that  $A \in L^2$  is a scale invariant drift in  $\mathbf{R}^2$  in the sense that if  $v(x)$

solves (1.7), then  $v_r(x) := v(rx)$  satisfies  $\Delta v_r + A_r \nabla v_r = 0$  in  $B_{8/r}$  with  $A_r(x) = rA(rx)$  and

$$\|A\|_{L^2(B_8)} = \|A_r\|_{L^2(B_{8/r})}.$$

It is clear that  $v(z) = \exp(-|z|^{-\epsilon})$  for  $\epsilon > 0$  is an easy counterexample of SUCP for  $A \in L^p$  with  $p < 2$ . For the dimension  $n \geq 3$ , Kim [Ki89] proved that SUCP holds for (1.7) when  $A \in L^p_{loc}$  with  $p = (3n - 2)/2$  and Wolff [Wo90] improved the exponent to  $p = \max\{n, (3n - 4)/2\}$ . On the other hand, if  $n \geq 5$ , counterexamples to the SUCP with  $A \in L^n_{loc}$  were given by Wolff in [Wo94] (or see [Wo93]). Counterexamples of the unique continuation property (UCP) for (1.7) with  $A \in L^p$ ,  $p < 2$ , or  $A \in L^2_{weak}$ , weak  $L^2$  space, were constructed by Mandache [Ma02] and Koch-Tataru [KT02], respectively. We also would like to mention that a counterexample of UCP for the Schrödinger operator  $\Delta u + Vu = 0$  with  $V \in L^1$  was constructed by Kenig and Nadirashvili [KN00] for dimension  $n \geq 2$ . For  $n = 2$  and  $A \in L^2$ , it seems likely that a variant of the Carleman estimate proved in Kim’s thesis for  $n \geq 3$  [Ki89, Theorem 3] is available for  $n = 2$  and the SUCP will follow from it (see the remark in [Wo90, Page 156]). Here we provide an explicit proof of the SUCP for (1.7) in two dimensions, where  $A \in L^2_{loc}$  is a real-valued vector. Using the same method, we also study the SUCP for

$$(1.9) \quad \Delta v + \nabla \cdot (Av) = 0 \quad \text{in } \mathbf{R}^2,$$

where  $A$  is a real-valued vector with bounded  $L^2_{loc}$  norm. We would like to remark that our proofs work for Equations (1.7) and (1.9), but not corresponding differential inequalities.

The structure of the paper is as follows. In Section 2, we consider the case where  $2 < p < \infty$ . The case of  $p = 2$  is treated in Section 3. We study the SUCP for (1.9) in Section 4. Throughout the paper,  $C$  stands for an absolute constant whose dependence will be specified if necessary. Its value may vary from line to line.

## 2. The case of $2 < p < \infty$

We consider any solution  $v \in W^{2,p}_{loc}(B_8)$  to Equation (1.7) with  $A = (A_1, A_2)$  satisfying (1.8). Denote  $g = v_x - iv_y$ . It is easy to see that

$$(2.1) \quad \begin{aligned} \bar{\partial}g &= \frac{1}{2}\Delta v = -\frac{1}{2}(A_1\partial_x v + A_2\partial_y v) \\ &= -\frac{1}{4}(A_1 + iA_2)g - \frac{1}{4}(A_1 - iA_2)\bar{g}, \end{aligned}$$

where in the last step we used that  $v$  is real. As usual, we denote  $\bar{\partial} = (\partial_x + i\partial_y)/2$ . Let us define

$$\alpha(z) = \begin{cases} -\frac{1}{4}(A_1 + iA_2) - \frac{1}{4}(A_1 - iA_2)\frac{\bar{g}}{g} & \text{if } g \neq 0, \\ 0 & \text{if } g = 0, \end{cases}$$

then (2.1) can be written as

$$(2.2) \quad \bar{\partial}g = \alpha g \quad \text{in } B_8.$$

Therefore, any solution of (2.2) is represented by

$$(2.3) \quad g = \exp(w)h \quad \text{in } B_8,$$

where  $h$  is holomorphic in  $B_8$  and

$$(2.4) \quad w(z) = -\frac{1}{\pi} \int_{B_8} \frac{\alpha(\xi)}{\xi - z} d\xi,$$

i.e.,  $w$  is the Cauchy transform of  $\alpha$ .

