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Critical exponent and bottom of the spectrum in pinched negative curvature

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In this note, we present a new proof of the celebrated theorem of Patterson-Sullivan which relates the critical exponent of a hyperbolic manifold and the bottom of its spectrum. The proof extends to manifolds with pinched negative curvatures. It provides a sufficient criterion for the existence of isolated eigenvalues for the Laplacian on geometrically finite manifolds with pinched negative curvatures.

1. Introduction

Let M be a complete manifold with pinched negative sectional curvatures, and $\tilde{M}$ be its universal covering. We denote by $\Gamma = \pi_1(M)$ its fundamental group, seen as a group of isometries of $\tilde{M}$. Let δ_Γ be the critical exponent of the Poincaré serie of Γ, let $\lambda_0(M)$ be the bottom of the spectrum of the Riemannian Laplacian Δ on M, and $\lambda_0^{ess}(M)$ the bottom of its essential spectrum.

When the sectional curvatures of M are constant, these three constants are related by the following well-known theorem, due to Patterson for surfaces and to Sullivan ([Sul87]) in higher dimensions.

Theorem 1 (Patterson-Sullivan). *Let $M = \mathbb{H}^{n+1}/\Gamma$ be a geometrically finite hyperbolic manifold of dimension $n+1$.*

1) *If $\delta_\Gamma > \frac{n}{2}$, then $\lambda_0(M) = \delta_\Gamma(n - \delta_\Gamma)$ and it is an isolated eigenvalue.*
2) *If $\delta_\Gamma \leq \frac{n}{2}$, then $\lambda_0(M) = \lambda_0^{ess}(M) = \frac{n^2}{4}$.*

The classical proof of this result combines potential theory and the use of Patterson-Sullivan measures. In this note, we provide a simpler proof which does not require potential theory. This proof allows us to extend the previous result to all hyperbolic manifolds *whose Bowen-Margulis measure is finite.* This terminology is defined in Section 2. As a consequence of the proof, we get a characterization of the first eigenfunction in the case of finite Bowen-Margulis measure. Combined with Sullivan's work, it also shows that when the Bowen-Margulis measure of a hyperbolic manifold M is infinite, then $\lambda_0(M) = \lambda_0^{ess}(M)$, and the bottom of the spectrum is not an eigenvalue. We prove these results in Section 3.

In Section 4, we adapt this proof to get a link between the critical exponent and the bottom of the spectrum on manifolds with pinched negative curvatures whose Bowen-Margulis measure is finite. It applies in particular to geometrically finite manifolds: on such manifolds, it provides a dynamical simple sufficient criterion for the bottom of the spectrum of the Laplacian to be an isolated eigenvalue. Such a criterion had remained unknown on manifolds with variable curvatures.

2. Conformal densities and Bowen-Margulis measure

Let $\tilde{M}$ be a complete Riemannian manifold whose sectional curvatures satisfy $K \leq -a^2 < 0$, and let Γ be a non-elementary discrete group of isometries acting on $\tilde{M}$.

Let $\partial\tilde{M}$ be the visual boundary (or boundary at infinity) of $\tilde{M}$, and d the the Riemannian distance on $\tilde{M}$. For all $x, y, \xi \in \tilde{M}$, we will write

$$\mathcal{B}_\xi(x, y) = d(\xi, x) - d(\xi, y).$$

This expression extends continuously to $\xi \in \partial\tilde{M}$; in this case $\mathcal{B}_\xi$ is called the *Busemann function* based at ξ.

Definition 2.1. *A* conformal density of dimension δ *is a family of positive finite measures $\mu = (\mu_x)_{x\in\tilde{M}}$, whose support is contained in $\partial\tilde{M}$, absolutely continuous with respect to each other, with Radon-Nikodym derivatives given by*

$$\frac{d\mu_{x'}}{d\mu_x}(\xi) = e^{-\delta\mathcal{B}_\xi(x',x)} \qquad (x, x' \in \tilde{M}, \xi \in \partial\tilde{M}).$$

If moreover $\gamma_*\mu_x = \mu_{\gamma x}$ for all $\gamma \in \Gamma$ and all $x \in \tilde{M}$, we will say that μ is invariant by Γ.

