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Rojtman's theorem for normal schemes

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We show that Rojtman's theorem holds for normal schemes: For every reduced normal scheme X of finite type over an algebraically closed field k, the torsion subgroup of the zero'th Suslin homology is isomorphic to the torsion subgroup of the k-rational points of the albanese variety of X (the universal object for morphisms to semi-abelian varieties).

1. Introduction

By a classical theorem of Abel and Jacobi, there is an isomorphism between the Chow group of zero-cycles of degree 0 and the rational points of the Jacobian variety

$$CH_0(C)^0 \xrightarrow{\sim} \mathrm{Jac}_C(k)$$

for every smooth projective curve C over an algebraically closed field. Rosenlicht gave a generalization to smooth (open) curves, comparing a Chow group with modulus to the generalized Jacobian variety [19], an extension of an abelian variety by a torus.

If X is a smooth and projective scheme of higher dimension, it is natural generalization is to study the albanese map

$$\mathrm{alb}_X : CH_0(X)^0 \to \mathrm{Alb}_X(k).$$

It is surjective, but can have a large kernel if X has dimension at least 2 [16]. However, Rojtman [18] proved that alb_X induces an isomorphism of torsion subgroups away from the characteristic. A cohomological proof of Rojtman's theorem has been given by Bloch [5], and Milne [14] proved the analogous statement for the p-part in characteristic p.

Rojtman's theorem has been generalized in several directions. If X is projective, then, using an improved duality theorem, the method of Bloch and Milne carries over to generalize Rojtman's theorem to normal schemes

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[6]. If X is an open subscheme of a smooth projective scheme, then Spiess-Szamuely [22] showed that away from the characteristic, there is an isomorphism

$$\mathrm{alb}_X : {}_{\mathrm{tor}}H_0^S(X,\mathbb{Z})^0 \to {}_{\mathrm{tor}}\,\mathrm{Alb}_X(k).$$

Here the left hand side is Suslin homology, and the right hand side is Serre's albanese variety [20], the universal semi-abelian variety (i.e. extension of an abelian variety by a torus) to which X maps. This theorem was generalized by Barbieri-Viale and Kahn [1, Cor. 14.5.3] to normal schemes in characteristic 0. The theorem of Spiess-Szamuely holds for the p-part in characteristic p under resolution of singularities [7]. In this paper, we give a different proof of the theorem of Barbieri-Viale and Kahn which also gives the result in characteristic p, including the p-part:

Theorem 1.1. *Let X be a reduced normal scheme, separated and of finite type over an algebraically closed field k of characteristic $p \geq 0$. Then the albanese map induces an isomorphism*

$$_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \xrightarrow{\sim} {}_{\mathrm{tor}}\,\mathrm{Alb}_X(k)$$

up to p-torsion groups, and $H_1^S(X,\mathbb{Z}) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$ for $l \neq p$. Under resolution of singularities for schemes of dimension at most $\dim X$*, the restriction on the characteristic is unnecessary.*

We note that there are also generalizations of Rojtman's theorem comparing the cohomological Chow group (as defined by Levine-Weibel) with Lang's albanese variety (universal for rational maps to abelian varieties) see [2][4][12][13].

The idea of our proof is to work with the *albanese scheme* $\mathcal{A}_X$ introduced by Ramachandran. It is universal for maps from X to *locally semi-abelian schemes*, i.e. group schemes locally of finite type whose connected component is a semi-abelian variety and whose group of components is a lattice. The connected component of $\mathcal{A}_X$ is isomorphic to the albanese variety for every choice of a base-point of X. Using a suggestion of Ramachandran, we prove

Theorem 1.2. *Let X be a reduced, semi-normal, connected variety over a perfect field, and $a : X_\bullet \to X$ be a 1-truncated proper hypercover X such that $X_0 \to X$ is proper and generically etale. Then the albanese scheme $\mathcal{A}_X$ of X is the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$, where $d = (\delta_0)_* - (\delta_1)_*$ for $\delta_i : X_1 \to X_0$ the two face maps.*

On the other hand, for each prime l different from the characteristic of k, Suslin homology tensored with $\mathbb{Z}_{(l)}$ can be calculated by a proper ldh-covering [9] (and by a hyperenvelope at the characteristic). This allows us to prove the main theorem by reducing to the theorem of Spiess-Szamuely.

If X is not normal, then Rojtman's theorem is wrong even for curves, as one sees by taking an elliptic curve and identifying 0 with a non-torsion point [6]. However, we propose a statement in terms of a hypercover which could serve as a generalization of Rojtman's theorem, and prove it for curves. The statement for curves has he following explicit version:

Theorem 1.3. *Let C be a reduced semi-normal curve over an algebraically closed field with normalization $\tilde{C}$. Then the albanese map induces an isomorphism*

$$H_0^S(C,\mathbb{Z}) \cong \mathbb{Z}[\pi_0(C)] \oplus \mathcal{A}^0_{\tilde{C}}(k)/H_1(D_{C_\bullet},\mathbb{Z})$$

Here $H_1(D_{C_\bullet},\mathbb{Z})$ is a free abelian group dual to $H^1_{\text{ét}}(C,\mathbb{Z})$, and $\mathcal{A}^0_{\tilde{C}}$ the connected component of the albanese scheme. Dividing the rational points $\mathcal{A}^0_{\tilde{C}}(k)$ by a free subgroup makes the torsion subgroup larger, explaining the example in [6].