From (1.8) and the definition of  $\alpha$ , we have that

$$\|\alpha\|_{L^p(B_8)} \leq K$$

with  $2 < p$ . In view of the mapping properties of the Cauchy transform (see for example [Ve62]), we see that

$$(2.5) \quad |w(z)| \leq CK \quad \text{for } z \in B_8,$$

where  $C$  depends on  $p$ . Since  $h$  is holomorphic in  $B_8$ , Hadamard's three circle theorem implies

$$\|h\|_{L^\infty(B_1)} \leq \|h\|_{L^\infty(B_{r/4})}^\theta \|h\|_{L^\infty(B_6)}^{1-\theta},$$

where we choose  $r/4 < 1$  and

$$\theta = \frac{\log(6)}{\log(24/r)}.$$

Standard interior estimates imply that

$$(2.6) \quad \|h\|_{L^\infty(B_1)} \leq C(r^{-1}\|h\|_{L^2(B_{r/2})})^\theta \|h\|_{L^2(B_7)}^{1-\theta}.$$

On the other hand, it is not hard to prove that  $v$  of (1.7) satisfies the following Caccioppoli's inequality

$$(2.7) \quad \int_{B_r} |\nabla v|^2 \leq \frac{C \|A\|_{L^p(B_8)}^2}{(\rho - r)^2} \|v\|_{L^\infty(B_\rho)}^2 \leq \frac{CK^2}{(\rho - r)^2} \|v\|_{L^\infty(B_\rho)}^2, \quad 0 < r < \rho < 8,$$

where  $C$  depends on  $p$ . The derivation of (2.7) follows from the standard procedure using a cutoff function. We omit the details here. Combining (2.3), (2.6) and (2.7), we have that

$$(2.8) \quad \exp(-CK) \|\nabla v\|_{L^\infty(B_1)} \leq C(r^{-1} \exp(CK) \|v\|_{L^\infty(B_r)})^\theta \|v\|_{L^\infty(B_8)}^{1-\theta}.$$

Based on (2.8), we immediately prove

**Theorem 2.1.** *Let  $v \in W_{loc}^{2,p}(B_8)$  be a real solution of (1.7) with  $A$  satisfying (1.8). Assume that  $v$  satisfies  $|v(z)| \leq C_0$  for all  $z \in B_8$  and  $\sup_{B_1} |\nabla v(z)| \geq 1$ . Then for  $r < 1/e$ ,*

$$(2.9) \quad \|v\|_{L^\infty(B_r)} \geq r^{C_1 + C_2 K},$$

where  $C_1$  depends on  $C_0$  and  $C_2$  depends on  $p$ .

From Theorem 2.1, we can easily derive the following quantitative uniqueness estimate, which is (i) of Theorem 1.1.

**Corollary 2.2.** *Let  $u \in W_{loc}^{2,p}(\mathbf{R}^2)$  be a real solution of (1.1) with  $|u(z)| \leq C_0$  and  $|\nabla u(0)| = 1$ . Assume that*

$$\|W\|_{L^p(\mathbf{R}^2)} \leq \tilde{K}.$$

Then

$$(2.10) \quad \inf_{|z_0|=R} \sup_{|z-z_0|<1} |u(z)| \geq \exp(-CR^{1-2/p} \log R)$$

for  $R \gg 1$ , where  $C$  depends on  $p, \tilde{K}$  and  $C_0$ .

*Proof.* We use the scaling argument in [BK05]. Precisely, let  $|z_0| = R$  with  $R \gg 1$ , and define  $u_R(z) = u(R(z + z_0/R))$ . Then  $u_R$  satisfies

$$\Delta u_R + W_R \cdot \nabla u_R = 0 \quad \text{in } B_8,$$

where  $W_R(z) = RW(R(z + z_0/R))$ . It is clear that

$$\left( \int_{B_8} |W_R|^p \right)^{1/p} \leq R^{1-2/p} \left( \int_{\mathbf{R}^2} |W|^p \right)^{1/p} \leq \tilde{K} R^{1-2/p}.$$

Also, we observe that

$$|\nabla u_R(-z_0/R)| = R|\nabla u(0)| = R > 1.$$

Taking  $K = \tilde{K}R^{1-2/p}$  and  $r = R^{-1}$ , estimate (2.9) yields (2.10). □

### 3. The case of $p = 2$

Likewise, we consider the local problem (1.7). Here we assume that

$$(3.1) \quad \|A\|_{L^2(B_8)} \leq K.$$

We first establish an estimate of the maximal vanishing order of  $v$  to (1.7) under the assumption (3.1).

