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A comparison principle for solutions to the Ricci flow

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In this note we derive a principle that enables us to bound solutions to the Ricci flow on a metric level.

1. Introduction

Barrier principles are a basic tool in the study of geometric flows. For example, if we have two solutions to the mean curvature flow with the property that one is contained in the other initially, then the maximum principle guarantees that this remains so for all subsequent times. It is a natural question whether there is a similar barrier principle for the Ricci flow. More specifically, if g_t is a family of metrics evolving by Ricci flow, we want to find conditions that ensure that the evolving metrics g_t stay above some family of comparison metrics. This is a non-trivial problem: indeed, while the mean curvature flow for hypersurfaces comes down to a scalar equation, the Ricci flow is a weakly parabolic system.

The following is the main result of this note: Let M^m, N^n be smooth manifolds and consider families $(g_t)_{t\in[0,T)}$ and $(h_t)_{t\in[0,T)}$ on M and N. We assume that g_t evolves by the Ricci flow $\partial_t g_t = -2\,\mathrm{Ric}_{g_t}$ and that the family h_t satisfies the following condition: For every point $p \in N$ and every two unit vectors $u, v \subset T_pN$ we have

$$\partial_t h_t(u,u) \leq -2(m-1)\max\{0, \sec_{h_t}(u\wedge v)\}. \tag{1.1}$$

Examples for such families are time-invariant metrics of non-positive sectional curvatures or solutions to the Ricci flow on which the sectional curvatures are non-negative and $m \leq n$.

Assume that $f_t : (M, g_t) \to (N, h_t)$, $t \in [0,T)$ is a family of smooth maps that evolve by harmonic map heat flow; that is

$$\frac{\partial}{\partial t} f_t = \tau_{g_t,h_t}(f_t). \tag{1.2}$$

Here, $\tau_{g,h}$ denotes the tension field, i.e.

$$\tau_{g,h}(f) = \sum_{k=1}^{m} (D_{e_k} df)(e_k) =: \triangle f, \tag{1.3}$$

where $\{e_k\}$ is a local orthonormal frame on (M, g) and D denotes the connection on $T^*M \otimes f^*TN$ induced by the Levi-Civita connections on (M, g) and (N, h).

Our theorem is then

Theorem 1.1. *Assume that M is closed, $(g_t)_{t\in[0,T)}$ is a Ricci flow, $(h_t)_{t\in[0,T)}$ satisfies the assumption involving equation (1.1) and that (M, h_t) is complete for all $t \in [0, T)$. Let $f : (M, g_0) \to (N, h_0)$ be a smooth and 1-Lipschitz map.*

Then the harmonic map heat flow equation (1.2) can be solved with the initial condition $f_0 = f$ for all times $t \in [0, T)$ and the Lipschitz constant of f_t is non-increasing in t.

In other words, if the initial map $f_0 : (M, g_0) \to (N, h_0)$ is distance-decreasing, then $f_t : (M, g_t) \to (N, h_t)$ is distance-decreasing for all $t \geq 0$. Note that we allow the dimension of N to be smaller than the dimension of M. The proof of Theorem 1.1 uses the maximum principle, and follows earlier work of Eells and Sampson [3]. We note that the harmonic map heat flow also plays a role in the shorttime existence theory for Ricci flow (see [1], [2], [4]).

A special case of this theorem is the following

Corollary 1.2. *Assume that M is closed, $(g_t)_{t\in[0,T)}$ is a Ricci flow and assume that (N, h) is a complete Riemannian manifold of non-positive sectional curvature. Let $f : (M, g_0) \to (N, h)$ be a smooth map.*

Then the harmonic map heat flow equation (1.2) with static target (N, h) can be solved for the initial condition $f_0 = f$ for all times $t \in [0, T)$ and the Lipschitz constant of f_t is non-increasing in t.

Remark 1.3. In the case in which $(N, h_t) = (\mathbb{R}, g_{eucl})$, the differential df_t evolves by the following heat equation

$$D_{\frac{\partial}{\partial t}} df_t = \triangle df_t,$$

where $D_{\frac{\partial}{\partial t}}$ represents the connection as in Uhlenbeck's trick, i.e. $D_{\frac{\partial}{\partial t}} df_t = \frac{\partial}{\partial t} df_t + df \circ \mathrm{Ric}$. In this setting, Theorem 1.1 follows directly via the maximum principle.

