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A constant coefficient Legendre-Hadamard system with no coercive constant coefficient quadratic form over $W^{1,2}$

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A family of linear homogeneous 2nd order strongly elliptic symmetric systems with real constant coefficients, and bounded nonsmooth convex domains Ω are constructed in $\mathbb{R}^6$ so that the systems have no constant coefficient coercive integro-differential quadratic forms over the Sobolev spaces $W^{1,2}(\Omega)$. The construction is deduced from the model construction for a 4th order scalar case [Ver14]. The latter is stated and parts of its proof discussed, one particular being the utility of having noncoercive formally positive forms as a starting point. An application of Macaulay's determinantal ideals to the noncoerciveness of formally positive forms for systems is then given.

Denote points of $\mathbb{R}^6$ by $X = (X_1, \dots, X_6)$, the gradient operator by $\partial = (\partial_1, \dots, \partial_6)$ with derivatives $\partial_\alpha = \frac{\partial}{\partial X_\alpha}$, and let $\triangle = \partial_1^2 + \cdots + \partial_6^2$ denote the Laplace operator.

In the following 6×6 systems of scalar valued equations the lowercase letters u, v, w, x, y, z represent the components of a complex (or real) valued vector field U, while the subscripts denote second order derivatives. Alternatively, the dependent variables are also written in uppercase with subscripts denoting components $U = (U_1, \dots, U_6) = (u, v, w, x, y, z)$. For each $0 < \gamma < 1/3$ then, consider the linear symmetric second order system of homogeneous equations,

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$$
\begin{aligned}
\triangle u - 2w_{13} - x_{14} - y_{15} - z_{16} &= 0, \\
\triangle v - 2w_{23} - x_{24} - y_{25} - z_{26} &= 0, \\
-2u_{13} - 2v_{23} + (2\triangle - \partial_1^2 - \partial_2^2)w - (\gamma+1)x_{34} & \\
-x_{56} - (\gamma+1)y_{35} - y_{46} - (\gamma+1)z_{36} - z_{45} &= 0, \\
-u_{14} - v_{24} - (\gamma+1)w_{34} - w_{56} + \triangle x - (1-\gamma^2)x_{44} & \\
+x_{55} - (1-\gamma^2)y_{45} - (1-\gamma^2)z_{46} &= 0, \\
-u_{15} - v_{25} - (\gamma+1)w_{35} - w_{46} - (1-\gamma^2)x_{45} & \\
+\triangle y - (1-\gamma^2)y_{55} + y_{66} - (1-\gamma^2)z_{56} &= 0, \\
-u_{16} - v_{26} - (\gamma+1)w_{36} - w_{45} - (1-\gamma^2)x_{46} & \\
-(1-\gamma^2)y_{56} + \triangle z + z_{44} - (1-\gamma^2)z_{66} &= 0.
\end{aligned} \tag{0.1}
$$

Denote the 6×6 systems of differential operators for these equations

$$
L_\gamma(\partial) = \left(L_\gamma^{jk}(\partial)\right)_{1 \le j,k \le 6}. \tag{0.2}
$$

In any open set $\Omega \subset \mathbb{R}^6$ the formally positive quadratic integro-differential forms

$$
\begin{aligned}
G_\gamma[U] = \int_\Omega & |u_1 + v_2 - w_3|^2 + |w_3 - \gamma x_4 - \gamma y_5 - \gamma z_6|^2 + |w_4 - y_6|^2 \\
& + |w_5 - z_4|^2 + |w_6 - x_5|^2 + |z_4 - x_6|^2 + |x_5 - y_4|^2 \\
& + |y_6 - z_5|^2 + |w_4 - x_3|^2 + |w_5 - y_3|^2 + |w_6 - z_3|^2 \\
& + |v_1 - u_2|^2 + |w_1 - u_3|^2 + |x_1 - u_4|^2 + |y_1 - u_5|^2 \\
& + |z_1 - u_6|^2 + |w_2 - v_3|^2 + |x_2 - v_4|^2 \\
& + |y_2 - v_5|^2 + |z_2 - v_6|^2 dX
\end{aligned} \tag{0.3}
$$

are associated with these systems by $G_\gamma[U] = -\sum_{j,k} \int_\Omega \overline{U_j} L_\gamma^{jk} U_k dX$ whenever U is twice continuously differentiable and vanishes at the boundary. The L_γ are *strongly elliptic*, i.e. each satisfies the Legendre-Hadamard condition $\overline{\eta}^{\mathsf{T}} L_\gamma(\xi)\eta \ge E_\gamma |\xi|^2 |\eta|^2,\ \xi \in \mathbb{R}^n,\ \eta \in \mathbb{C}^m$ for a constant $E_\gamma > 0$, as will be shown Section 1.1. Here η^{T} is the row vector that is the transpose of η.

For each $M > 0$ and $T > 0$ define the bounded convex domains of $\mathbb{R}^6$

$$
\Omega = \Omega_{M,T} = \left\{ X : T\sqrt{X_4^2 + X_5^2 + X_6^2} < X_3 < \frac{1 - \sqrt{X_1^2 + X_2^2}}{M} \right\}. \tag{0.4}
$$

Theorem 0.1. *Let $2 \leq M \leq 2\sqrt{2}$. For each $0 < \gamma < 1/3$*

(i) *The system of operators* (0.1) (0.2) L_γ *satisfies the Legendre-Hadamard condition.*

(ii) *The associated formally positive form* (0.3) G_γ *is not coercive over the vector valued Sobolev spaces* $W^{1,2}(\Omega)$ *for any bounded Lipschitz domain* Ω.

(iii) *For each* $L\gamma$ *there is no associated constant complex coefficient Hermitian quadratic integro-differential form, formally positive or not, that is coercive over* $W^{1,2}(\Omega_{M,T})$ *when* T *is taken large enough, depending only on* γ *and* M*, for the convex domains* (0.4) $\Omega_{M,T}$.

A quadratic integro-differential form $A[U]=\sum_{j,k,\alpha,\beta}\int_\Omega \partial_\alpha \overline{U_j} a^{jk}_{\alpha\beta}\partial_\beta U_k dX$ is termed *Hermitian* when it is derived from a bilinear form (1.3) that is anti-linear in one variable and linear in the other; sometimes called a sesquilinear form. In particular no symmetry condition, such as the Hermitian symmetric condition on the *matrices* $\overline{a_{\alpha\beta}}^{\mathsf{T}} = a_{\beta\alpha}$, is being assumed for the coefficients in (iii) of Theorem 0.1. Saying that a constant coefficient form $A[U]$ in (ii) or (iii) is *associated* to L_γ means that the coefficients of A satisfy $\sum_{\alpha,\beta} a_{\alpha\beta}\partial_\alpha\partial_\beta = L_\gamma$ or that $\sum_{\alpha,\beta} a_{\alpha\beta}\partial_\alpha\partial_\beta$ is an equivalent system by linear change of equations and dependent variables (see Section 2). In no case do the particular system coefficients of (0.1) uniquely determine the coefficients $a_{\alpha\beta}$ of the forms associated to L_γ.

For a quadratic form $A[V]$ to be *coercive* over $W^{1,2}(\Omega)$ when Ω is any bounded Lipschitz domain it suffices that there exists a constant $c > 0$ and another constant c_0 such that

$$ReA[V] + c_0 \int_\Omega |V|^2 dX \geq c\|V\|_1^2 \quad \text{for all } V \in W^{1,2}(\Omega).$$

As proved in [Aro61] this formulation of *the coercive inequality* with $ReA[V]$ in place of $|A[V]|$ is equivalent to the apparently weaker formulation of Definition 1.2 below; see Remark 1.5.