**Theorem 3.1.** *Let  $v \in W_{loc}^{2,2}(B_8)$  be a real solution of (1.7) with  $A$  satisfying (3.1). Assume that  $v$  satisfies  $|v(z)| \leq M$  ( $M \geq 1$ ) for all  $z \in B_8$  and*

$$\|\nabla v\|_{L^2(B_{6/5})} \geq 1.$$

*Then for  $r < C_1$*

$$(3.2) \quad \|v\|_{L^\infty(B_r)} \geq M^{-C_2} r^{C_3 \log M} r^{C_4 K^2},$$

*where  $C_1, \dots, C_4$  are absolute positive constants.*

The proof of Theorem 3.1 is more involved. Note that the formula (2.2) remains valid, i.e.,

$$(3.3) \quad \bar{\partial}g = \alpha g \quad \text{in } B_8,$$

and

$$\|\alpha\|_{L^2(B_8)} \leq K.$$

Likewise, let

$$w(z) = \frac{1}{\pi} \int_{B_8} \frac{\alpha(\xi)}{\xi - z} d\xi,$$

then any solution of (3.3) is represented by

$$g(z) = \exp(-w(z))h(z) \quad \text{for } z \in B_8$$

where  $h$  is holomorphic in  $B_8$ . It is not hard to see that

$$\|w\|_{W^{1,2}(B_8)} = \|w\|_{L^2(B_8)} + \|\nabla w\|_{L^2(B_8)} \leq CK.$$

In the sequel, we need to estimate  $\int_{B_r} \exp(2|w|)$  for  $r \leq 2$ . For this end, we recall the following Trudinger’s Sobolev embedding theorem in the plane [St72], [Tr67]. Assume that  $f \in W^{1,2}(B_1)$  and  $\|f\|_{W^{1,2}(B_1)} \leq 1$ , then there exist two absolute constants  $\tilde{\alpha}_*$  and  $\tilde{C}_*$  such that

$$\int_{B_1} \exp(\tilde{\alpha}_* f^2) \leq \tilde{C}_*.$$

By Poincaré’s inequality, we immediately obtain that

**Corollary 3.2.** *If  $f \in W^{1,2}(B_1)$ ,  $\int_{B_1} f = 0$ , and  $\|\nabla f\|_{L^2(B_1)} \leq 1$ , then there exist  $\alpha_*$  and  $C_*$  such that*

$$\int_{B_1} \exp(\alpha_* f^2) \leq C_*.$$

Our task now is to prove

**Lemma 3.3.** *For  $q > 0$  and  $0 < r \leq 2$ , we have that*

$$(3.4) \quad \frac{1}{|B_r|} \int_{B_r} \exp(q|w|) \leq Cr^{-qCK} \exp(qCK + q^2CK^2).$$

*Proof.* By a scaling argument, we can deduce from Corollary 3.2 that if  $f \in W^{1,2}(B_r)$ ,  $\int_{B_r} f = 0$ , and  $\|\nabla f\|_{L^2(B_r)} \leq 1$ , then

$$(3.5) \quad \frac{1}{r^2} \int_{B_r} \exp(\alpha_* f^2) \leq C_*.$$

To verify (3.5), we define  $f_r(x) = f(rx)$  for  $x \in B_1$  and observe that

$$\int_{B_1} |\nabla f_r|^2 = \int_{B_r} |\nabla f|^2.$$



Then (3.5) follows directly from Corollary 3.2.