2. An evolution equation for the differential

Consider the pull back $T^{spat}(N \times [0,T))$ of the tangent bundle TN onto the space-time $N \times [0,T)$ via the obvious projection $N \times [0,T) \to N$. This is the bundle of *spatial vector fields.* The family of metrics $(h_t)_{t\in[0,T)}$ induces a metric on $T^{spat}(N \times [0,T))$. Consider the Levi-Civita connection D of h_t on each time-slice $N \times \{t\}$. We can extend D to a connection on $T^{spat}(N \times [0,T))$ via

$$D_{\frac{\partial}{\partial t}} X = \frac{\partial}{\partial t} X + \frac{1}{2} (\partial_t h) (X)$$

for every section X of $T^{spat}(N \times [0,T))$. Then D is a metric connection since

$$\frac{\partial}{\partial t} h(X,Y) = h(D_{\frac{\partial}{\partial t}} X, Y) + h(X, D_{\frac{\partial}{\partial t}} Y).$$

Similarly, we define a metric connection D on the dual $T^{spat*}(M \times [0,T))$ via

$$D_{\frac{\partial}{\partial t}} \alpha = \frac{\partial}{\partial_t} \alpha + \alpha \circ \mathrm{Ric}^M$$

for every section α of $T^{spat*}(M \times [0,T))$. This construction is known as Uhlenbeck's trick.

Assume for now that $(f_t)_{t\in[0,T)}$ is a solution to the harmonic map heat flow equation (1.2) and consider the induced map $f : M \times [0,T) \to N \times [0,T)$ that acts as the identity on the second factors. Then the vector bundle

$$E = T^{spat*}(M \times [0,T)) \otimes f^* T^{spat}(N \times [0,T))$$

over $M \times [0,T)$ has an induced (metric) connection D. In the following we will view df_t as a section of E. Note that the choice of D allows us to study the evolution of the eigenvalues of df_t in a convenient way, e.g. $D_{\frac{\partial}{\partial t}} df_t = 0$ implies that the eigenvalues of df_t don't change up to first order.

We will now compute the evolution equation for df_t. We first find that for any stationary vector field X on M

$$\begin{aligned}(D_{\frac{\partial}{\partial t}} df_t)(X) &= D_{\frac{\partial}{\partial t} + \frac{\partial f}{\partial_t}} (df_t(X)) + df_t(\mathrm{Ric}^M(X)) \\ &= D_{df_t(X)} \frac{\partial f}{\partial t} + \frac{1}{2} (\partial_t h) (df_t(X)) + df_t(\mathrm{Ric}^M(X)).\end{aligned}$$

Here we have used that $df(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + \frac{\partial f}{\partial t}$. Note that the term $D_{df_t(X)} \frac{\partial f}{\partial t}$, being a section of $T^{spat}(N \times [0,T))$, can be viewed as a section of the pull back $f^*T^{spat}(N \times [0,T))$ in which case we can write $D_X \frac{\partial f}{\partial t}$. By (1.3) we have

$$D_X \frac{\partial f}{\partial t} = \sum_{k=1}^{m} (D^2_{X,e_k} df_t)(e_k).$$

From now on, we will only work on a fixed time-slice $M \times \{t\}$ and we will drop the t-index. Note that we have the following identity, which follows from the fact that the Levi-Civita connection on N is torsion free

$$(D_A df)(B) = (D_B df)(A).$$

Hence

$$\begin{aligned} D_X \frac{\partial f}{\partial t} &= \sum_{k=1}^{m} (D^2_{X,e_k} df)(e_k) \\ &= \sum_{k=1}^{m} \Big((D^2_{e_k,X} df)(e_k) + R^{f^*TN}(e_k, X)(df(e_k)) - df(R^M(e_k,X)e_k) \Big) \\ &= \sum_{k=1}^{m} \Big((D^2_{e_k,e_k} df)(X) + R^{f^*TN}(e_k, X)(df(e_k)) \Big) - df(\mathrm{Ric}^M(X)). \end{aligned}$$

Putting everything together yields

Lemma 2.1. *The differential df_t satisfies the following evolution equation*

$$\Big((D_{\frac{\partial}{\partial t}} - \triangle) df_t \Big)(X) = \frac{1}{2} (\partial_t h)(df_t(X)) + \sum_{k=1}^{m} R^N(df_t(e_k), df_t(X))(df_t(e_k)).$$

3. Proof of the Theorem

We will use the following Lemma.