Perhaps the most widely studied Legendre-Hadamard systems are the $n \times n$ systems of elastostatics. A. Korn (see [Fri47] p. 443) formulated his famous inequality in order to apply variational methods to these systems. Korn's inequality is sometimes stated simply as coerciveness (see [KO89] p. 485, for example). It's original formulation is a more precise coerciveness [Fri47] p. 446, $\sum_{k,l}\int_\Omega(\partial_k V_l - \partial_l V_k)^2 \leq C\sum_{k,l}\int_\Omega(\partial_k V_l + \partial_l V_k)^2$ under the conditions $\sum_{k,l}\int_\Omega \partial_k V_l - \partial_l V_k = 0$, $k,l = 1,\dots,n$. When the Lamé constants that determine each elastostatics system are taken as $\mu > 0$ and $\lambda \geq -2\mu/n$ one obtains formally positive associated quadratic forms

$$(0.5)\qquad \begin{aligned} A_{\lambda,\mu}[V] &= \int_\Omega \lambda|divV|^2 + \frac{\mu}{2}\sum_{k,l}|\partial_k V_l + \partial_l V_k|^2 \\ &= \int_\Omega (\lambda + \frac{2\mu}{n})|divV|^2 + \frac{\mu}{n}\sum_{k,l}|\partial_k V_k - \partial_l V_l|^2 \\ &\quad + \frac{\mu}{2}\sum_{k\neq l}|\partial_k V_l + \partial_l V_k|^2 \end{aligned}$$

that yield the naturally occurring Neumann boundary operators (see Kupradze's book [Kup65]) of *elastic traction*, $\lambda divV\mathbf{n} + \mu(\partial V + \partial V^{\mathsf{T}})\mathbf{n}$. Here ∂V^{T} is the transpose of the *differential matrix* ∂V, and $\mathbf{n}$ denotes the outer unit normal vector at the boundary. The forms $A_{\lambda,\mu}[V]$ are functions of the symmetric part of the differential matrices. Korn's inequality shows that the forms dominate the anti-symmetric part and thus satisfy the coercive inequality when $\lambda > -2\mu/n$. This in turn yields existence of solutions with prescribed traction boundary values either by minimization of the forms or through the Lax-Milgram theorem. A form $A_{\lambda,\mu}[V]$ may be modified to be formally positive in each entry of the differential matrix, while remaining associated to the same elastostatics system, by adding a positive multiple of the null form (1.4) $Re\sum_{k,l}\int_\Omega \partial_k\overline{V_k}\partial_l V_l - \partial_k\overline{V_l}\partial_l V_k$, obviating Korn's inequality. However, the new form no longer yields the traction boundary operator of interest in elastic theory.

For the systems L_γ, part (ii) Theorem 0.1 says that no inequality of Korn's type can exist to show that the linear combinations of differential matrix entries, present in the associated form (0.3) $G_\gamma[U]$, dominate each entry of the differential matrix ∂U. Part (iii) says, when considering the systems in domains with corners, that in general no inequality of Korn's type can exist for *any* constant coefficient quadratic form associated to the L_γ. In general, classical Hilbert space methods, for solving Neumann problems derived only from constant coefficient forms for constant coefficient 2nd order systems, must completely break down when applied to systems such as the L_γ in Lipschitz domains.

Theorem 0.1 will be derived from the following theorem. Letters α, β used as *superscripts* on ∂ denote multi-indices. Since the parameters M and T measure the severity of the interior angles about the boundary disc

$\{(X_1, X_2, 0, 0, 0, 0) : X_1^2 + X_2^2 \leq 1\}$ of $\Omega_{M,T}$, they will be called *Lipschitz constants.*

Theorem 0.2. *[Ver14] For each elliptic constant coefficient 4th order operator*

$$\left(\frac{1}{4}(\partial_1^2 + \partial_2^2) - \partial_3^2\right)^2 + \left(\partial_3^2 - \gamma(\partial_4^2 + \partial_5^2 + \partial_6^2)\right)^2 + (\partial_3\partial_4 - \partial_5\partial_6)^2 + (\partial_3\partial_5 - \partial_6\partial_4)^2 + (\partial_3\partial_6 - \partial_4\partial_5)^2, \quad 0 < \gamma < 1/3, \tag{0.6}$$

and each Lipschitz constant M, $1 \leq M \leq \sqrt{2}$, there is a real number $T(\gamma, M)$ so that for all Lipschitz constants $T > T(\gamma, M)$ there is no associated constant complex coefficient Hermitian quadratic form

$$A[v] = \sum_{|\alpha|\leq 2, |\beta|\leq 2} \int_{\Omega_{M,T}} \partial^\alpha \overline{v} a_{\alpha\beta} \partial^\beta v dX, \tag{0.7}$$

that is coercive over the Sobolev spaces of functions with square integrable derivatives up to order 2 in the bounded convex domains $\Omega_{M,T}$.

Theorem 0.2 remains true after scaling $X_j \to 2X_j$ in the variables X_3, X_4, X_5, X_6 and replacing $\Omega_{M,T}$ with $\Omega_{2M,T}$, $1 \leq M \leq \sqrt{2}$, whence the interval of Theorem 0.1. In both theorems, because of the compactness of the interval, T may be taken large enough depending only on γ. With the scaling, the five 2nd order operators squared in (0.6) are seen to correspond to the 1st five squares of (0.3) when $(u, v, w, x, y, z) = U$ is the gradient of a scalar valued function $(U_1, \ldots, U_6) = \partial f$.

Besides the breakdown of classical Hilbert space methods, both theorems exhibit the general lack of coercive *Rellich identities* for real symmetric strongly elliptic operators on the boundaries of Lipschitz (in fact, convex Lipschitz) domains when only constant coefficient forms are employed. See the proof from the introduction of [Ver14]; for the history and subsequent more recent applications of coercive Rellich identities to strong pointwise boundary theory on Lipschitz boundaries, see Kenig's book [Ken94].

All forms *directly associated* to L_γ are shown to be noncoercive in Section 1. A remark summarizes the strategy behind the proof of Theorem 0.2. Forms *secondarily associated* to L_γ will be defined in Section 2 and also shown to be noncoercive over $W^{1,2}(\Omega_{M,T})$. More background on Theorem 0.2, comparisons with closely related examples of D. Serre [Ser83] and F. J. Terpstra [Ter39], and an application of F. S. Macaulay's determinantal ideals to coerciveness for strongly elliptic systems with formally positive forms are found in Section 3.

1. Proof of Theorem 0.1

1.1. In $\mathbb{R}^n$ let $L = \left(L^{jk}\right)_{1\le j,k\le m}$ be a homogeneous system of linear complex constant coefficient second order operators

$$L^{jk}(\partial) = \sum_{1\le\alpha,\beta\le n} a^{jk}_{\alpha\beta}\partial_\alpha\partial_\beta,\ 1 \le j,k \le m. \tag{1.1}$$

We also write $L = L(\partial) = \sum_{1\le\alpha,\beta\le n} a_{\alpha\beta}\partial_\alpha\partial_\beta$ where the $a_{\alpha\beta}$ are $m\times m$ matrices.