Let us define  $w_r(x) = w(x) - \bar{w}_r$ , where  $\bar{w}_r = \frac{1}{|B_r|} \int_{B_r} w$ . We first consider the case when  $\|\nabla w_r\|_{L^2(B_r)} > 0$ . We can write

$$(3.6) \quad \int_{B_r} \exp(q|w_r|) = \int_{B_r} \exp\left(q\|\nabla w_r\|_{L^2(B_r)} \cdot \left| \frac{w_r}{\|\nabla w_r\|_{L^2(B_r)}} \right| \right) = \int_{B_r} \exp(a|f|),$$

where

$$a = q\|\nabla w_r\|_{L^2(B_r)} \quad \text{and} \quad f = \frac{w_r}{\|\nabla w_r\|_{L^2(B_r)}}.$$

Note that  $\|\nabla w_r\|_{L^2(B_r)} \leq CK$ . It is helpful to study the function  $e^{ax}$  for  $x > 0$ . We first consider the case when  $ax \leq \alpha_* x^2$ , i.e.,  $x \geq a/\alpha_*$ . In this case, it is trivial that  $e^{ax} \leq e^{\alpha_* x^2}$ . In the case when  $x \leq a/\alpha_*$ , we have  $e^{ax} \leq e^{a^2/\alpha_*}$ . Consequently, we obtain that

$$e^{ax} \leq e^{\alpha_* x^2} + e^{a^2/\alpha_*}, \quad x > 0.$$

Therefore, it follows from (3.5) and (3.6) that

$$(3.7) \quad \begin{aligned} \int_{B_r} \exp(q|w_r|) &\leq \int_{B_r} \exp(\alpha_* |f|^2) + \int_{B_r} \exp(a^2/\alpha_*) \\ &\leq (C_* + \exp(a^2/\alpha_*))r^2 \leq Cr^2 \exp(q^2CK^2). \end{aligned}$$

Next we want to estimate  $|\bar{w}_r|$ .

**Claim 3.4.**

$$|\bar{w}_r| \leq CK \log(1/r) + CK.$$

*Proof.* Note that

$$\begin{aligned} |\bar{w}_r - \bar{w}_{2r}| &= \left| \frac{1}{|B_r|} \int_{B_r} w - \bar{w}_{2r} \right| \leq \frac{1}{|B_r|} \int_{B_r} |w - \bar{w}_{2r}| \\ &\leq \frac{C}{|B_{2r}|} \int_{B_{2r}} |w - \bar{w}_{2r}| \leq C \left( \frac{1}{|B_{2r}|} \int_{B_{2r}} |w - \bar{w}_{2r}|^2 \right)^{1/2} \\ &\leq C \left( \int_{B_{2r}} |\nabla w|^2 \right)^{1/2} \leq CK. \end{aligned}$$

It is clear that

$$(3.8) \quad |\bar{w}_r| \leq |\bar{w}_r - \bar{w}_{2r}| + |\bar{w}_{2r} - \bar{w}_{4r}| + \dots + |\bar{w}_{2^k r}|.$$

We now choose

$$k = \left\lfloor \frac{1}{\log 2} \log\left(\frac{1}{r}\right) \right\rfloor + 1 \leq C \log\left(\frac{1}{r}\right),$$

where  $\lfloor \cdot \rfloor$  is the floor function. With the choice of  $k$ , we can see that

$$1 \leq 2^k r \leq 2.$$

Each term of (3.8) is bounded by  $CK$ . The claim follows immediately.  $\square$

It is clear that Claim 3.4 implies

$$(3.9) \quad \exp(q|\bar{w}_r|) \leq \exp(qCK)r^{-qCK}.$$

Combining (3.7) and (3.9) yields

$$\begin{aligned} \int_{B_r} \exp(q|w|) &\leq \int_{B_r} \exp(q|w - \bar{w}_r|) \exp(q|\bar{w}_r|) \\ &\leq Cr^{2-qCK} \exp(qCK + q^2CK^2). \end{aligned}$$

Now if  $\|\nabla w_r\|_{L^2(B_r)} = 0$ , then  $w(x) \equiv \bar{w}_r$  in  $B_r$ . Hence, we have

$$\int_{B_r} \exp(q|w|) = \int_{B_r} \exp(q|\bar{w}_r|) \leq Cr^{2-qCK} \exp(qCK).$$