Let δ_Γ be the critical exponent of Γ, defined by

$$\delta_\Gamma = \inf\left\{\delta \geq 0; \sum_{\gamma\in\Gamma} e^{-\delta d(o,\gamma o)} < \infty\right\}.$$

This does not depends on the base point o. The following result is due to Patterson in the case of the hyperbolic half-plane; however its proof extends easily in all dimension and variable negative curvatures (see for instance [Rob11]).

Proposition 2.1. *Let $\tilde{M}$ be a complete simply connected Riemannian manifold whose sectional curvatures satisfy $K \leq -a^2 < 0$. Let Γ be a non-elementary discrete group of isometries acting on $\tilde{M}$. Then there exists a conformal density of dimension δ_Γ invariant by Γ. In particular, $\delta_\Gamma > 0$.*

Moreover, if Γ is not cocompact, there exist a conformal density of any dimension *$\delta \geq \delta_\Gamma$ invariant by Γ.*

Let $S\tilde{M}$ be the unit tangent bundle of $\tilde{M}$, $\pi : S\tilde{M} \to \tilde{M}$ be the canonical projection, and $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow on $S\tilde{M}$. For any $u \in S\tilde{M}$, we will write $g^{-\infty}(u)$ and $g^{+\infty}(u)$ the extreme points at infinity of the geodesic $\gamma_u(t) = \pi g^t u$ whose tangent vector at $t = 0$ is u. Let $o \in \tilde{M}$ be a base point, fixed once for all. The unit tangent bundle $S\tilde{M}$ can be identified with $\big[(\partial\tilde{M} \times \partial\tilde{M})\backslash\mathrm{Diag}\big] \times \mathbb{R}$, through the map

$$S\tilde{M} \ni u \mapsto \mathcal{I}_o(u) = (g^{-\infty}u, g^{+\infty}u, \mathcal{B}_{g^{-\infty}u}(\pi u, o)).$$

In this identification, the real coordinate depends on the chosen base point $o \in \tilde{M}$. In such coordinates, the geodesic flow acts as follows : for all $u = (\xi, \eta, s)$ and all $t \in \mathbb{R}$,

$$g^t(\xi, \eta, s) = (\xi, \eta, s + t).$$

Given a conformal density μ of dimension δ invariant by Γ, we will call *Bowen-Margulis measure associated to* μ the locally finite Radon measure m^μ on $S\tilde{M}$ given by

$$dm^\mu(u) = e^{\delta\mathcal{B}_\xi(\pi u, o) + \delta\mathcal{B}_\eta(\pi u, o)} d\mu_o(\xi) d\mu_o(\eta) ds \text{ where } u = (\xi, \eta, s),$$

where $o \in \tilde{M}$ is any base point. By construction, m^μ does not depends on o, and is invariant by Γ and by the geodesic flow. Therefore, it induces a (locally finite) measure on the unit tangent bundle SM of $M := \tilde{M}/\Gamma$, which we will denote by m^μ_Γ. We will say that M has *finite Bowen-Margulis measure* if the total mass of m^μ_Γ is finite for some μ.

When the total mass of m^μ_Γ is finite, one can show that there exists a unique conformal density μ of dimension δ_Γ invariant by Γ, up to normalization. The normalized measure m^μ_Γ, which we will denote by m_Γ, is then the *Bowen-Margulis measure for the geodesic flow* in the classical meaning of this terminology : it is the unique probability measure invariant by the geodesic flow and with maximal entropy. We refer the reader to [Rob03] and [O-P04] for more details.

When M is convex-cocompact (with variable negative curvatures), its Bowen-Margulis measure is finite: it is a locally finite measure supported by a compact set. A geometrically finite real, complex or quaternionic hyperbolic manifold has finite Bowen-Margulis measure (cf. [Sul79] and [C-I99]). There exists also real hyperbolic manifolds whose fundamental group is not finitely generated (hence which are not geometrically finite) whose Bowen-Margulis measure is finite, cf. [Pei]. On the other hand, there are examples of geometrically finite manifolds with pinched negative curvatures whose Bowen-Margulis measure is not finite, cf. [D-O-P].

3. Critical exponent and bottom of the spectrum on hyperbolic manifolds

We now present a complete elementary proof of the following extension of the Patterson-Sullivan theorem quoted in our introduction.