The operator L satisfies the *Legendre-Hadamard condition* (strong ellipticity) if and only if there is a complex constant $m\times m$ matrix Θ and a constant $E > 0$ such that

$$Re\ \overline{\eta}^{\mathsf{T}}\Theta L(\xi)\eta \ge E|\xi|^2|\eta|^2, \quad \xi \in \mathbb{R}^n,\ \eta\in\mathbb{C}^m. \tag{1.2}$$

When $L(\xi)$ is real, as $L_\gamma(\xi)$, and Θ equals the identity matrix, then restricting η to $\mathbb{R}^m$ is equivalent to (1.2).

Given any open subset $\Omega\subset\mathbb{R}^n$ and operator L as in (1.1) the coefficients of L uniquely determine a *Hermitian bilinear form*, anti-linear in the left variable and linear in the right,

$$\begin{aligned} A[V,U] &= \sum_{1\le j,k\le m}\ \sum_{1\le\alpha,\beta\le n} \int_\Omega \partial_\alpha\overline{V_j}a^{jk}_{\alpha\beta}\partial_\beta U_k dX \\ &= \sum_{1\le\alpha,\beta\le n}\int_\Omega \partial_\alpha\overline{V}^{\mathsf{T}}a_{\alpha\beta}\partial_\beta U dX. \end{aligned} \tag{1.3}$$

The Hermitian bilinear form in turn uniquely determines the Hermitian quadratic form $A[V] = A[V,V]$. Equivalently the coefficients of L uniquely determine the quadratic form $A[V]$ which by the *polarization identity* $A[V,U] = \frac{1}{4}\sum_{l=0}^{3} i^l A[i^l V + U]$ uniquely determines the Hermitian bilinear form (1.3). See [Aro61].

We say that any other complex constant coefficient bilinear form $B[V,U] = \sum_{1\le\alpha,\beta\le n}\int_\Omega \partial_\alpha\overline{V}^{\mathsf{T}}b_{\alpha\beta}\partial_\beta U dX$ is *directly associated* to L if and only if $L(\partial) = \sum_{1\le\alpha,\beta\le n} b_{\alpha\beta}\partial_\alpha\partial_\beta$. Thus B must differ from A by a null form

$B - A =$

$$N[V,U] = \sum_{1\leq\alpha,\beta\leq n} \int_\Omega \partial_\alpha \overline{V}^{\mathsf{T}} n_{\alpha\beta}\partial_\beta U dX \tag{1.4}$$

where one defines a *null form* to be any bilinear form (1.4) satisfying $\sum_{1\leq\alpha,\beta\leq n} n_{\alpha\beta}\partial_\alpha\partial_\beta = 0$. Since $\sum_{1\leq\alpha,\beta\leq n} \Theta n_{\alpha\beta}\partial_\alpha\partial_\beta = 0$ will hold, the *symbol* $\sum \overline{\eta}^{\mathsf{T}}\Theta a_{\alpha\beta}\xi_\alpha\xi_\beta\eta$ for ΘL can be recovered from the integrand of any directly associated bilinear form for ΘL by substituting $\partial = \xi$ and $U = V = \eta$. In particular, by using the associated forms G_γ over all $\xi \in \mathbb{R}^6 \setminus \{0\}$ and $\eta = U = (u, v, w, x, y, z) \in \mathbb{C}^6 \setminus \{0\}$, and with Θ the identity, the system L_γ is seen to satisfy the Legendre-Hadamard condition if and only if

$$\begin{aligned} &|u\xi_1 + v\xi_2 - w\xi_3|^2 + |w\xi_3 - \gamma x\xi_4 - \gamma y\xi_5 - \gamma z\xi_6|^2 + |w\xi_4 - y\xi_6|^2 \\ &+ |w\xi_5 - z\xi_4|^2 + |w\xi_6 - x\xi_5|^2 + |z\xi_4 - x\xi_6|^2 + |x\xi_5 - y\xi_4|^2 \\ &+ |y\xi_6 - z\xi_5|^2 + |w\xi_4 - x\xi_3|^2 + |w\xi_5 - y\xi_3|^2 + |w\xi_6 - z\xi_3|^2 \\ &+ |v\xi_1 - u\xi_2|^2 + |w\xi_1 - u\xi_3|^2 + |x\xi_1 - u\xi_4|^2 + |y\xi_1 - u\xi_5|^2 \\ &+ |z\xi_1 - u\xi_6|^2 + |w\xi_2 - v\xi_3|^2 + |x\xi_2 - v\xi_4|^2 \\ &+ |y\xi_2 - v\xi_5|^2 + |z\xi_2 - v\xi_6|^2 \end{aligned} \tag{1.5}$$

never vanishes.

Equivalently, the system L_γ will satisfy the Legendre-Hadamard condition if and only if the matrix $\mathcal{M}$ with entries

$$\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_1 & -\xi_2 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_2 & \xi_1 & 0 & 0 & 0 & 0 & -\xi_3 & -\xi_4 & -\xi_5 & -\xi_6 \\
\xi_3 & \xi_4 & \xi_5 & \xi_6 & 0 & 0 & 0 & \xi_4 & \xi_5 & \xi_6 & -\xi_3 & 0 & \xi_1 & 0 & 0 & 0 & \xi_2 & 0 & 0 & 0 \\
-\gamma\xi_4 & 0 & 0 & -\xi_5 & -\xi_6 & \xi_5 & 0 & -\xi_3 & 0 & 0 & 0 & 0 & 0 & \xi_1 & 0 & 0 & 0 & \xi_2 & 0 & 0 \\
-\gamma\xi_5 & -\xi_6 & 0 & 0 & 0 & -\xi_4 & \xi_6 & 0 & -\xi_3 & 0 & 0 & 0 & 0 & 0 & \xi_1 & 0 & 0 & 0 & \xi_2 & 0 \\
-\gamma\xi_6 & 0 & -\xi_4 & 0 & \xi_4 & 0 & -\xi_5 & 0 & 0 & -\xi_3 & 0 & 0 & 0 & 0 & 0 & \xi_1 & 0 & 0 & 0 & \xi_2
\end{array}$$

is rank 6 for all $\xi \in \mathbb{R}^6 \setminus \{0\}$, i.e. if and only if $\eta^{\mathsf{T}}\mathcal{M}$ never vanishes. Here the first quadratic squared in (1.5) has been moved so as to be represented by the 11th column of $\mathcal{M}$, allowing $\mathcal{M}$ to be viewed as a block matrix $\begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{B} & \mathcal{D} \end{pmatrix}$ with both $0 = \mathcal{A}$ and $\mathcal{C}$ of size 2×10; both $\mathcal{B}$ and $\mathcal{D}$ of size 4×10.

If $\xi_3 = \xi_4 = \xi_5 = \xi_6 = 0$ and $(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$, then $\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix}$ has rank 6 as then does $\mathcal{M}$. If $(\xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^4 \setminus \{0\}$, then $\mathcal{C}$ has rank 2 and it suffices to show that $\mathcal{B}$ has rank 4 in order to prove that the systems L_γ satisfy the Legendre-Hadamard condition.

Lemma 1.1. *For $0 < \gamma < 1/3$ and $(\xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^4 \setminus \{0\}$ the matrix $\mathcal{B}$ has rank* 4.