The derivation of (3.4) is now completed.  $\square$

As above, we will apply Hadamard's three circle theorem to  $h = \exp(w)g$  with  $r_2 = 6/5$ ,  $r_3 = 2$ , and  $r_1 = r/4 < 6/5$ , i.e.,

$$(3.10) \quad \|\exp(w)g\|_{L^\infty(B_{r_2})} \leq \|\exp(w)g\|_{L^\infty(B_{r_1})}^\theta \|\exp(w)g\|_{L^\infty(B_{r_3})}^{1-\theta},$$

where

$$(3.11) \quad \theta = \frac{\log(10/6)}{\log(8/r)}.$$

We will estimate the terms on both sides of (3.10). We begin with the terms on the right hand side. Note that  $v$  here also satisfies Caccioppoli's estimate (2.7) for  $p = 2$ . On the other hand, using the Poisson kernel of the

unit disc, it is easy to see that for any holomorphic function  $h$

$$\|h\|_{L^\infty(B_{r/2})} \leq C \frac{1}{|B_r|} \int_{B_r} |h|.$$

Putting all estimates together and in view of  $g = v_x - iv_y$ , we have that

$$\begin{aligned} (3.12) \quad & \|\exp(w)g\|_{L^\infty(B_{r/4})} \\ &= \|h\|_{L^\infty(B_{r/4})} \leq \frac{C}{|B_{r/2}|} \int_{B_{r/2}} |\exp(w)g| \\ &\leq C \left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} \exp(2|w|) \right)^{\frac{1}{2}} \left( \frac{1}{|B_{r/2}|} \int_{B_{r/2}} |\nabla v|^2 \right)^{\frac{1}{2}} \\ &\leq Cr^{-CK} \exp(CK^2) \|\nabla v\|_{L^2(B_{r/2})} \\ &\leq CKr^{-CK} \exp(CK^2) \|v\|_{L^\infty(B_r)} \\ &\leq C^{CK^2} r^{-CK} \|v\|_{L^\infty(B_r)}, \end{aligned}$$

where we used (3.4) with  $q = 2$  in the third inequality and Caccioppoli's estimate in the fourth inequality. Using (3.12) on the right hand side of (3.10) gives

$$\begin{aligned} (3.13) \quad & \|\exp(w)g\|_{L^\infty(B_{r/4})}^\theta \|\exp(w)g\|_{L^\infty(B_2)}^{1-\theta} \\ &\leq (C^{CK^2} r^{-CK} \|v\|_{L^\infty(B_r)})^\theta (C^{CK^2} 8^{-CK} \|v\|_{L^\infty(B_8)})^{1-\theta} \\ &\leq MC^{CK^2} (C^{CK^2} r^{-CK} \|v\|_{L^\infty(B_r)})^\theta. \end{aligned}$$

We now turn to the estimate of  $\|\exp(w)g\|_{L^\infty(B_{r/2})} = \|\exp(w)g\|_{L^\infty(B_{6/5})}$  on the left side of (3.10). From (3.4) with  $q = 4$  and  $r = 6/5$ , it is readily seen that

$$\begin{aligned} (3.14) \quad & 1 \leq \|\nabla v\|_{L^2(B_{6/5})} = \|g\|_{L^2(B_{6/5})} = \|\exp(-w)h\|_{L^2(B_{6/5})} \\ &\leq \|\exp(|w|)\|_{L^4(B_{6/5})} \|h\|_{L^4(B_{6/5})} \leq C^{CK^2} \|h\|_{L^\infty(B_{6/5})}. \end{aligned}$$

Combining (3.13), (3.14) and the form of  $\theta$  (see (3.11)), we immediately arrive at the estimate (3.2). The proof of Theorem 3.1 is completed.

Now we can put everything together to prove (ii) of Theorem 1.1.