The results of this paper were reported on in [8].

Notation: For abelian group A, we write $A_{(l)}$ for $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}$, $A[l]$ for the subgroup of l-power torsion elements, and ${}_{\text{tor}}A = \oplus_l A[l]$ for the subgroup of torsion elements.

2. The Albanese scheme

Throughout this paper, k is a perfect field, and Sch/k the category of schemes locally of finite type over k. The following discussion is based on work of Ramachandran [17], the notation follows Kahn-Sujatha [11]. A *semi-abelian variety* is an extension of an abelian variety by a torus, and a *locally semi-abelian scheme* is a commutative group scheme such that the scheme of components $\pi_0(\mathcal{A})$ is a lattice D, and the connected component $\mathcal{A}^0$ is a semi-abelian variety. For a homomorphism between semi-abelian varieties $d : \mathcal{A}^0_1 \to \mathcal{A}^0_0$, the image of d is a closed subvariety, the quotient $\mathcal{A}^0_1/d\mathcal{A}^0_0$ is a again a semi-abelian variety, and, if k is algebraically closed, $\mathcal{A}^0_1(k)/d\mathcal{A}^0_0(k)$ is isomorphic to $\big(\mathcal{A}^0_1/d\mathcal{A}^0_0\big)(k)$.

Lemma 2.1. *Let $d : \mathcal{A}_1 \to \mathcal{A}_0$ be a homomorphism of locally semi-abelian schemes and $\mathcal{A}$ be the largest locally semi-abelian scheme quotient of the* *presheaf quotient $\mathcal{A}_0/d\mathcal{A}_1$. Then $\mathcal{A}^0$ is the largest semi-abelian variety quotient of*

$$\mathcal{A}_0^0/\big(d\mathcal{A}_1 \cap \mathcal{A}_0^0\big) \cong \mathcal{A}_0^0/\langle d\mathcal{A}_1^0, \delta \ker(D_1 \to D_0)\rangle.$$

Here δ is the map from the snake Lemma in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}_1^0 & \longrightarrow & \mathcal{A}_1 & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}_0^0 & \longrightarrow & \mathcal{A}_0 & \longrightarrow & D_0 & \longrightarrow & 0. \end{array} \tag{1}$$

Proof. This follows from the definitions and the exact sequence of presheaves on Sch/k:

$$\ker(D_1 \to D_0) \xrightarrow{\delta} \mathcal{A}_0^0/d\mathcal{A}_1^0 \to \mathcal{A}_0/d\mathcal{A}_1 \to D_0/D_1 \to 0. \qquad \square$$

Consider the site $(Sch/k)_{fl}$ with the fppf-topology. Let Z_X be the sheaf associated to the free abelian group on the presheaf $U \mapsto \mathbb{Z}[\mathrm{Hom}_k(U, X)]$ represented by X. The *albanese scheme* $u_X : X \to \mathcal{A}_X$ is the universal object for morphisms from Z_X to sheaves represented by locally semi-abelian schemes. For reduced schemes of finite type over k, the albanese scheme exists [17, Thm.1.11], and the assignment $X \to \mathcal{A}_X$ is a covariant functor. By definition, there is an exact sequence

$$0 \to \mathcal{A}_X^0 \to \mathcal{A}_X \to D_X \to 0,$$

and D_X is the flat sheaf associated to the free presheaf $U \mapsto \mathbb{Z}\,\mathrm{Hom}(U, \pi_0(X))$, where $\pi_0(X)$ is the largest quotient scheme of X etale over k. For example, $D_X \cong \mathbb{Z}$ if X is geometrically connected. For a connected scheme X, the scheme $\mathcal{A}_X^0$ is isomorphic to the usual Albanese variety Alb_X for every choice of a base-point x_0, because the map $X \to \mathcal{A}_X^0$, $x \mapsto u_X(x) - u_X(x_0)$ factors through Alb_X by the universal property.

Recall that Suslin homology is the homology of the complex $C_*(X)$ which in degree i consists of the free abelian group generated by closed irreducible subschemes of $X \times \Delta^i$ which map finitely and surjectively onto Δ^i. The boundary maps are the alternating sums of pull-back maps to the faces. Let $H_0^S(X, \mathbb{Z})^0$ be the kernel of the canonical degree map (induced by covariant functoriality) $H_0^S(X, \mathbb{Z}) \to H_0^S(\pi_0(X), \mathbb{Z}) \cong D_X$.