Lemma 3.1. *Assume that $(f_t)_{t \in [0,T)}$, $T < \infty$ is a solution to the harmonic map heat flow equation (1.2) and assume that $|df_t|$ is uniformly bounded on $M \times [0,T)$. Moreover, assume that (g_t) and (h_t) are smooth up to time T. Then all higher derivatives of f_t are uniformly bounded on $M \times [0,T)$ as well.*

Proof. Differentiating the evolution equation from Lemma 2.1 $k-1$ times yields

$$(D_{\frac{\partial}{\partial t}} - \triangle) d^k f_t = \sum_{\substack{i_0+\cdots+i_l=k \\ i_1\neq 0}} (D^{i_0}(\partial_t h)) * d^{i_1} f_t * \cdots * d^{i_l} f_t + \sum_{\substack{i_0+\cdots+i_l=k+2 \\ i_1,i_2,i_3\neq 0}} D^{i_0} R^N * d^{i_1} f_t * \cdots * d^{i_l} f_t.$$

Now assume by induction that $|df_t|, \ldots, |d^{k-1} f_t| < C$ on $M \times [0,T)$. Then we can estimate

$$(\partial_t - \triangle)|d^k f_t| \leq C'\big(1 + |d^k f_t|\big).$$

It follows by the maximum principle, that $|d^k f_t|$ is uniformly bounded on $M \times [0,T)$ as well. □

Proof of Theorem 1.1. The theorem follows with the help of the maximum principle. We may assume without loss of generality that the Lipschitz constant $\lambda(0)$ of f_0 is < 1. Note hereby that we may replace h_t by $(1-\varepsilon^2-\varepsilon t)h_t$ for small $\varepsilon > 0$.

Let $[0,T_1) \subset [0,T)$ be the maximal time-interval on which we can solve the harmonic map heat flow equation (1.2) and let $\lambda(t)$ be the Lipschitz constant of f_t.

We first show that $\lambda(t)$ is non-increasing in t. Pick $t < T_1$ such that $\lambda(t) < 1$ and consider a point $p \in M$ and a unit vector $u \in T_pM$ for which $|df_t(u)| = \lambda(t)$.

Then $\langle \triangle df_t(u), df_t(u)\rangle \leq 0$ and hence

$$\begin{aligned}
\lambda'(t) &= \langle (D_{\frac{\partial}{\partial t}} df_t)(u), df_t(u)\rangle \\
&\leq \left\langle \frac{1}{2}(\partial_t h)(df_t(u)) + \sum_{k=1}^m R^N(df_t(e_k), df_t(u))df_t(e_k), df_t(u)\right\rangle \\
&= \frac{1}{2}(\partial_t h)(df_t(u), df_t(u)) + \sum_{k=1}^m \sec_{h_t}(df_t(u)\wedge df_t(e_k)) \cdot |df_t(u)\wedge df_t(e_k)|^2 \\
&\leq \frac{1}{2}(\partial_t h)(df_t(u), df_t(u)) + \sum_{k=1}^m \max\{0, \sec_{h_t}(df_t(u)\wedge df_t(e_k))\} \leq 0
\end{aligned}$$

Here we have used that $|df_t(u)\wedge df_t(e_k)| \leq \lambda^2(t) \leq 1$. It follows that the condition $\lambda(t) < 1$ is preserved and hence $\lambda(t)$ is non-decreasing on $[0,T_1)$.

It remains to show that $T_1 = T$. Assume that $T_1 < T$. Then by Lemma 3.1 and the fact that $|df_t|$ is uniformly bounded on $M \times [0, T_1)$, we conclude that f_t is smooth up to time T_1. However, this contradicts the maximality of T_1. □

References

[1] S. Brendle, *Ricci Flow and the Sphere Theorem.* Graduate Studies in Mathematics, Vol. 111, American Mathematical Society (2010).

[2] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors.* J. Diff. Geom., **18**, 157–162 (1983).

[3] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math., **86**, 109–160 (1964).

[4] R. Hamilton, *The formation of singularities in the Ricci flow.* Surveys in Differential Geometry, Vol. II, 7–136, International Press, Somerville MA (1995).

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