*Proof of (ii) of Theorem 1.1.* Let  $|z_0| = R \gg 1$  and  $v(z) = u(R(z + z_0/R))$ . Then  $v$  solves (1.7) and with  $A(z) = RW(R(z + z_0/R))$ . Note that

$$\|A\|_{L^2(B_8)} \leq K$$

since  $\|W\|_{L^2(\mathbf{R}^2)} \leq K$ . The assumption of  $u$  implies  $\|v\|_{L^\infty(B_8)} \leq 9^m R^m$ , i.e.,  $M = 9^m R^m$ . On the other hand, we can see that for  $\tilde{z}_0 = -z_0/R$  ( $|\tilde{z}_0| = 1$ )

$$1 \leq \|\nabla u\|_{L^2(B_1)} = \|\nabla v\|_{L^2(B_{1/R}(\tilde{z}_0))} \leq \|\nabla v\|_{L^2(B_{6/5})}$$

provided  $R$  is large. Therefore, letting  $r = 1/R$  in (3.2), we obtain that

$$\begin{aligned} \|u\|_{L^\infty(B_1(z_0))} &= \|v\|_{L^\infty(B_r(0))} \geq (9^m R^m)^{-C_2} R^{-C_3 \log(9^m R^m)} R^{-C_4 K^2}, \\ &\geq \tilde{C}_1 \exp(-\tilde{C}_2(\log R)^2) \end{aligned}$$

provided  $R$  is sufficiently large, where  $\tilde{C}_1$  depends on  $m$  and  $\tilde{C}_2$  depends on  $m$  and  $K$ . □

Note that  $v(z) - v(0)$  is also a solution of (1.7). Thus the estimate of vanishing order (3.2) remains valid for  $v(z) - v(0)$ . Consequently, we obtain the following (SUCP) result.

**Corollary 3.5.** *Assume that  $\Omega$  is an open connected domain of  $\mathbf{R}^2$ . Let  $v \in W_{loc}^{2,2}(\Omega)$  be any solution of*

$$\Delta v + A \cdot \nabla v = 0 \quad \text{in } \Omega,$$

with real-valued drift  $A \in L^2(\Omega)$ , then  $v$  satisfies (SUCP), namely, if for some  $z_0 \in \Omega$

$$|v(z) - v(z_0)| = \mathcal{O}(|z - z_0|^N) \quad \text{for all } N \in \mathbb{N}, \text{ as } |z - z_0| \rightarrow 0,$$

i.e., if for  $N \in \mathbb{N}$ , there exist  $C_N > 0$  and  $r_N > 0$  such that

$$|v(z) - v(z_0)| \leq C_N |z - z_0|^N, \quad \forall |z - z_0| < r_N,$$

then  $v(z) \equiv v(z_0)$  for all  $z \in \Omega$ .

*Proof.* It suffices to consider a real solution  $v$ . First assume that  $z_0 = 0$  and  $B_8 \subset \Omega$ . We can always assume this by translation and scaling. Note that  $\|A\|_{L^2(B_8)}$  is finite. If  $v(z) \not\equiv v(0)$  in  $B_{6/5}$ , then  $\|\nabla v\|_{L^2(B_{6/5})} \geq C$  for some  $C > 0$ . The estimate (3.2) implies that  $v(z) - v(0)$  cannot vanish at 0 to infinite order. Therefore, we must have  $v(z) = v(0)$  for all  $z \in B_{6/5}$ . A chain of balls argument then finishes the proof. □

### 4. SUCP for an equation of divergence form

In this section, we would like to prove the SUCP for solutions of

$$(4.1) \quad \Delta v + \nabla \cdot (Av) = 0 \quad \text{in } \Omega,$$

where  $\Omega \subset \mathbf{R}^2$  is an open connected domain and  $A = (A_1, A_2)$  is a real-valued vector satisfying

$$(4.2) \quad \|A\|_{L^2(\Omega)} \leq C_0.$$

In other words, we will show that

**Theorem 4.1.** *Let  $v \in W_{loc}^{1,2}(\Omega)$  be any solution of (4.1). Let  $z_0 \in \Omega$  and*

$$|v(z)| = \mathcal{O}(|z - z_0|^N) \quad \text{as } |z - z_0| \rightarrow 0$$

for all  $N > 0$ , then  $v \equiv 0$  in  $\Omega$ .