Theorem 3.1. *Let $M = \mathbb{H}^{n+1}/\Gamma$ be a $(n+1)$-hyperbolic manifold, and μ be a conformal density on $\partial\mathbb{H}^{n+1}$, invariant by Γ and of dimension δ_Γ the critical exponent of Γ. For all $x \in M$, let us denote by $\phi_\mu(x) = \|\mu_{\tilde{x}}\|$ the total mass of the measure $\mu_{\tilde{x}}$, where $\tilde{x} \in \mathbb{H}^{n+1}$ is any lift of x.*

Assume that the Bowen-Margulis measure m_Γ^μ associated to μ has finite mass. Then the following holds.

1) *If $\delta_\Gamma > \frac{n}{2}$, the bottom of the spectrum of Δ satisfies $\lambda_0(M) = \delta_\Gamma(n - \delta_\Gamma)$. It is an eigenvalue, associated to the eigenfunction $\phi_\mu \in L^2(M)$. Moreover,*

$$\|\phi_\mu\|^2_{L^2(M)} = \|m_\Gamma^\mu\| \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\delta_\Gamma}}.$$

2) *If $\delta_\Gamma \leq \frac{n}{2}$, then $\lambda_0(M) = \lambda_0^{ess}(M) = \frac{n^2}{4}$.*

When the (some) Bowen-Margulis measure has infinite mass, this theorem does not apply. We will see in Corollary 3.7 that, in this case, the bottom of the spectrum is not associated to a $L^2(M)$ eigenfunction.

Theorem 3.1 will follow from the four propositions which we present now.

Proposition 3.2. *Let M be a Riemannian manifold. Assume that for some $\lambda \geq 0$, there exist a positive smooth map $\phi : M \to (0, \infty)$ satisfying*

$$\Delta\phi \geq \lambda\phi.$$

Then $\lambda_0(M) \geq \lambda$.

The following formula, known as *Barta's trick*, gives a simple proof of this proposition.

Lemma 3.3 (Barta's trick). *Let $u, \phi : M \to \mathbb{R}$ be $\mathcal{C}^2$ maps, the support of u being compact. We have*

$$\int_M \|\nabla(u\phi)\|^2 \, dV = \int_M \phi^2 \|\nabla u\|^2 \, dV + \int_M u^2 \phi \Delta\phi dV.$$

Proof. An immediate computation shows that

$$\|\nabla(u\phi)\|^2 = \phi^2 \|\nabla u\|^2 + \frac{1}{2}\langle\nabla(u^2)|\nabla(\phi^2)\rangle + u^2 \|\nabla\phi\|^2 .$$

Moreover, by Stokes' formula,

$$\int_M \langle\nabla(u^2)|\nabla(\phi^2)\rangle dV = \int_M u^2 \Delta(\phi^2) dV.$$

Eventually, $\frac{1}{2}\Delta(\phi^2) = \phi\Delta\phi - \|\nabla\phi\|^2$. Gathering these three equalities gives Barta's trick. □

Proof of Proposition 3.2. Let $f : M \to \mathbb{R}$ be any $\mathcal{C}^2$ map with compact support. Since $\phi > 0$, we can apply Barta's trick with $u = f/\phi$. This yields

$$\int_M \|\nabla f\|^2 \, dV \geq \int_M \frac{f^2}{\phi} \Delta\phi dV \geq \lambda \int_M f^2 dV.$$

Remembering that $\lambda_0(M)$ is the infimum of Rayleigh quotients, this implies that $\lambda_0(M) \geq \lambda$. □

Using potential theory, it is shown in [Sul87] that actually,

$$\lambda_0(M) = \sup\{\lambda > 0; \exists\phi > 0, \Delta\phi \geq \lambda\phi\}.$$

We will not use this fact here.