Proof. Consider first the case when none of $\xi_3, \xi_4, \xi_5, \xi_6$ is zero. Let [1 2 3 4] denote the 4×4 submatrix of $\mathcal{B}$ consisting of columns 1,2,3 and 4, etc. Then $\det[2\ 8\ 9\ 10] = \xi_3^2(\xi_5\xi_6 - \xi_4\xi_3)$, $\det[3\ 8\ 9\ 10] = \xi_3^2(\xi_6\xi_4 - \xi_5\xi_3)$ and $\det[4\ 8\ 9\ 10] = \xi_3^2(\xi_4\xi_5 - \xi_6\xi_3)$. For $\mathcal{B}$ not to be rank 4 all of these must vanish. A couple calculations show that then $\xi_3^2 = \xi_4^2 = \xi_5^2 = \xi_6^2$ must hold. But then $\det[1\ 8\ 9\ 10] = \xi_3^2(-\xi_3^2 + \gamma\xi_4^2 + \gamma\xi_5^2 + \gamma\xi_6^2) \neq 0$ since $\gamma < 1/3$. Thus $\mathcal{B}$ has rank 4 in this case.

When only $\xi_3 = 0$, then $\det[2\ 3\ 4\ 8] \neq 0$. When only $\xi_4 = 0$, when only $\xi_5 = 0$ or when only $\xi_6 = 0$, then $\det[1\ 2\ 3\ 4] \neq 0$ for each.

When $\xi_3 = \xi_4 = 0$ only, then $\det[2\ 3\ 4\ 7] \neq 0$. When $\xi_4 = \xi_5 = 0$ only, then $\det[2\ 4\ 5\ 10] \neq 0$ and similarly for the remaining four choices of two zeros.

Similarly for the four choices of exactly one nonzero variable. □

By this lemma and the preceding argument each L_γ satisfies the Legendre-Hadamard condition.

1.2. The Sobolev space $W^{k,2}(\Omega)$ of vector valued functions with complex valued components that have square integrable weak derivatives up to order k is a Hilbert space with inner product $(V, U)_k = \sum_{|\alpha|\leq k} \int_\Omega \partial^\alpha \overline{V}^{\mathsf{T}} \partial^\alpha U dX$ and norm $\|V\|_k = \sqrt{(V, V)_k}$. Semi-norms are defined by

$$|V|_j^2 = \sum_{|\alpha|=k} \int_\Omega \partial^\alpha \overline{V}^{\mathsf{T}} \partial^\alpha V dX, \quad j = 0, \ldots, k.$$

The more general definition of coerciveness given in [Aro61] employs a *completely continuous quadratic form* in place of the familiar lower order term to be found on the left of (1.6) below. However, in any domain in which the *Rellich compactness theorem* holds, (i.e. the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ when, for example, Ω is a bounded Lipschitz domain or is any bounded domain with the segment property; see [Agm10] pp. 11, 24), replacing the completely continuous form with the squared L^2 norm gives a definition that is equivalent to Aronszajn's (see the introduction of [Ver14]).

Definition 1.2. [Aro61] p. 38. In domains for which the Rellich compactness theorem holds the Hermitian bilinear form (1.3) will be *coercive over* $W^{1,2}(\Omega)$ if the corresponding quadratic form $A[V] = A[V, V]$ satisfies

$$|A[V]| + c_0 \int_\Omega |V|^2 dX \geq c\|V\|_1^2, \quad c > 0 \text{ and } c_0 \in \mathbb{R} \tag{1.6}$$

for some constants c, c_0 independent of V. Equivalently, the norm on the right may be replaced with the semi-norm $|V|_1$.

Imitating [Agm10] p. 113, for $\lambda \in \mathbb{C}$ and $X \in \mathbb{R}^6$ put $u(X) = e^{\lambda(X_1+iX_2)}$, $v(X) = ie^{\lambda(X_1+iX_2)}$ and $w(X) = x(X) = y(X) = z(X) = 0$ in the quadratic form G_γ. One sees that G_γ vanishes identically. Given any $c > 0$ and c_0, letting $A = G_\gamma$ and taking $|\lambda|$ large enough the coercive estimate (1.6) cannot hold in any bounded domain for which the Rellich compactness theorem holds.

Remark 1.3. Both Definition 1.2 and its generalization, when the square L^2-norm is replaced with some other completely continuous quadratic form, have the consequence of yielding the *strongly coercive estimate* $|A[V]| \geq \frac{c}{2}\|v\|_1^2$ for all V in a *subspace of finite codimension.* All linear combinations of the above exponential solutions, as $\lambda \in \mathbb{C}$ varies, form an infinite dimensional subspace of $W^{1,2}(\Omega)$ for any bounded open set $\Omega \subset \mathbb{R}^6$. That the forms G_γ vanish identically on this infinite dimensional subspace then contradicts the strong coercive estimate. Therefore the G_γ fail to be coercive in any bounded open set.

Alternatively, J. Nečas's extension to systems [HN70] p. 310 of the Aronszajn-Smith condition [Aro54] [Smi61] for the coerciveness of formally positive forms in Lipschitz domains, when specialized to $m \times m$ homogeneous systems, is

Theorem 1.4. N. Aronszajn, K. T. Smith, J. Nečas. *Let $p_1, p_2, \ldots$ denote a finite number of column vectors of length m with polynomial components $p_{jk} \in \mathbb{C}[X_1, \ldots, X_n]$, $j = 1, \ldots, m$, all homogeneous of degree d, and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then a necessary and sufficient condition for the coercive estimate over all m-vector valued $V \in W^{d,2}(\Omega)$,*

$$\sum_k \int_\Omega |p_k^T(\partial)V|^2 dX + c_0 \int_\Omega |V|^2 dX \geq c\|V\|_d^2, \quad c > 0,\ c_0 \in \mathbb{R}, \tag{1.7}$$

is that the matrix of homogeneous polynomials $(p_{jk}(\xi))$ have rank m for all $\xi \in \mathbb{C}^n \setminus \{0\}$.

The above 6×20 matrix for each form G_γ fails to be rank 6 when $\xi = (1, i, 0, 0, 0, 0)$.

When $m = 1$ Aronszajn's condition in Theorem 1.4 is commonly stated, "the polynomials $p_1, p_2, \dots$ have no common nontrivial zeros." The extension of the Aronszajn-Smith condition to systems in Lipschitz domains was proved independently in [Smi70] p. 74.

1.3. Theorem 0.2 will be used to show that for each $0 < \gamma < 1/3$ there is a T large enough, depending only on γ, so that every quadratic form $G_\gamma + N$ directly associated to L_γ, where N is any null form (1.4), fails to be coercive over the Sobolev spaces $W^{1,2}(\Omega_{M,T})$ whenever $2 \le M \le 2\sqrt{2}$.

Let $f(X)$ denote any complex valued function in the scalar Sobolev space $W^{2,2}(\Omega)$. Taking $(u, v, w, x, y, z) = grad\, f = \partial f$ in the quadratic form G_γ one obtains

$$G_\gamma[\partial f] = \int_\Omega |f_{11} + f_{22} - f_{33}|^2 + |f_{33} - \gamma f_{44} - \gamma f_{55} - \gamma f_{66}|^2 + |f_{34} - f_{56}|^2 + |f_{35} - f_{64}|^2 + |f_{36} - f_{45}|^2 dX$$

which is seen to be a quadratic form associated with the elliptic 4th order scalar operator

$$(1.8) \qquad \left(\partial_1^2 + \partial_2^2 - \partial_3^2\right)^2 + \left(\partial_3^2 - \gamma(\partial_4^2 + \partial_5^2 + \partial_6^2)\right)^2 + (\partial_3\partial_4 - \partial_5\partial_6)^2 + (\partial_3\partial_5 - \partial_6\partial_4)^2 + (\partial_3\partial_6 - \partial_4\partial_5)^2.$$

By Theorem 0.2 and the remarks that follow it, (1.8) has no associated constant coefficient Hermitian integro-differential form that is coercive over the scalar spaces $W^{2,2}(\Omega_{M,T})$ whenever T is large enough depending on γ. As with a system L and (1.4), two quadratic forms associated to the same scalar opertor must differ by a null form. Therefore, letting α and β denote *multi-indices* of orders less than or equal to 2 and given any *complex numbers* $n_{\alpha\beta}$ satisfying $\sum_{|\alpha|\le 2, |\beta|\le 2} n_{\alpha\beta}\partial^{\alpha+\beta} = 0$, defining a *scalar null form* over $W^{2,2}(\Omega)$, the quadratic form

$$(1.9) \qquad G_\gamma[\partial f] + \sum_{|\alpha|\le 2, |\beta|\le 2} \int_\Omega \partial^\alpha \overline{f} n_{\alpha\beta} \partial^\beta f dX$$

fails to be coercive over $W^{2,2}(\Omega_{M,T})$.