*Proof.* As before, it suffices to consider a real solution  $v$ . We first assume  $z_0 = 0$ ,  $B_8 \subset \Omega$  and consider

$$(4.3) \quad \Delta v + \nabla \cdot (Av) = 0 \quad \text{in } B_8.$$

Since (4.3) is of divergence form, there exists  $\tilde{v}$  with  $\tilde{v}(0) = 0$  such that

$$(4.4) \quad \begin{cases} \partial_y \tilde{v} = \partial_x v + A_1 v, \\ -\partial_x \tilde{v} = \partial_y v + A_2 v. \end{cases}$$

Let  $f = v + i\tilde{v}$ , then  $f$  satisfies

$$(4.5) \quad \bar{\partial} f = \frac{1}{2}(A_1 + iA_2)v = \frac{1}{4}(A_1 + iA_2)(f + \bar{f}) = \alpha f,$$

where

$$\alpha = \begin{cases} \frac{1}{4}(A_1 + iA_2)(1 + \frac{\bar{f}}{f}) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

It follows from (4.2) that

$$(4.6) \quad \|\alpha\|_{L^2(B_8)} \leq C_0.$$

Any solution of (4.5) in  $B_8$  is written as  $f = \exp(-w)h$ , where  $h$  is holomorphic in  $B_8$  and

$$w(z) = \frac{1}{\pi} \int_{B_8} \frac{\alpha(\xi)}{\xi - z} d\xi.$$

As before, we have that

$$\|w\|_{W^{1,2}(B_8)} \leq C,$$

where  $C$  depends on  $C_0$ .

Applying Hadamard's three circle theorem to  $h = \exp(w)f$  with  $r_1 = r/4 < 1, r_2 = 1, r_3 = 2$ , we have that

$$(4.7) \quad \|\exp(w)f\|_{L^\infty(B_1)} \leq \|\exp(w)f\|_{L^\infty(B_{r/4})}^\theta \|\exp(w)f\|_{L^\infty(B_2)}^{1-\theta},$$

where

$$\theta = \theta(r) = \frac{\log 2}{\log(8/r)}.$$

As in the estimate (3.14), we can see that

$$(4.8) \quad \begin{aligned} \|v\|_{L^2(B_1)} &\leq \|f\|_{L^2(B_1)} = \|\exp(-w)h\|_{L^2(B_1)} \\ &\leq \|\exp(|w|)\|_{L^4(B_1)} \|h\|_{L^4(B_1)} \leq C \|h\|_{L^\infty(B_1)}. \end{aligned}$$

This estimate will give us a lower bound on the right hand side of (4.7).

It is not hard to prove that a Caccioppoli's type inequality holds for the solution  $v$  of (4.3), i.e., for  $r < \rho < 8$ , we have

$$(4.9) \quad \int_{B_r} |\nabla v|^2 \leq \frac{C}{(\rho - r)^2} \|v\|_{L^\infty(B_\rho)}^2.$$

As in the derivation of (3.12), we can obtain that

$$(4.10) \quad \begin{aligned} \|\exp(w)f\|_{L^\infty(B_{r/4})} &= \|h\|_{L^\infty(B_{r/4})} \leq \frac{C}{|B_{r/2}|} \int_{B_{r/2}} |\exp(w)f| \\ &\leq C \left( \int_{B_{r/2}} \exp(2|w|) \right)^{\frac{1}{2}} \left( \int_{B_{r/2}} |f|^2 \right)^{\frac{1}{2}} \\ &\leq Cr^{-C} \left( \int_{B_{r/2}} |f|^2 \right)^{\frac{1}{2}} \\ &\leq Cr^{-C} (\|v\|_{L^2(B_{r/2})} + \|\tilde{v}\|_{L^2(B_{r/2})}), \end{aligned}$$

where  $0 < r < 8$ . We now need to estimate  $\|\tilde{v}\|_{L^2(B_{r/2})}$  in (4.10). To this end, we can use (4.4) and (4.9) to compute