Lemma 2.2. *The albanese map induces a map from Suslin-homology to the albanese scheme such that the following diagram is commutative:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_0^S(X,\mathbb{Z})^0 & \longrightarrow & H_0^S(X,\mathbb{Z}) & \longrightarrow & D_X & \longrightarrow & 0 \\
 & & \downarrow & & \mathrm{alb}_X \downarrow & & \Vert & & \\
0 & \longrightarrow & \mathcal{A}_X^0(k) & \longrightarrow & \mathcal{A}_X(k) & \longrightarrow & D_X & \longrightarrow & 0
\end{array} \tag{2}
$$

Proof. Extending the map u_X to formal linear combinations, we obtain a map from zero-cycles on X to $\mathcal{A}_X$, and the Lemma follows exactly as in [22, Lemma 3.1]. □

2.1. Proper hypercovers

Recall that a simplicial object and an n-truncated simplicial object in a category $\mathcal{C}$ is a contravariant functor from the category of finite totally ordered sets $\boldsymbol{\Delta}$ and the category $\boldsymbol{\Delta}^{\leq n}$ of finite totally ordered sets of order at most $n+1$ to $\mathcal{C}$, respectively. If $\mathcal{C}$ has finite limits, then the restriction functor i_n^* from simplicial sets to n-truncated simplicial sets has a right adjoint $(i_n)_*$, and we denote the composition $(i_n)_* i_n^*$ by cosk_n. A proper hypercover $X_\bullet \to X$ is an augmented simplicial scheme $X_\bullet$ such that the adjunction maps $X_{i+1} \to (\mathrm{cosk}_i X_\bullet)_{i+1}$ are proper and surjective, and n-truncated proper hypercovers are defined similarly. For example, a 1-truncated proper hypercover is a diagram

$$X_1 \rightrightarrows X_0 \to X$$

such that $a\delta_0 = a\delta_1$, and such that $(\delta_0, \delta_1) : X_1 \to X_0 \times_X X_0$ is proper and surjective, together with a section $s : X_0 \to X_1$ to δ_0 and δ_1.

Proof of Theorem 1.2. Let $d = \delta_{0*} - \delta_{1*} : \mathcal{A}_{X_1} \to \mathcal{A}_{X_0}$, and let $\mathcal{A}$ be the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$. Consider the following commutative diagram

$$
\begin{array}{ccccc}
X_1 & \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} & X_0 & \xrightarrow{a} & X \\
u_1 \downarrow & & u_0 \downarrow & & u_X \downarrow \\
\mathcal{A}_{X_1} & \overset{\delta_{0*}}{\underset{\delta_{1*}}{\rightrightarrows}} & \mathcal{A}_{X_0} & \xrightarrow{a_*} & \mathcal{A}_X,
\end{array}
$$

and denote the composition $X_0 \overset{u_0}{\to} \mathcal{A}_{X_0} \to \mathcal{A}$ by u'. Since $a\delta_0 = a\delta_1$, the canonical map $\mathcal{A}_{X_0} \to \mathcal{A}_X$ factors through $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$, hence through $\mathcal{A}$. It suffices to show that the induced map

$$\mathcal{A} \to \mathcal{A}_X$$

is an isomorphism of locally semi-abelian schemes. Since a is surjective, so is the composition $\mathcal{A}_{X_0} \to \mathcal{A} \to \mathcal{A}_X$, and it suffices to show that the map $u_X : X \to \mathcal{A}_X$ factors through $\mathcal{A}$. Let $V \subseteq X$ be a dense open subset such that $V_0 = a^{-1}V \to V$ is etale and surjective, hence faithfully flat and thus a universal epimorphism, i.e. for any scheme T, the following sequence is an equalizer:

$$\mathrm{Hom}(V,T) \to \mathrm{Hom}(V_0,T) \rightrightarrows \mathrm{Hom}(V_0 \times_V V_0, T).$$

Since $V_0 \to V$ is etale and V is reduced, $V_0 \times_V V_0$ is reduced, hence after decreasing V further, we can assume that the map $V_1 = V \times_X X_1 \to V \times_X (X_0 \times X_0) \cong V_0 \times_V V_0$ induced by (δ_0, δ_1) is (faithfully) flat. Indeed being flat is an open condition, and since the target is reduced, surjectivity implies flatness at the generic points. Consequently $\mathrm{Hom}(V_0 \times_V V_0, T) \to \mathrm{Hom}(V_1, T)$ is injective for any scheme T.

Taking $T = \mathcal{A}$ and $u'|_{V_0} \in \mathrm{Hom}(V_0, \mathcal{A})$, the two pull-back maps to $\mathrm{Hom}(V_1, \mathcal{A})$, hence the pull-back maps to $\mathrm{Hom}(V_0 \times_V V_0, \mathcal{A})$ agree, so there is a unique map $v \in \mathrm{Hom}(V, \mathcal{A})$ with $u'|_{V_0} = v \circ a|_{V_0}$. It suffices to extend v to X. Consider the graphs $\Gamma_v \subset V \times \mathcal{A}$ and $\Gamma_{u'} \subset X_0 \times \mathcal{A}$ of v and u', respectively, and let C be the reduced image of $\Gamma_{u'}$ in $X \times \mathcal{A}$. Consider the following diagram

$$\begin{array}{ccccc}
\Gamma_v & \longrightarrow & V \times \mathcal{A} & \longrightarrow & V \\
{\scriptstyle incl}\downarrow & & {\scriptstyle incl}\downarrow & & {\scriptstyle incl}\downarrow \\
C & \longrightarrow & X \times \mathcal{A} & \longrightarrow & X \\
{\scriptstyle \alpha}\uparrow & & {\scriptstyle a'}\uparrow & & {\scriptstyle a}\uparrow \\
\Gamma_{u'} & \longrightarrow & X_0 \times \mathcal{A} & \longrightarrow & X_0.
\end{array}$$

Since the upper and lower horizontal compositions are isomorphisms, α is proper (and surjective), and C is closed in $X \times \mathcal{A}$. It suffices to show that the middle composition $f : C \to X$ is an isomorphism. Comparing to the upper row we see that f is birational and proper, and by maximality of the semi-normalization it suffices to show that it is a bijection on K-rational points for any field extension $k \subseteq K$.