Proposition 3.4. *Let $M = \mathbb{H}^{n+1}/\Gamma$ be a $(n+1)$-hyperbolic manifold, and μ be a conformal density on $\partial\mathbb{H}^{n+1}$, invariant by Γ and of dimension $\delta \geq \delta_\Gamma$. For any $x \in M$, we write $\phi_\mu(x) = \|\mu_{\tilde{x}}\|$ the mass of the measure $\mu_{\tilde{x}}$ where $\tilde{x} \in \mathbb{H}^{n+1}$ is any lift of x. Then*

$$\Delta\phi_\mu = \delta(n-\delta)\phi_\mu.$$

Proof. We identify $\mathbb{H}^{n+1}$ to $\mathbb{R}^{+*} \times \mathbb{R}^n$, and call $(x_0, x_1, \ldots, x_n)$ the induced coordinates on $\mathbb{H}^{n+1}$. Let $o = (1, 0, \ldots, 0) \in \mathbb{H}^{n+1}$, and for all $\xi \in \partial\mathbb{H}^{n+1}$, let ϕ_ξ be the map defined on $\mathbb{H}^{n+1}$ by $\phi_\xi(x) = e^{\delta\mathcal{B}_\xi(o,x)}$. Since $\|\mu_x\| = \int \phi_\xi(x) d\mu_o(\xi)$, it is enough to show that for all $\xi \in \partial\mathbb{H}^{n+1}$, we have $\Delta\phi_\xi = \delta(n-\delta)\phi_\xi$. Since this is invariant by isometries, we can assume that $\xi = \infty$.

The hyperbolic Laplacian on the upper-half-space model can be expressed as $\Delta = -x_0^2\big(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \cdots\big) + (n-1)x_0\frac{\partial}{\partial x_0}$, and $\phi_\infty = x_0^\delta$. A simple computation gives hence $\Delta\phi_\infty = \delta(n-\delta)\phi_\infty$. □

From Propositions 2.1, 3.2 and 3.4, we get (without any assumption on the mass of Bowen-Margulis measure) $\lambda_0(\mathbb{H}^{n+1}/\Gamma) \geq \delta_\Gamma(n-\delta_\Gamma)$.

Proposition 3.5. *Let $M = \tilde{M}/\Gamma$ be a complete Riemannian manifold satisfying $K \leq -a^2$, where $\tilde{M}$ is its universal cover. Let $\mu = (\mu_x)_{x\in\tilde{M}}$ be a conformal density of dimension δ, invariant by Γ, whose associated Bowen-Margulis measure m^μ has finite total mass. Then*

$$\int_D \|\mu_x\|^2 \, dV(x) = \int_{SD} dm^\mu(v) \int_{H^+(v)} e^{\delta\mathcal{B}_{v^+}(\pi v, \pi u)} dV_{H^+(v)}(u),$$

where $\pi : S\tilde{M} \to \tilde{M}$ is the canonical projection, $D \subset \tilde{M}$ is any Borelian fundamental domain for Γ, $SD = \pi^{-1}(D)$, and for any vector $v \in S\tilde{M}$, we write $v^+ = g^{+\infty}v \in \partial\tilde{M}$ for the positive extremity of the geodesic γ_v, $H^+(v)$ for the unstable horosphere *of v, which is the horosphere centered in $v^- = g^{-\infty}v$ through v, and eventually $dV_{H^+(v)}$ is the measure induced on $H^+(v)$ by the Riemannian volume dV on $\tilde{M}$.*

Proof. Let $o \in \tilde{M}$ be a fixed base point. We first write

$$\int_D \|\mu_x\|^2 \, dV(x) = \int_{\partial\mathbb{H}^{n+1}\times\partial\mathbb{H}^{n+1}} d\mu_o(\xi) d\mu_o(\eta) \int_{\tilde{M}} e^{\delta\mathcal{B}_\xi(o,x)} e^{\delta\mathcal{B}_\eta(o,x)} \mathbb{I}_D(x) dV(x).$$

In the above integral, let $v_{\xi,\eta}$ be the unit vector in $S\tilde{M}$ such that $v_{\xi,\eta}^- = \xi$, $v_{\xi,\eta}^+ = \eta$ and $\mathcal{B}_\xi(o, \pi v_{\xi,\eta}) = 0$. We identify $\tilde{M}$ with the weakly unstable manifold of $v_{\xi,\eta}$. Decomposing the volume form according to this identification, the integral becomes

$$\int_{\partial\mathbb{H}^{n+1\,2}} d\mu_o(\xi)d\mu_o(\eta)\int_{\mathbb{R}} dt\int_{H^+(g^t v_{\xi,\eta})} e^{\delta(\mathcal{B}_\xi(o,\pi u)+\mathcal{B}_\eta(o,\pi u))}\mathbb{I}_D(\pi u)dV_{H^+(g^t v_{\xi,\eta})}(u).$$