Applying the *system null forms* (1.4) to *grad f* yields

$$N[\partial f] = \sum_{1\leq j,k\leq 6} \sum_{1\leq \alpha,\beta\leq 6} \int_\Omega \partial_\alpha \partial_j \overline{f} n^{jk}_{\alpha\beta} \partial_\beta \partial_k f dX$$

where $\sum_{1\leq\alpha,\beta\leq 6} n^{jk}_{\alpha\beta}\partial_\alpha\partial_\beta = 0$ for each (j,k) and thus also

$$\sum_{1\leq j,k\leq 6} \sum_{1\leq \alpha,\beta\leq 6} n^{jk}_{\alpha\beta}\partial_\alpha\partial_\beta\partial_j\partial_k = 0$$

making $N[\partial f]$ a scalar null form over $W^{2,2}(\Omega)$. Consequently, if $G_\gamma[V] + N[V]$ were coercive (1.6) over vector valued $W^{1,2}(\Omega_{M,T})$, then $G_\gamma[\partial f] + N[\partial f]$ would be a coercive form associated to the 4th order operator (1.8) over all scalar $f \in W^{2,2}(\Omega_{M,T})$,

$$|G_\gamma[\partial f] + N[\partial f]| + c_0 \int_{\Omega_{M,T}} |\partial f|^2 dX \geq c|f|_2^2,$$

(equivalently

$$|G_\gamma[\partial f] + N[\partial f]| + c_0' \int_{\Omega_{M,T}} |f|^2 dX \geq c'|f|_2^2$$

by applying Gagliardo-Nirenberg inequalities, e.g. [Agm10]), contradicting Theorem 0.2.

It follows that for each Legendre-Hadamard system $L\gamma$ there is no coercive constant coefficient directly associated quadratic integro-differential form over $W^{1,2}(\Omega_{M,T})$ when T, depending on γ, is large enough.

Remark 1.5. A general argument of Aronszajn [Aro61] pp. 38–39 on the equivalence of coercive conditions for Hermitian quadratic forms shows that the coerciveness condition (1.6) holds if and only if there is a fixed angle θ so that $Re(e^{i\theta}A[V]) + c_0' \int_\Omega |V|^2 dX \geq c'\|V\|_1^2$, $c' > 0$ and $c_0' \in \mathbb{R}$, holds over $W^{1,2}(\Omega)$. If A is associated to a real coefficient system L, as is the case here, the Hermitian symmetric quadratic form $Re(e^{i\theta}A[V])$ will be associated to $\cos\theta L$. When the quadratic form (1.9) is coercive over scalar $W^{2,2}(\Omega)$, Aronszajn's argument likewise yields an angle such that $A_\theta[f] = \cos\theta\, G_\gamma[\partial f] + \sum_{|\alpha|\leq 2,|\beta|\leq 2} \int_\Omega \partial^\alpha \overline{f}\, \widetilde{n}_{\alpha\beta} \partial^\beta f dX$ satisfies the coercive condition $A_\theta[f] + c_0' \int_\Omega |f|^2 dX \geq c'|f|_2^2$ over $W^{2,2}(\Omega)$ where the new null form has Hermitian symmetric coefficients $\widetilde{n}_{\alpha\beta}(\theta) = \frac{1}{2}(e^{i\theta} n_{\alpha\beta} + e^{-i\theta}\overline{n_{\beta\alpha}})$. The form A_θ is necessarily a *strongly elliptic quadratic form* [Agm10] pp. 62–63, and because

$G_\gamma[\partial f]$ is associated with the elliptic (1.8) it follows that $\cos\theta > 0$ and may be divided through the coercive inequality. See Sections 1.2 and 2 of [Ver14]. Consequently

A quadratic form (1.9) *is coercive according to Definition 1.2 only if there is a Hermitian symmetric form* (1.9) *satisfying*

$$G_\gamma[\partial f] + \sum_{|\alpha|\le 2, |\beta|\le 2} \int_\Omega \partial^\alpha \overline{f} n_{\alpha\beta} \partial^\beta f dX + c_0 \int_\Omega |f|^2 dX \ge c|f|_2^2.$$

The proof of noncoerciveness in [Ver14] for suitable M, T and for every form (1.9) then proceeds by showing

For every constant coefficient Hermitian symmetric quadratic form (1.9) *associated to the 4th order* (1.8) *there is an infinite dimensional subspace of* $W^{2,2}(\Omega_{M,T})$, *depending on the null form in* (1.9), *such that for all* f *in this subspace* $G_\gamma[\partial f] = 0$ *and* $\sum_{|\alpha|\le 2, |\beta|\le 2} \int_{\Omega_{M,T}} \partial^\alpha \overline{f} n_{\alpha\beta} \partial^\beta f dX \le 0$.

2. Continuation of proof–Linear change of equations and dependent variables

A system (1.1) L will have a form that is coercive over the space of $W^{1,2}$ functions with compact support in Ω (Gårding's inequality) if and only if L satisfies the Legendre-Hadamard inequality (1.2). As that inequality can be made false by choice of the matrix Θ and *vice versa*, so too can Gårding's inequality be made false and *vice versa.* Consequently for completeness of the nonexistence argument for coercive constant coefficient forms we consider the following.

Suppose that the solution spaces $\mathcal{S}_L^f$ and $\mathcal{S}_M^g$ to the linear equations $LU = f$ and $MV = g$, where L and M are systems (1.1), are mapped isomorphically by a matrix $R : \mathbb{C}^m \to \mathbb{C}^m$. In this case the existence of a coercive form for one system could yield estimates for solutions for the other even though the latter still has no coercive form. Here we first show that this kind of solution space isomorphism is always the result of a linear transformation of the equations and a linear transformation of dependent variables. Secondly, in the case of such transformations applied to the L_γ, no constant coefficient coercive form exists for the transformed system.

Define a constant coefficient bilinear form

$$B[V, U] = \sum_{1\le\alpha,\beta\le n} \int_\Omega \partial_\alpha \overline{V}^{\mathsf{T}} b_{\alpha\beta} \partial_\beta U dX$$

to be *secondarily associated* to L in (1.1) if and only if

$$\sum_{1\le\alpha,\beta\le n} b_{\alpha\beta}\partial_\alpha\partial_\beta = SLR$$

where S and R are *invertible* constant coefficient matrices $\mathbb{C}^m \to \mathbb{C}^m$.

Let $\mathcal{S}_L$ and $\mathcal{S}_M$ denote the space of vector valued solutions to the homogeneous equations $LU = 0$ and $MV = 0$ respectively. One sees, when $LU = f$ and $MV = g$ have solutions, that $R : \mathcal{S}_M^g \to \mathcal{S}_L^f$ is an isomorphism if and only if $R : \mathcal{S}_M \to \mathcal{S}_L$ is an isomorphism.