Let $x_1, x_2 \in C(K)$ be two points with $f(x_1) = f(x_2)$, and $\tilde{x}_i$ be lifts to $\Gamma_{u'} \cong X_0$. We claim that the image $u'(\tilde{x}_i) \in \mathcal{A}$ is independent of i. Indeed, since $X_1 \twoheadrightarrow X_0 \times_X X_0$, there is a point $t \in X_1$ such that $\delta_i(t) = \tilde{x}_i$, hence $u'(\tilde{x}_1) = u'(\tilde{x}_2)$. So $a'(\tilde{x}_1, u'(\tilde{x}_1)) = a'(\tilde{x}_2, u'(\tilde{x}_2))$ and hence $x_1 = \alpha(\tilde{x}_1) = \alpha(\tilde{x}_2) = x_2$ because $C \to X \times \mathcal{A}$ is a closed embedding. □

Remark. The statement of the theorem is wrong if X is not semi-normal. For example, let C be $\mathbb{G}_m$ with a cusp at the point 1 and $p : \mathbb{G}_m \to C$ be the normalization. Then the simplicial albanese variety defined above is isomorphic to $\mathbb{G}_m$, but the (naive) albanese variety of C is trivial. Indeed, being a quotient of $\mathbb{G}_m$, it is either trivial or isomorphic to $\mathbb{G}_m$. If it was isomorphic to $\mathbb{G}_m$, then from the commutativity of

$$\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow[\sim]{u_{\mathbb{G}_m}} & \mathbb{G}_m \\
{\scriptstyle p}\big\downarrow & & \big\downarrow \\
C & \xrightarrow{u_C} & \mathbb{G}_m
\end{array}$$

we see that the composition $u_C p : \mathbb{G}_m \to \mathbb{G}_m$ is an isogeny of some degree m, hence every closed point has m inverse images. But as a dominant map to a regular curve, u_C is flat, and over the origin, p has degree 2, a contradiction.

3. Rojtman's theorem

From now on we assume that our base field is algebraically closed. Consider a 2-truncated simplicial scheme $X_\bullet$ together with the corresponding map of locally semi-abelian schemes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{A}^0_{X_2} & \longrightarrow & \mathcal{A}_{X_2} & \longrightarrow & D_{X_2} & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle d}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0_{X_1} & \longrightarrow & \mathcal{A}_{X_1} & \longrightarrow & D_{X_1} & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle d}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{A}^0_{X_0} & \longrightarrow & \mathcal{A}_{X_0} & \longrightarrow & D_{X_0} & \longrightarrow & 0
\end{array} \tag{3}$$

with d is the alternating sum of the maps induced by the face maps $X_i \to X_{i-1}$. Taking k-rational points, we obtain an analogous diagram of abelian groups. We define $H_i(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}/\mathbb{Z}) = \mathrm{Tor}_i(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}/\mathbb{Z})$ to be the homology of the double complex $\mathcal{A}_{X_\bullet}(k) \to \mathcal{A}_{X_\bullet}(k)_{\mathbb{Q}}$. It is easy to see that $H_0(D_{X_\bullet}, \mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$ is free, so that the exact sequence of k-rational points

$$H_1(D_{X_\bullet},\mathbb{Z}) \xrightarrow{\delta} \mathcal{A}^0_{X_0}(k)/d\mathcal{A}^0_{X_1}(k) \\ \to \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) \to H_0(D_{X_\bullet},\mathbb{Z}) \to 0 \tag{4}$$

gives an isomorphism of abelian groups

$$_{\mathrm{tor}}\big(\mathcal{A}^0_{X_0}(k)/(d\mathcal{A}^0_{X_1}(k)+\operatorname{im}\delta)\big) \cong {}_{\mathrm{tor}}\big(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)\big).$$

We follow [17] and [3] in defining the Albanese 1-motive $M(X_\bullet)$ of $X_\bullet$ as

$$H_1(D_{X_\bullet},\mathbb{Z})/\mathrm{tor} \xrightarrow{\delta} \mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}.$$

Its homology with $\mathbb{Q}_l/\mathbb{Z}_l$-coefficients

$$H_1(M(X_\bullet),\mathbb{Q}_l/\mathbb{Z}_l) := H_1(M(X_\bullet)_{(l)} \to M(X_\bullet)_{\mathbb{Q}})$$

sits in an exact sequence

$$0 \to (\mathcal{A}^0_{X_0}(k)/d\mathcal{A}^0_{X_1}(k))[l] \\ \to H_1(M(X_\bullet),\mathbb{Q}_l/\mathbb{Z}_l) \to H_1(D_{X_\bullet},\mathbb{Z})\otimes\mathbb{Q}_l/\mathbb{Z}_l \to 0. \tag{5}$$

Theorem 3.1. *Let X be separated and of finite type over the algebraically closed field k of characteristic $p \geq 0$, and let $X_\bullet$ be a 2-truncated proper hypercover of X consisting of normal schemes. Then we have canonical isomorphisms*

$$H^S_1(X,\mathbb{Q}_l/\mathbb{Z}_l) \cong H_1(\mathcal{A}_{X_\bullet}(k),\mathbb{Q}_l/\mathbb{Z}_l) \cong H_1(M(X_\bullet),\mathbb{Q}_l/\mathbb{Z}_l)$$

if either $l \neq p$, or if resolution of singularities exists for schemes of dimension at most $\dim X$.