Now, using the definition of m^μ, the fact that $\mathcal{B}_\xi(o,\pi u)=\mathcal{B}_\xi(o,\pi g^t v_{\xi,\eta})$ and the identity $\mathcal{B}_\eta(o,\pi v)+\mathcal{B}_\eta(\pi v,\pi u)=\mathcal{B}_\eta(o,\pi u)$, we get

$$\int_D \|\mu_x\|^2\, dV(x) = \int_{S\tilde{M}} dm^\mu(v)\int_{H^+(v)} e^{\delta\mathcal{B}_{v^+}(\pi v,\pi u)}\mathbb{I}_{SD}(u)dV_{H^+(v)}(u).$$

Eventually, since $1=\sum_{\gamma\in\Gamma}\mathbb{I}_{SD}(\gamma v)$, and since m^μ is invariant by Γ, this gives the announced formula. □

When $\tilde{M}$ is the hyperbolic space, Proposition 3.5 gives an explicit formula for the L^2-norm of $x\mapsto\|\mu_x\|$ using the following fact.

Proposition 3.6. *Let $v\in S\mathbb{H}^{n+1}$ and $\delta>0$, then*

$$\int_{H^+(v)} e^{\delta\mathcal{B}_{v^+}(\pi v,\pi u)}dV_{H^+(v)}(u)=\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^\delta}.$$

Hence, this integral is finite if and only if $\delta>\frac{n}{2}$.

Proof. Since this integral is invariant by the isometry group of $\mathbb{H}^{n+1}$, we can assume that v is such that $v^-=\infty$, $v^+=0$ and $\pi v=o$, where o is the point $(1,0,\dots,0)$ when $\mathbb{H}^{n+1}$ is identified to $\mathbb{R}^{+*}\times\mathbb{R}^n$. Identifying $H^+(v)$ to $\mathbb{R}^n$ through $u\mapsto x$, where $(1,x)\in\mathbb{H}^{n+1}$ are the coordinates of πu, the volume element $dV_{H^+(v)}$ is precisely the Lebesgue measure dx. Eventually, for all $z=(z_0,z_1,...,z_n)\in\mathbb{H}^{n+1}$, $e^{\mathcal{B}_\infty(o,z)}=z_0$. Therefore, using the isometry $z\mapsto\frac{z}{|z|^2}$ which exchanges 0 and ∞ and fixes o, we get $e^{\mathcal{B}_0(o,\pi u)}=\frac{1}{1+|x|^2}$. □

Proof of Theorem 3.1. (1) We assume $\delta_\Gamma>\frac{n}{2}$.

From Proposition 3.4, we get $\Delta\phi_\mu=\delta_\Gamma(n-\delta_\Gamma)\phi_\mu$. Therefore, Proposition 3.2, implies that $\lambda_0(M)\geq\delta_\Gamma(n-\delta_\Gamma)$.

Moreover, Propositions 3.5 and 3.6 show that $\phi_\mu\in L^2(M)$; they even provide an exact formula for $\|\phi_\mu\|_{L^2(M)}$. Hence, $\delta_\Gamma(n-\delta_\Gamma)$ is an eigenvalue in the L^2-spectrum of the Laplacian on M, with eigenfunction ϕ_μ. This gives $\lambda_0(M)\leq\delta_\Gamma(n-\delta_\Gamma)$.

(2) Assume now that $\delta_\Gamma\leq\frac{n}{2}$.

Since Γ is not cocompact, we can find a conformal density μ of dimension $\frac{n}{2}$ invariant by Γ (proposition 2.1). Let ϕ_μ be the map associated to μ as in

Proposition 3.4. From Propositions 3.4 and 3.2, we get that $\lambda_0(M) \geq \frac{n^2}{4}$. Moreover, $\lambda_0(M) \leq \lambda_0^{ess}(M) \leq \frac{n^2}{4}$, from Theorem 1 of [Bro]. $\square$

When the Bowen-Margulis measures have infinite mass, our proof of Theorem 3.1 only gives

$$\lambda_0(M) \geq \delta_\Gamma(n-\delta_\Gamma) \text{ if } \delta_\Gamma > \frac{n}{2}, \quad \text{otherwise } \lambda_0(M) = \frac{n^2}{4}.$$

Nevertheless, combining our study with the results of [Sul87], we get the following corollary.