In the case where L and M are linear constant coefficient scalar ($m = 1$) differential operators, the isomorphism R can be normalized to be multiplication by 1. Then $Lu = 0$ *and* $Mv = 0$ *locally having the same solution space is equivalent to* M *being a nonzero constant multiple of* L. To see this one notes that $L = M$ if and only if $(LX^\alpha)_{|X=0} = (MX^\alpha)_{|X=0}$ if and only if $LX^\alpha = MX^\alpha$ if and only if $L(X - X_0)^\alpha = M(X - X_0)^\alpha$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ and $X, X_0 \in \mathbb{R}^n$. If there were no complex number s such that $sL = M$ then there would be a multi-index so that $LX^\alpha \neq MX^\alpha$, with both sides nonzero since $\mathcal{S}_L = \mathcal{S}_M$. Now the operator inequality $(MX^\alpha)L \neq (LX^\alpha)M$ follows because there is no s. Consequently there is β such that $(MX^\alpha)LX^\beta \neq (LX^\alpha)MX^\beta$. Thus $(MX^\alpha)X^\beta - (MX^\beta)X^\alpha \in \mathcal{S}_M \setminus \mathcal{S}_L$, a contradiction. Therefore $sL = M$ for some s. If $s = 0$, then $\mathcal{S}_L = \mathcal{S}_M$ implies $LX^\alpha = 0$ for all α, whence $L = 0$ also.

For systems

Lemma 2.1. *Let* $R : \mathbb{C}^m \to \mathbb{C}^m$ *be linear. Given two* $m \times m$ *systems* (1.1) L *and* M*, the mapping* $R : \mathcal{S}_M \to \mathcal{S}_L$ *is invertible if and only if* $R : \mathbb{C}^m \to \mathbb{C}^m$ *is invertible and there exists an invertible linear* $S : \mathbb{C}^m \to \mathbb{C}^m$ *such that* $SLR = M$.

Proof. Sufficiency follows readily. The invertibility of R on the solution spaces necessarily implies invertibility in $\mathbb{C}^m$ since constant vectors are solutions to the systems (1.1). Thus LR may be replaced with L, it may be assumed that $\mathcal{S}_L = \mathcal{S}_M$, and it suffices to find an invertible S such that $SL = M$ in order to show necessity.

By first using vectors with one nonzero component, two systems $L = M$ if and only if $LU = MU$ whenever U is a vector of monomials X^α. If no invertible S exists there is then a vector U^1 with degree 2 monomial components such that $LU^1 \neq MU^1$ with both sides nonzero constant vectors since $\mathcal{S}_L = \mathcal{S}_M$. Let S^1 be an invertible matrix such that

$$S^1LU^1 = MU^1. \tag{2.1}$$

Since $S^1L \neq M$ there is a vector U^2 such that

$$S^1LU^2 \neq MU^2, \tag{2.2}$$

both sides nonzero constant vectors. The vectors MU^1 and MU^2 are linearly independent since $s_1MU^1 + s_2MU^2 = 0$ implies, by (2.1) and (2.2), that $S^1L(s_1U^1 + s_2U^2) \neq 0$ if $s_2 \neq 0$, contradicting $\mathcal{S}_L = \mathcal{S}_M$, or implies $s_1 = 0$ if $s_2 = 0$. Similarly S^1LU^1 and S^1LU^2 are linearly independent. Thus there is an invertible S^2 such that $S^2S^1LU^j = MU^j$, $j = 1, 2$. These arguments can be continued until an invertible $S = S^m \cdots S^1$ is obtained with $SLU^j = MU^j$, $1 \leq j \leq m$, and $\{MU^1, \ldots, MU^m\}$ linearly independent.

Let $\mathcal{U}$ denote the matrix with columns $U^1, \ldots, U^m$ and $\mathcal{M}$ the matrix with columns $MU^1, \ldots, MU^m$. Then $\mathcal{M}$ is invertible and applying the systems to each column of $\mathcal{U}$,

$$SL\mathcal{U} = M\mathcal{U} = \mathcal{M}. \tag{2.3}$$

Now for any vector V of monomials, $M(V - \mathcal{U}\mathcal{M}^{-1}MV) = 0$ so that $SL(V - \mathcal{U}\mathcal{M}^{-1}MV) = 0$ which by (2.3) implies $MV = SLV$ and therefore $M = SL$. □

In the case of the L_γ the formally positive quadratic form (0.3) and polarization yields a bilinear form written variously as

$$\begin{aligned} G_\gamma[V, U] &= \sum_{1 \leq \alpha, \beta \leq 6} \int_\Omega \partial_\alpha \overline{V}^{\mathsf{T}} a_{\alpha\beta} \partial_\beta U \, dX \\ &= \sum_{I=1}^{20} \int_\Omega \Big(\sum_{\substack{1 \leq j \leq 6 \\ 1 \leq \alpha \leq 6}} \overline{a^j_{I\alpha}} \partial_\alpha \overline{V_j} \Big) \Big(\sum_{\substack{1 \leq k \leq 6 \\ 1 \leq \beta \leq 6}} a^k_{I\beta} \partial_\beta U_k \Big) dX. \end{aligned} \tag{2.4}$$

Let $N[V] = N[V, V]$ denote any quadratic null form (1.4). By the association of operators and forms (1.1) (1.3) and the first equality of (2.4), any quadratic form $A[V]$ directly associated to an $S^{\mathsf{T}} L_\gamma R$, i.e. any secondarily associated to L_γ, can be written $A[V] = G_\gamma[SV, RV] + N[V]$. Define a null form $N_{R^{-1}}$ by replacing $n_{\alpha\beta}$ in (1.4) with ${R^{-1}}^{\mathsf{T}} n_{\alpha\beta} R^{-1}$. Then $A[V] = G_\gamma[SV, RV] + N_{R^{-1}}[RV]$.

By Aronszajn's argument in Remark 1.5 $A[V]$ is coercive only if there is θ so that $Re(e^{i\theta} A[V]) + c_0 \int_\Omega |V|^2 dX \geq c|V|_1^2$ holds over $W^{1,2}(\Omega)$. Put $V = R^{-1}\partial f$ for f in scalar $W^{2,2}(\Omega)$. As described in Remark 1.5 the proof in [Ver14] shows there is an infinite dimensional subspace of f so that the Hermitian symmetric null form $Re(e^{i\theta} N_{R^{-1}}[\partial f]) \leq 0$ for all f in the subspace, and so that $G_\gamma[\partial f]$ vanishes for all f in the subspace. The formal positivity of G_γ (2.4) shows that $G_\gamma[SR^{-1}\partial f, \partial f]$ also vanishes. Consequently, if $A[V]$ is coercive, then for an infinite dimensional subspace of $V \in W^{1,2}(\Omega_{M,T})$ it follows that $c_0 \int_\Omega |V|^2 dX \geq c|V|_1^2$ contradicting Rellich compactness and proving that no secondarily associated form to L_γ can be coercive.

3. Comments and comparisons with examples from the literature

Both the scalar example (1.8) and the system example possess formally positive forms. This part of the construction is useful because in general it is difficult to determine whether or not a form is coercive. The formally positive forms are an exception because of the algebraic characterizations of coerciveness of Aronszajn [Aro54], Smith [Smi61] and Nečas [HN70] seen in Theorem 1.4 above. By constructing operators with unique formally positive forms that in addition are noncoercive forms, one has a reasonable chance of showing all associated forms noncoercive. That this cannot be done without considering the regularity of the domain Ω is demonstrated in [Ver12] where the operators (1.8) are shown to possess *indefinite* constant coefficient coercive forms in C^2 domains while having no formally positive coercive forms.