Proof. Recall that for $l \neq p$, an n-truncated l-hyperenvelope (hyperenvelope) is a n-truncated proper hypercover $X_\bullet \to X$ satisfying the following condition: for any $i \leq n$ and any point in the target of $X_{i+1} \to (\operatorname{cosk}_i X_\bullet)_{i+1}$, there is a point mapping to it such that the extension of residue fields is finite of order prime to l (trivial), respectively. We introduce the convention that for $l = p$, an n-truncated l-hyperenvelope means an n-truncated hyperenvelope.

We first prove the theorem in case that $X_\bullet$ is a 2-truncated proper l-hyperenvelope of X which is contained as an open subscheme in a 2-truncated simplicial scheme $\bar{X}_\bullet$ consisting of smooth, projective schemes.

Let X be a smooth scheme embedded in a smooth projective scheme $\bar{X}$. Let B be the image of d_1 in the Suslin complex $C_2(X) \overset{d_1}{\to} C_1(X) \overset{d_0}{\to} C_0(X)$, and consider the double complex $S(X)$ given by

$$\begin{array}{ccc}
(C_1(X)/B)_{(l)}/\mathrm{tor} & \xrightarrow{d_0} & C_0(X)_{(l)} \\
\downarrow & & \downarrow \\
(C_1(X)/B)_{\mathbb{Q}} & \xrightarrow{d_0} & C_0(X)_{\mathbb{Q}}
\end{array}$$

Clearly $H_i(S(X)) \cong H_i^S(X, \mathbb{Q}_l/\mathbb{Z}_l)$ for $i \le 1$, and since the left vertical map is injective, we obtain $H_i(S(X)) = 0$ for $i \ge 2$.

Let $T(X)$ be the complex $\mathcal{A}_X(k)_{(l)} \to \mathcal{A}_X(k)_{\mathbb{Q}}$. The albanese map induces a map of complexes $S(X) \to T(X)$, which we claim to be a quasi-isomorphism. Indeed, in degree 0 both groups are isomorphic to $D_X \otimes \mathbb{Q}_l/\mathbb{Z}_l$, and the map is compatible with this isomorphism by Lemma 2.1. By our hypothesis on X, the theorem of Spiess-Szamuely [22] implies that $H_1^S(X, \mathbb{Z}) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$, hence in degree 1 the map induces the isomorphism $H_0^S(X, \mathbb{Z})[l] \cong \mathcal{A}_X(k)[l]$.

By the above discussion, the rows of complexes coming from the hypercover,

$$\begin{array}{ccccc}
S(X_2) & \longrightarrow & S(X_1) & \longrightarrow & S(X_0) \\
\downarrow & & \downarrow & & \downarrow \\
T(X_2) & \longrightarrow & T(X_1) & \longrightarrow & T(X_0).
\end{array}$$

are quasi-isomorphic. By [9], the upper row calculates $H_1^S(X, \mathbb{Q}_l/\mathbb{Z}_l)$, whereas the lower row calculates $H_1(\mathcal{A}_{X_\bullet}(k), \mathbb{Q}_l/\mathbb{Z}_l)$ because $X_\bullet$ is an l-hyperenvelope.

We claim that the canonical map $\alpha : M(X_\bullet) \to \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)$ induces the second isomorphism. If we first take vertical homology in the double complex $T(X_\bullet)$, we obtain as the E_1-page

$$\begin{array}{ccccc}
\mathcal{A}_{X_2}(k)[l] & \longrightarrow & \mathcal{A}_{X_1}(k)[l] & \longrightarrow & \mathcal{A}_{X_0}(k)[l] \\
D_{X_2} \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & D_{X_1} \otimes \mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & D_{X_0} \otimes \mathbb{Q}_l/\mathbb{Z}_l.
\end{array}$$

Now

$${}_{\mathrm{tor}}\mathcal{A}_{X_0}(k)/d_{\mathrm{tor}}\mathcal{A}_{X_1}(k) = {}_{\mathrm{tor}}\mathcal{A}^0_{X_0}(k)/d_{\mathrm{tor}}\mathcal{A}^0_{X_1}(k) \cong {}_{\mathrm{tor}}\big(\mathcal{A}^0_{X_0}(k)/d\mathcal{A}^0_{X_1}(k)\big)$$

on the one hand, and since $H_0(D_{X_\bullet}, \mathbb{Z})$ as well as the D_{X_i} are torsion free,

$$H_1(D_{X_\bullet}) \otimes \mathbb{Q}_l/\mathbb{Z}_l \xrightarrow{\sim} H_1(D_{X_\bullet}, \mathbb{Q}_l/\mathbb{Z}_l) = H_1(D_{X_\bullet} \otimes \mathbb{Q}_l/\mathbb{Z}_l).$$