Corollary 3.7. *Let $M = \mathbb{H}^{n+1}/\Gamma$ be a hyperbolic manifold whose Bowen-Margulis measures have infinite mass. Then $\lambda_0(M) = \lambda_0^{ess}(M)$ and it is not a $L^2(M)$-eigenvalue of the Laplacian.*

Proof. If $\delta_\Gamma \leq \frac{n}{2}$, then we always have $\lambda_0(M) = \lambda_0^{ess}(M) = \frac{n^2}{4}$, by the same proof as in 3.1 (2).

Assume now that $\delta_\Gamma > \frac{n}{2}$. Then, by Theorem 2.17 of [Sul87], we still have $\lambda_0(M) = \delta_\Gamma(n-\delta_\Gamma)$. Let μ be a conformal density of dimension δ_Γ, invariant by Γ, and let $\phi(x) = \|\mu_{\tilde{x}}\|$ be the mass of the Patterson-Sullivan measure at any lift $\tilde{x}$ of $x \in M$. From Proposition 3.4, we have $\Delta\phi = \lambda_0(M)\phi$, and from Propositions 3.5 and 3.6, $\phi \notin L^2(M)$ since $\|m_\Gamma^\mu\| = \infty$.

Let $\psi \in \mathcal{H}^1(M)$ be such that $\Delta\psi = \lambda_0(M)\psi$. We write $u = \psi/\phi$. Let $x_0 \in M$ be any fixed point, for all $R > 0$ let $\rho_R : M \to [0,1]$ be a smooth cut-off function such that $\mathrm{Supp}\rho_R \subset B(x_0, R+1)$, $\rho_R|_{B(x_0,R} \equiv 1$ and on $B(x_0, R+1)\backslash B(x_0,R)$, $\|\nabla\rho_R\| \leq C$ where $C > 0$ is independent of R.

For all $R > 0$, since $\rho_R u\phi = \rho_R\psi$ has compact support, Barta's trick (Lemma 3.3) gives

$$\int_M \|\nabla(\rho_R u\phi)\|^2 = \int_M \phi^2 \|\nabla\rho_R u\|^2 + \lambda_0 \int \rho_R^2 u^2\phi^2$$

and

$$\int_M \|\nabla(\rho_R\psi)\|^2 = \int_M \psi^2 \|\nabla\rho_R\|^2 + \lambda_0 \int \rho_R^2\psi^2.$$

Therefore,

$$\int_M \phi^2 \|\nabla\rho_R u\|^2 = \int_M \psi^2 \|\nabla\rho_R\|^2$$

which gives eventually

$$\int_{B(x_0,R)} \phi^2 \|\nabla u\|^2 + \int_{B(x_0,R+1)\backslash B(x_0,R)} \phi^2 \|\nabla\rho_R u\|^2 \leq C \int_{B(x_0,R+1)\backslash B(x_0,R)} \psi^2.$$

Since $\psi \in L^2(M)$, the right-hand side in the above equality tends to zero when $R \to \infty$. Therefore, u is constant on any ball of M. Since $\psi = u\phi$, and $\phi \notin L^2(M)$, this implies $u \equiv 0$ and $\psi \equiv 0$. Therefore, the bottom of the spectrum $\lambda_0(M) = \delta_\Gamma(n - \delta_\Gamma)$ is not associated to any L^2-eigenfunction of the Laplacian, which implies $\lambda_0(M) = \lambda_0^{ess}(M)$. □

This corollary is the only point of this note where we use the results of [Sul87] coming from potential theory.

4. Bottom of the spectrum and critical exponent in variable curvature

The relationship between potential theory and critical exponent is specific to locally symmetric manifolds. However, the geometric proof presented above does extend to manifolds with pinched negative curvatures. This yields the following generalisation of Patterson-Sullivan Theorem .