A quadratic form $A[V]$ (1.3) for a system L is formally positive if and only if, fixing a linear ordering of the monomials $\partial_\alpha V_j$, the coefficients $a^{jk}_{\alpha\beta}$ form a $mn \times mn$ *nonnegative* Hermitian symmetric matrix. Consequently, each $m \times m$ matrix $a_{\alpha\beta}$ satisfies $\overline{a_{\alpha\beta}}^{\mathsf{T}} = a_{\beta\alpha}$, $A[V]$ is said to be Hermitian symmetric and L is said to be *formally self-adjoint* $\int_\Omega \overline{V}^{\mathsf{T}} LU = \int_\Omega \overline{LV}^{\mathsf{T}} U$, (compactly supported U). Two distinct formally positive forms for L must have distinct $mn \times mn$ nonnegative matrices and must differ by a null form (1.4) ($n_{\alpha\beta} + n_{\beta\alpha} = 0$ for each α, β) with these same symmetry properties. When the system L has real coefficients, the form $A_{real}[V] = \sum_{1\leq\alpha,\beta\leq n} \int_\Omega \partial_\alpha \overline{V}^{\mathsf{T}} (Re\, a_{\alpha\beta}) \partial_\beta V dX$ will also be associated to L, with $A_{real}[V] = A_{real}[Re\, V] + A_{real}[Im\, V]$ whenever $Re\, a^{\mathsf{T}}_{\alpha\beta} = Re\, a_{\beta\alpha}$. If A is formally positive, so too will be A_{real}, with its $mn \times mn$ matrix real, symmetric and *positive semi-definite* (no *negative* eigenvalues). If A is any Hermitian symmetric coercive form, A_{real} will also be coercive. This follows because noncoerciveness of symmetric Hermitian forms is equivalent to failure of the coercive estimate on an infinite dimensional subspace of real functions, and because $A_{real}[V] = A[V]$ when V is real. (The implication, if A_{real} coercive then A coercive, is false.) Thus when L is real, having no coercive real coefficient forms implies having no coercive Hermitian symmetric forms. Aronszajn's argument in Remark 1.5 then implies no coercive forms. We can say that a real system has a *unique* formally positive form when it is associated with exactly one positive semi-definite $mn \times mn$ matrix.

The problems of determining when real polynomials can be written as sums of squares (Hilbert [Hil88]) and when real scalar operators have formally positive forms are the same. In the context of the former, real symmetric matrices that represent real polynomials as quadratic forms are called *Gram matrices* when they are positive semi-definite, i.e. when the polynomials can be written as a sum of squares. See [CLR95]. We will also call an $mn \times mn$ symmetric matrix associated to a real system L a Gram matrix when it is positive semi-definite.

With the substitution $V = \partial f$ the 36×36 Gram matrix associated with L_γ and $G_\gamma[V]$ induces a linear order on the multi-indices of order 2 in $\mathbb{R}^6$ and yields a 21×21 Gram matrix for (1.8) and $G_\gamma[\partial f]$. Thinking of (1.8) as a homogeneous polynomial in $\mathbb{R}[X_1, \ldots, X_6]$ it is proved in [Ver10] that this 21×21 matrix is the unique Gram matrix representing the sum of squares (1.8) as a quadratic form in the monomials X^α, $|\alpha| = 2$. Whether or not the corresponding 36×36 Gram matrix here is unique is not clear. For example, the null form $Re \int \partial_2 \overline{V_1} \partial_3 V_2 - \partial_3 \overline{V_1} \partial_2 V_2 - \partial_1 \overline{V_2} \partial_2 V_3 + \partial_2 \overline{V_2} \partial_1 V_3$ vanishes when $V = \partial f$. Nor is it true that uniqueness of a Gram matrix for a 2nd order system implies uniqueness of the Gram matrix for a corresponding 4th order scalar equation. This is seen in the following example.

D. Serre [Ser83] pp. 193, 195 constructs a real system in $\mathbb{R}^3$ with a unique Gram matrix and therefore unique formally positive form

$$S[u,v,w] = \int_\Omega |v_2|^2 + |w_3|^2 + |u_1 - w_2 - v_3|^2 + |v_1 + w_1 - u_3|^2 + |u_2 - w_1 - u_3|^2 dX.$$

By Theorem 1.4 $S[u,v,w]$ is coercive over $W^{1,2}(\Omega)$. For $\epsilon > 0$ small enough, so too will be the forms $S_\epsilon[u,v,w] = S[u,v,w] - \epsilon \int |\partial u|^2 + |\partial v|^2 + |\partial w|^2$. Uniqueness of the Gram matrix can only occur when it is not *positive definite*. Otherwise it can be perturbed to another Gram matrix by a null form. Therefore S_ϵ cannot be formally positive and is indefinite. For ϵ larger, but still small enough to retain strong ellipticity, it is possible that S_ϵ might not be coercive. One could then try to construct a domain of $\mathbb{R}^3$ in which none of the associate forms for the system associated to an S_ϵ are coercive.

Quadratic forms for elliptic 4th order scalar operators are obtained from the strongly elliptic S and S_ϵ by the substitution $\partial f = (u,v,w)$. By invoking Hilbert's result that *all* positive definite 4th degree homogeneous real polynomials of $\mathbb{R}^3$ are sums of squares, one sees that formally positive forms for these scalar operators are coercive and nonunique (e.g., [Ver10] p. 238).

A classic geometric construction of Terpstra [Ter39] begins with nonelliptic but formally positive forms for systems in $\mathbb{R}^3$.

Consider four planes through the origin in general position, i.e. four lines in general position in the projective plane $\mathbb{P}^2$ (see the exposition of Terpstra's result in [Qua10]) $l_1(X) = l_2(X) = l_3(X) = l_4(X) = 0,\ X \in \mathbb{R}^3$. Redefining one of $l_1(X)$ or $l_2(X)$ if need be by multiplying it by -1, one obtains $l_1(X)l_2(X) > 0$ for $0 \neq X$ on the line of intersection $l_3(X) = l_4(X) = 0$. Likewise $l_3(X)l_4(X) > 0$ on the line of intersection $l_1(X) = l_2(X) = 0$. Terpstra proves the *biquadratic forms* (quadratic in η for each ξ, and in ξ for each η)

$$\begin{aligned} F_a(\eta;\xi) = {} & l_1^2(\eta)l_1^2(\xi) + l_2^2(\eta)l_2^2(\xi) + l_3^2(\eta)l_3^2(\xi) + l_4^2(\eta)l_4^2(\xi) \\ & + a l_1(\eta)l_2(\eta)l_3(\xi)l_4(\xi), \quad \eta \in \mathbb{R}^3, \xi \in \mathbb{R}^3 \end{aligned}$$

are (i) positive semi-definite for $0 < a$ small enough, and are (ii) never a sum of squares for $0 < a < \infty$.