Hence α induces a diagram of short exact sequences

$$
\begin{array}{ccccc}
(\mathcal{A}^0_{X_0}(k)/d\mathcal{A}^0_{X_1}(k))[l] & \longrightarrow & H_1(M(X_\bullet),\mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & H_1(D_{X_\bullet},\mathbb{Z})\otimes\mathbb{Q}_l/\mathbb{Z}_l \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_{X_0}(k)[l]/d\mathcal{A}_{X_1}(k)[l] & \longrightarrow & H_1(\mathcal{A}_{X_\bullet}(k),\mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & H_1(D_{X_\bullet}\otimes\mathbb{Q}_l/\mathbb{Z}_l).
\end{array} \tag{6}
$$

in which the outer maps are isomorphisms, and hence so is the middle map. This ends the proof of Theorem 3.1 in the case that $X_\bullet$ is a 2-truncated l-hyperenvelope contained as an open simplicial scheme in a simplicial scheme $\bar{X}_\bullet$ consisting of smooth, projective schemes.

By Gabber's refinement of de Jong's theorem on alterations [10] (or assuming resolution of singularities for schemes of dimension at most $\dim X$ if $l = p$), any scheme over a perfect field admits an l-hyperenvelope $X_\bullet$ consisting of smooth schemes which can be embedded into smooth projective schemes (see [9, Thm.1.5]). Hence for *any* reduced semi-normal scheme, the albanese map induces a map of short exact coefficient sequences

$$
\begin{array}{ccccc}
H_1^S(X)\otimes\mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & H_1^S(X,\mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & H_0^S(X,\mathbb{Z})[l] \\
\downarrow & & \| & & \downarrow \\
H_1(\mathcal{A}_{X_\bullet}(k))\otimes\mathbb{Q}_l/\mathbb{Z}_l & \longrightarrow & H_1(\mathcal{A}_{X_\bullet}(k),\mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & (\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))[l],
\end{array} \tag{7}
$$

and hence a surjection

$$\mathrm{alb}_X : H_0^S(X,\mathbb{Z})[l] \to (\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))[l]$$

if either $l \neq p$ or if resolution of singularities exists. We next show that for *normal* X, the left hand terms of (7) vanish, so that this surjection is an isomorphism.

Recall that $H^i_{\text{ét}}(X_\bullet,\mathcal{F}^\bullet)$ is the derived functor of the global section functor $\mathcal{F}^\bullet \mapsto \ker \delta_0^* - \delta_1^* : \mathcal{F}^0(X_0) \to \mathcal{F}^1(X_1)$ from simplicial sheaves of abelian groups on $X_\bullet$ to abelian groups.

Proposition 3.2. *If $a : X_\bullet \to X$ is a 2-truncated proper hypercover by normal schemes, then*

$$H^i_{\text{ét}}(X,\mathbb{Z}) \cong H^i_{\text{ét}}(X_\bullet,\mathbb{Z}) \cong H^i(D_{X_\bullet},\mathbb{Z})$$

for $i \leq 1$. In particular, $H_1(D_{X_\bullet},\mathbb{Z})$ is finite for normal X.

Proof. We can assume that X is connected. Consider the functor $a_* : \mathcal{F}^\bullet \mapsto \ker \mathcal{F}^0 \xrightarrow{\delta_0^*-\delta_1^*} \mathcal{F}^1$ from simplicial sheaves of abelian groups on $X_\bullet$ to sheaves of abelian groups on X, and let Ra_* be the total derived functor. If $a_p : X_p \to X$ is the canonical map, we have the spectral sequence

$$E_1^{p,q} = R^q(a_p)_*\mathcal{F}^p \Rightarrow R^{p+q}a_*\mathcal{F}^\bullet.$$

Since $H^i_{\text{ét}}(X_\bullet, \mathcal{F}^\bullet) = H^i_{\text{ét}}(X, Ra_*\mathcal{F}^\bullet)$, it suffices to show that $a_*\mathbb{Z} \cong \mathbb{Z}$ and $R^1a_*\mathbb{Z} = 0$. But $H^1_{\text{ét}}(Y, \mathbb{Z}) = 0$ for normal schemes Y, so we have $R^1(a_p)_*\mathbb{Z} = 0$ for all p, and it suffices to show that

$$0 \to \mathbb{Z} \to (a_0)_*\mathbb{Z} \to (a_1)_*\mathbb{Z} \to (a_2)_*\mathbb{Z} \to \cdots$$

is an exact sequence of sheaves on X. By the proper base-change theorem for H^0 and constant sheaves [15, II Remark 3.8], we can assume that X is the spectrum of a separably closed field. In this case, the hypercover has a section, and the sequence splits. The second isomorphism follows because both groups are calculated by the complex $\mathbb{Z}^{\pi_0(X_\bullet)}$ in degrees at most 1.