$$\begin{aligned}
 (4.11) \quad \int_{B_{r/2}} |\tilde{v}(x)|^2 &= \int_{B_{r/2}} |\tilde{v}(x) - \tilde{v}(0)|^2 = \int_{B_{r/2}} \left| \int_0^1 \nabla \tilde{v}(tx) \cdot x dt \right|^2 dx \\
 &\leq (r/2)^2 \int_{B_{r/2}} \int_0^1 |\nabla \tilde{v}(tx)|^2 dt dx \\
 &\leq Cr^3 \int_0^{r/2} \left\{ \frac{1}{|B_s|} \int_{B_s} |\nabla \tilde{v}(y)|^2 dy \right\} ds \\
 &\leq Cr^3 \int_0^{r/2} \left\{ \frac{1}{|B_s|} \int_{B_s} (|\nabla v(y)|^2 + |Av|^2) dy \right\} ds \\
 &\leq Cr^3 \int_0^{r/2} \left\{ \frac{\|v\|_{L^\infty(B_{2s})}^2}{s^2|B_s|} + \frac{\|v\|_{L^\infty(B_s)}^2}{|B_s|} \int_{B_s} |A|^2 dy \right\} ds \\
 &\leq Cr^3 \int_0^{r/2} \left\{ \frac{\|v\|_{L^\infty(B_{2s})}^2}{s^2|B_s|} + \frac{\|v\|_{L^\infty(B_s)}^2}{|B_s|} \right\} ds.
 \end{aligned}$$

The assumption that  $v$  vanishes at 0 to infinite order implies that there exist  $C_4 > 0$  and  $r_4 < 8$  such that

$$|v(z)| \leq C_4|z|^4, \quad \forall |z| < r_4.$$

The estimate (4.11) gives us

$$(4.12) \quad \int_{B_4} |\tilde{v}|^2 \leq C \left( \int_0^{r_4/2} + \int_{r_4/2}^8 \right) \left\{ \frac{\|v\|_{L^\infty(B_{2s})}^2}{s^2|B_s|} + \frac{\|v\|_{L^\infty(B_s)}^2}{|B_s|} \right\} ds \leq C.$$

Combining (4.10) and (4.12) yields

$$(4.13) \quad \|\exp(w)f\|_{L^\infty(B_2)}^{1-\theta} \leq C'$$

for all  $0 < \theta < 1$ , where  $C' > 0$ . Now if we assume that

$$(4.14) \quad \|v\|_{L^2(B_1)} \geq e^{-k}$$

for some  $k > 0$ , then we obtain from (4.7), (4.8), and (4.13) that

$$\tilde{C}r^{\tilde{C}k} \leq \|\exp(w)f\|_{L^\infty(B_{r/4})},$$

where  $\tilde{C}$  depends on  $C'$ . However, using the fact that  $v$  vanishes at 0 to infinite order, (4.10), (4.11), we have that there exist  $N_0 > \tilde{C}k$  and  $r_{N_0}$  so

that

$$\|\exp(w)f\|_{L^\infty(B_{r/4})} \leq C_{N_0} r^{N_0}$$

for all  $r < r_{N_0}$ . This leads to a contradiction. In other words, we must have  $\|v\|_{L^2(B_1)} < e^{-k}$  for all  $k > 0$  and hence  $v \equiv 0$  in  $B_1$ .

Now we consider the general case, i.e.,  $v$  vanishes at some  $z_0 \in \Omega$  to infinite order. We choose a  $r_0$  satisfying  $B_{8r_0}(z_0) \subset \Omega$ . We define  $\tilde{v}(z) = v(z_0 + r_0 z)$  and  $\tilde{A}(z) = r_0 A(z_0 + r_0 z)$ . Then

$$\Delta \tilde{v} + \nabla \cdot (\tilde{A} \tilde{v}) = 0 \quad \text{in } B_8$$

and

$$\int_{B_8} |\tilde{A}|^2 dz = \int_{B_{8r_0}(z_0)} |A|^2 dz \leq C_0.$$

Hence, we have that  $\tilde{v}(z) = 0$  in  $B_1$ , namely,  $v = 0$  in  $B_{r_0}(z_0)$ . Using similar arguments as in the proof of Corollary 3.5, we then conclude that  $v$  is identically zero in  $\Omega$ .  $\square$

### Acknowledgements

Carlos Kenig is supported in part by NSF Grant DMS-1265249 and DMS-0968472. Jenn-Nan Wang is supported in part by MOST Grants 102-2918-I-002-009 and 102-2115-M-002-009-MY3.

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RECEIVED JULY 16, 2014