As an example, $l_1 = X_1, l_2 = -X_2, l_3 = X_3, l_4 = X_1 + X_2 + X_3$ yields

$$\begin{aligned} F_a = {} & \eta_1^2\xi_1^2 + \eta_2^2\xi_2^2 + \eta_3^2\xi_3^2 + (\eta_1 + \eta_2 + \eta_3)^2(\xi_1 + \xi_2 + \xi_3)^2 \\ & - a\eta_1\eta_2\xi_3(\xi_1 + \xi_2 + \xi_3). \end{aligned}$$

Legendre-Hadamard fails as shown by $\eta = (0, 0, 1)$ and $\xi = (1, -1, 0)$. Using the non-negativity of an F_a and adding any positive multiple of $\eta_1^2\xi_3^2 + \eta_2^2\xi_3^2 + \eta_3^2\xi_1^2 + \eta_3^2\xi_2^2$ yields the symbol for a Legendre-Hadamard system. If the multiple is small enough, depending on a, the new biquadratic form remains not a sum of squares, i.e. the resulting strongly elliptic system has no formally positive forms. Aronszajn-Smith-Nečas cannot be applied to discern coerciveness. Nor is it clear that any positive multiple, however large, results in a sum of squares. On the other hand, adding instead the *fundamental form* $b|\eta|^2|\xi|^2$ for $b > 0$ large enough must yield a positive definite Gram matrix. Fixing $a > 0$ small enough $F_{a,b} = F_a(\eta;\xi) + b|\eta|^2|\xi|^2$ will be a positive definite (Legendre-Hadamard) bi-quadratic form for all $b > 0$ and there will exist $\beta = \beta(a) > 0$ defined by $\beta = \min\{b : F_{a,b} \text{ is a sum of squares}\}$. The Gram matrix for this first sum of squares $F_{a,\beta}$ will necessarily not be positive definite. In addition it will be the unique Gram matrix for $F_{a,\beta}$. This follows by applying Theorem 2.4 of [Ser83] that characterizes uniqueness of indefinite forms for 2nd order systems in the $m = n = 3$ case. Applying also Serre's Lemma 2.2 one can deduce that $F_{a,\beta}$ will either be a sum of $s = 5$ or a sum of $s = 4$ squares. As Serre's example $S[u, v, w]$ above shows, the sum of 5 squares can turn out coercive. Our last observation is

If an $F_{a,\beta}$ were to turn out to be a sum of 4 squares, then it would necessarily be noncoercive.

One might then attempt to construct a bounded domain of $\mathbb{R}^3$, similar to that in [Ver14] and this article, in which the resulting Legendre-Hadamard system has no associated coercive forms.

To prove the last observation, the Nečas matrix for testing coerciveness will be of size 3×4 with entries that are homogeneous polynomials of degree 1 in three variables. As noted at the end of Section 1.2, coerciveness requires this matrix to remain rank 3 when it is evaluated over $\xi \in \mathbb{C}^3 \setminus \{0\}$. F. S. Macaulay first proved bounds on the ranks of matrices with entries in polynomial rings in 1916 [Mac94] pp. 54–57. His theorem shows that there are always $\xi \in \mathbb{C}^3 \setminus \{0\}$ such that any 3×4 matrix, as just described, fails to be rank 3, establishing the observation.

However, some comments might be helpful. In Macaulay's work the term *rank* is not the rank here, the rank of a matrix. Let $\mathcal{M}$ be an $m \times s$ matrix with entries from a polynomial ring $\mathsf{k}[X_1, \ldots, X_n]$ with k a commutative field (see [vdW70] Chapter 16). It is rather what is now more commonly called the *height* or *codimension* of the ideal (the *determinantal ideal*) $I_m(\mathcal{M})$ in $\mathsf{k}[X_1, \ldots, X_n]$ that is generated by the $m \times m$ minors of $\mathcal{M}$. See [BV88] p. 10 or [Eis05] p. 221 respectively. It coincides with the *geometric codimension* of the set of common zeros of polynomials of $I_m(\mathcal{M})$, the *variety* of $I_m(\mathcal{M})$, when the field (extension field of k) over which the coordinates X_j, $1 \leq j \leq n$, are evaluated is *algebraically complete*. Specializing Macaulay's theorem,

If $\mathcal{M}$ is an $m \times s$ $(m \leq s)$ matrix with entries in the polynomial ring $R = k[X_1, \ldots, X_n]$ and $I_m(\mathcal{M}) \neq R$, then codim$(I_m(\mathcal{M}) \leq s - m + 1$.

In the case of the 3×4 matrices $\mathcal{M}$ of interest here with $\mathsf{k} = \mathbb{C}$ or $\mathbb{R}$, we have $m = n = 3$, $s = 4$ and the observation that the ideal $I_3(\mathcal{M})$ is a proper subset of the polynomial ring because the entries are homogeneous polynomials of degree 1. Therefore the bound on the codimension is 2 and the variety of common zeros in $\mathbb{C}^3$ of $I_3(\mathcal{M})$ is at least of dimension 1, showing that such $\mathcal{M}$ always have matrix rank less than 3 for some $\xi \in \mathbb{C}^3 \setminus \{0\}$. This last conclusion need not be true over $\mathbb{R}^3 \setminus \{0\}$ since $\mathbb{R}$ is not algebraically complete, permitting the formally positive forms that are sums of 4 squares to be elliptic while never coercive. For example, if $\mathcal{M}_{2\times 2} = \begin{pmatrix} X_1 & -X_2 \\ X_2 & X_1 \end{pmatrix}$ with $n = 2$, then $m = s = 2$, the bound on $I_2(\mathcal{M}_{2\times 2})$ is 1, and the variety $\{(\xi, \pm i\xi) : \xi \in \mathbb{R}\}$ is one dimensional in $\mathbb{C}^2$ while the determinant of $\mathcal{M}_{2\times 2}$ vanishes only at the origin in $\mathbb{R}^2$. In each case the codimension of the ideal $I = (X_1^2 + X_2^2)$ is equal to 1. This is seen as follows. In $\mathbb{R}[X_1, X_2]$ I is prime and $I \supset (0)$ is the longest descending chain of prime ideals for I. The codimension of a prime ideal is defined to be the greatest length of prime chains descending from it. The chain here is of length 1. In $\mathbb{C}[X_1, X_2]$ I is not prime. The codimension of I is then defined as the smallest of lengths of prime chains descending from prime ideals containing I. Thus $(X_1 + iX_2) \supset I$ is prime and has codimension 1 as then does I. The ring itself is not considered a prime ideal. See [Eis05] pp. 205–206. The matrix $\mathcal{M}_{2\times 2}$ yields the noncoercive formally positive form $\int_\Omega |u_1 + v_2|^2 + |u_2 - v_1|^2 dX$ associated to the strongly elliptic system $\begin{pmatrix} \triangle & 0 \\ 0 & \triangle \end{pmatrix}$. In contrast to the 3×4, $n = 3$ case just considered, the matrix with homogeneous entries $\mathcal{M}_{2\times 3} = \begin{pmatrix} X_1 & 0 & X_2 \\ 0 & X_2 & X_1 \end{pmatrix}$ likewise has Macaulay bound 2, but when $n = 2$ only the origin in $\mathbb{C}^2$ yields a matrix with rank less than 2. Accordingly the form $\int_\Omega |u_1|^2 + |v_2|^2 + |u_2 + v_1|^2 dX$ is coercive. This is the classical Korn's inequality for the system of elastostatics in the plane. The determinantal ideal $I_2(\mathcal{M}_{2\times 3}) = (X_1X_2, -X_2^2, X_1^2)$ is not prime. A prime containing it with smallest codimension is (X_1, X_2), and $(X_1, X_2) \supset (X_1) \supset (0)$ is a chain of length 2.

Macaulay's theorem can be used to show that noncoerciveness of formally positive forms is necessary when the number of squares s is small enough depending on m and n. It evidently cannot be used for this purpose when applied to the L_γ and forms (1.5).

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