For the final statement, $H^1_{\text{ét}}(X, \mathbb{Z}) = 0$ for normal X, hence the exact sequence

$$0 \to \operatorname{Ext}(H_{i-1}(D_{X_\bullet}, \mathbb{Z}), \mathbb{Z}) \to H^i(D_{X_\bullet}, \mathbb{Z}) \to \operatorname{Hom}(H_i(D_{X_\bullet}, \mathbb{Z}), \mathbb{Z}) \to 0$$

and finite generation of $H_1(D_{X_\bullet}, \mathbb{Z})$ shows that the group is finite. □

Lemma 3.3. *If $H_1(D_{X_\bullet}, \mathbb{Z}) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$, then $H_1(\mathcal{A}_{X_\bullet}(k)) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$.*

Proof. Since the groups $H_i(D_{X_i}, \mathbb{Z})$ are finitely generated, the hypothesis implies that $H_1(D_{X_\bullet}, \mathbb{Z})$ is finite, and we obtain a short exact sequence

$$H_1(\mathcal{A}^0_{X_\bullet}(k), \mathbb{Z}) \to H_1(\mathcal{A}_{X_\bullet}(k), \mathbb{Z}) \to (finite) \to 0.$$

The result follows because $H_1(\mathcal{A}^0_{X_\bullet}(k), \mathbb{Z})$ is an extension of a finite group by the (divisible) group of k-rational points of a semi-abelian variety, hence tensoring with $\mathbb{Q}_l/\mathbb{Z}_l$ annihilates it. □

Proof of Theorem 1.1. Consider the following diagram, where the vertical maps are induced by the albanese map:

$$\begin{array}{ccccc} H_1^S(X_\bullet, \mathbb{Q}_l/\mathbb{Z}_l) & \xrightarrow{\sim} & H_0^S(X_\bullet, \mathbb{Z})[l] & \xrightarrow{\sim} & H_0^S(X, \mathbb{Z})[l] \\ \sim\downarrow alb & & \downarrow alb & & \downarrow alb \\ H_1(\mathcal{A}_\bullet(k), \mathbb{Q}_l/\mathbb{Z}_l) & \xrightarrow{\sim} & (\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))[l] & \longrightarrow & \mathcal{A}_X(k)[l] \end{array} \tag{8}$$

The left vertical map and the left horizontal maps are isomorphisms by Theorem 3.1, Lemma 3.3 and (7), respectively. Recall the exact sequence of presheaves

$$H_1(D_{X_\bullet},\mathbb{Z}) \xrightarrow{\delta} \mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1} \to \mathcal{A}_{X_0}/d\mathcal{A}_{X_1} \to H_0(D_{X_\bullet},\mathbb{Z}) \to 0$$

from Lemma 2.1. From the exactness of the global section functor we obtain that $\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) \cong \big(\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}\big)(k)$, and since $H_0(D_{X_\bullet},\mathbb{Z})$ is torsion free, the right map in (8) factors through $\big(\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1} + \operatorname{im}\delta\big)(k)[l]$. The finiteness of $H_1(D_{X_\bullet},Z)$ implies that $\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1} + \operatorname{im}\delta$ is a semi-abelian variety, hence $\big(\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1} + \operatorname{im}\delta\big)(k)[l]$ is the l-torsion of the connected component of the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}/d\mathcal{A}_{X_1}$ by Lemma 2.1. By Proposition 1.2 this is the albanese scheme of X. □

To conclude the proof of Theorem 3.1 in the general case, we note that the only property of $X_\bullet$ used in the proof of Theorem 3.1 are $H_1^S(X_i,\mathbb{Z})\otimes\mathbb{Q}_l/\mathbb{Z}_l = 0$ and $H_0^S(X_i,\mathbb{Z})[l] \cong \mathcal{A}_{X_i}(k)[l]$. By Theorem 1.1, this hypothesis is satisfied for a hypercover by normal schemes. □

Remark. There is a canonical proper hypercover by normal schemes of a reduced semi-normal scheme X: Take $X_0 = \tilde{X}$ to be the normalization of X. The diagonal map $\tilde{X} \to (\tilde{X}\times_X \tilde{X})^{\mathrm{red}}$ is a closed immersion with a section between reduced schemes, so is an irreducible component. The normalization of $(\tilde{X}\times_X \tilde{X})^{\mathrm{red}}$ has $\tilde{X}$ as a connected component, and we let Z be its complement (of smaller dimension). Then

$$\tilde{X} \coprod Z \to \tilde{X} \to X$$

is a 1-truncated proper hypercover of X by normal schemes. Since the two projections from $\tilde{X}$ to itself are equal, the difference of the maps induced on albanese schemes is trivial, and the quotient of semi-abelian schemes is $\mathcal{A}_{\tilde{X}}(k)/d\mathcal{A}_Z(k)$ sitting in the exact sequence

$$\ker\big(D_Z \to D_{\tilde{X}}\big) \to \mathcal{A}^0_{\tilde{X}}/d\mathcal{A}^0_Z(k) \to \mathcal{A}_{\tilde{X}}(k)/d\mathcal{A}_Z(k) \to \mathbb{Z}^{\pi_0(X)} \to 0.$$

In particular, we obtain a surjection ${}_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \to {}_{\mathrm{tor}}(\mathcal{A}_{\tilde{X}}(k)/d\mathcal{A}_Z(k))$.

4. Curves

We start with an example showing that the statement of the main theorem fails for non-normal curves.

Let E be an elliptic curve and p be a closed point of E. Let N be the variety obtained by glueing the points 0 and p of E. A proper hypercover of

N in low degree is given by

$$E \times_N E \times_N E \to E \times_N E \xrightarrow{\delta_0,\delta_1} E \to N.$$

The middle term is isomorphic to $E \cup x \cup y$ where x and y correspond to the points $(0,p)$ and $(p,0)$ in the product, respectively. Similarly, the term on the left is isomorphic to E and 6 points corresponding to triples (x,y,z) with $x,y,z \in \{0,p\}$ and not all equal. The albanese schemes are

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & \mathcal{A}_2 & \longrightarrow & \mathbb{Z}^7 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \mathcal{A}_1 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle \delta_1}\downarrow{\scriptstyle \delta_0} & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \mathcal{A}_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

A calculation shows that $H_1(D_{X_\bullet}, \mathbb{Z}) = \mathbb{Z}$, and the sequence (4) becomes

$$H_1(\mathcal{A}_\bullet(k), \mathbb{Z}) \to \mathbb{Z} \xrightarrow{\delta} E(k) \to H_0(\mathcal{A}_\bullet(k), \mathbb{Z}) \to \mathbb{Z} \to 0,$$

where δ sends 1 to $p - 0$ on E. Now assume that p is not torsion. Then the albanese scheme of N is the largest locally semi-abelian scheme quotient of $\mathcal{A}_{X_0}$ modulo the subabelian variety generated by $\langle p \rangle$, hence it is isomorphic to $\mathbb{Z}$. The coranks of $H_1^S(N, \mathbb{Q}/\mathbb{Z})$ and of ${}_{\mathrm{tor}}H_0^S(N, \mathbb{Z})$ can be calculated to be 3, hence ${}_{\mathrm{tor}}H_0^S(N, \mathbb{Z})$ is not isomorphic to the torsion of the albanese variety. However ${}_{\mathrm{tor}}H_0^S(N, \mathbb{Z})$ isomorphic to the torsion of the abelian group quotient $H_0(\mathcal{A}_\bullet(k), \mathbb{Z}) = \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) \cong \big(\mathcal{A}^0_{X_0}/d\mathcal{A}^0_{X_1}\big)(k)/\operatorname{im}\delta$. In other words, taking the quotient in the category of locally semi-abelian schemes, and then taking rational points, does not give the correct answer, but taking rational points, and then dividing in the category of abelian groups, does. More generally:

Theorem 4.1. *Let X be a reduced semi-normal curve. Then the albanese map induces an isomorphism*

$$H_0^S(X, \mathbb{Z}) \cong \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k)$$

By (4), the right hand group is isomorphic to $H_0(D_{X_\bullet}, \mathbb{Z}) \oplus \mathcal{A}^0_{\tilde{X}}(k)/\operatorname{im} H_1(D_{X_\bullet}, \mathbb{Z})$, for $\tilde{X}$ the normalization of X, with $H_0(D_{X_\bullet}, \mathbb{Z}) \cong \mathbb{Z}^{\pi_0(X)}$ and $H_1(D_{X_\bullet}, \mathbb{Z})$ having the same rank as $H^1_{\text{ét}}(X, \mathbb{Z})$.

Proof. We can choose X_0 to be the normalization of X, and $X_1 = X_0 \coprod S$ with S of dimension 0. Since Suslin homology satisfies descent for hyperenvelopes [9], we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_0^S(X_1,\mathbb{Z}) & \longrightarrow & H_0^S(X_0,\mathbb{Z}) & \longrightarrow & H_0^S(X,\mathbb{Z}) & \longrightarrow & 0 \\ {\scriptstyle \mathrm{alb}_{X_1}}\downarrow & & {\scriptstyle \mathrm{alb}_{X_0}}\downarrow & & {\scriptstyle \mathrm{alb}_{X}}\downarrow & & \\ \mathcal{A}_{X_1}(k) & \xrightarrow{d} & \mathcal{A}_{X_0}(k) & \longrightarrow & \mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k) & \longrightarrow & 0. \end{array} \tag{9}$$

The left two vertical maps are isomorphisms by the Abel-Jacobi theorem stating that for a regular curve the albanese map is an isomorphism. Hence the right hand map is an isomorphism. □

Question. Does the analog statement hold in higher dimensions, i.e. is the surjection

$$\mathrm{alb}_X : {}_{\mathrm{tor}}H_0^S(X,\mathbb{Z}) \to {}_{\mathrm{tor}}(\mathcal{A}_{X_0}(k)/d\mathcal{A}_{X_1}(k))$$

an isomorphism for any reduced semi-normal scheme X?

The proof of Theorem 4.1 does not carry over, because there could be a uniquely divisible subgroup in the albanese kernel of X_0 which maps to a torsion divisible group in the albanese kernel of X.

On the other hand, the proof of Theorem 4.1 does carry over if the albanese map is an isomorphism (not only on the torsion part), as is known for the algebraic closure of finite fields, and expected for the algebraic closure of the field of rational numbers.

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