Theorem 4.1. *Let M be a Riemannian manifold whose sectional curvatures satisfy $-b^2 \le K \le -a^2$ and whose Bowen-Margulis measure is finite. Let δ_Γ be the critical exponent of its fundamental group Γ. The bottom of the spectrum of its Laplacian satisfies:*

1) *without anymore hypothesis,*

$$\delta_\Gamma(na - \delta_\Gamma) \le \lambda_0(M) \le \frac{(nb)^2}{4};$$

2) *if $\delta_\Gamma < \frac{na}{2}$,*

$$\frac{(na)^2}{4} \le \lambda_0(M) \le \frac{(nb)^2}{4};$$

3) *if $\delta_\Gamma > \frac{nb}{2}$,*

$$\delta_\Gamma(na - \delta_\Gamma) \le \lambda_0(M) \le \delta_\Gamma(nb - \delta_\Gamma).$$

On any complete Riemannian manifold with infinite volume and a lower-bound on Ricci curvature $Ric \ge -(n-1)b^2 g$, the bottom of the essential spectrum of the Laplacian satisfies $\lambda_0^{ess}(M) \le \frac{(nb)^2}{4}$ (cf. [Bro]). Therefore, following the proof of Theorem 3.1, it is enough to show the two following propositions.

Proposition 4.2. *Let M be any Riemannian $(n+1)$-manifold whose sectional curvatures satisfy $-b^2 \le K \le -a^2$, and $\tilde{M}$ its universal cover. Let μ*

be a conformal density of dimension $\delta \geq \delta_\Gamma$ invariant by Γ the fundamental group of M. Let ϕ_μ defined on M by $\phi_\mu(x) = \|\mu_{\tilde{x}}\|$ where $\tilde{x}$ is any lift of x. Then

$$\delta(na - \delta)\phi \leq \Delta\phi \leq \delta(nb - \delta)\phi.$$

Proof. As in the proof of 3.4, let us write $\phi_\xi(x) = e^{\delta\mathcal{B}_\xi(o,x)}$ ($\xi \in \partial\tilde{M}$, $x \in \tilde{M}$), where $o \in \tilde{M}$ is a fixed origin. Once again, since $\|\mu_x\| = \int_{\partial\tilde{M}} \phi_\xi(x) d\mu_o(\xi)$, it is enough to show that for all $\xi \in \partial\tilde{M}$,

$$\delta(na - \delta)\phi_\xi \leq \Delta\phi_\xi \leq \delta(nb - \delta)\phi_\xi. \tag{4.1}$$

For all $\xi \in \partial\tilde{M}$ and $x \in \tilde{M}$, let $(\gamma_{x,\xi}(t))_{t\in(0,\infty)}$ be the geodesic ray starting at x and ending in ξ such that for all $t \geq 0$, $\|\gamma'_{x,\xi}(t)\| = 1$. For any $v \in T_x\tilde{M}$, we write $Y^s_v(t)$ the *stable Jacobi field* along $\gamma_{x,\xi}$ with $Y^s_v(0) = v$ (cf. [H-IH], p. 482 for a definition of stable Jacobi fields). By Proposition 3.1 of [H-IH], we have

$$\nabla\mathcal{B}_\xi(.,o)(x) = -\gamma'_{x,\xi}(0) \quad \text{and} \quad \nabla_v\nabla\mathcal{B}_\xi(.,o)(x) = -(Y^s_v)'(0)$$

for all $v \in T_x\tilde{M}$. Moreover,

$$\Delta\mathcal{B}_\xi(.,o) = -\mathrm{Trace}(v \mapsto \nabla_v\nabla\mathcal{B}_\xi(.,o))(x).$$

Therefore, by Rauch comparison theorem (cf. [Klin] p. 216), we get

$$na \leq \Delta\mathcal{B}_\xi(.,o) \leq nb, \tag{4.2}$$

which implies (4.1). □

Proposition 4.3. *Let $\tilde{M}$ be a simply connected Riemannian $(n+1)$-manifold, whose sectional curvatures satisfy $-b^2 \leq K \leq -a^2$. Let $\delta > \frac{nb}{2}$. Then there exists $C > 0$, depending only on M and on δ, such that for all unit vector $v \in S\tilde{M}$,*

$$\int_{H^+(v)} e^{\delta\mathcal{B}_{v^+}(\pi v,\pi u)} dV_{H^+(v)}(u) \leq C.$$

(using the same notations as in Proposition 3.5.)

Proof. In this proof, we will write $C_1, \ldots$ for constants depending only on the curvatures bound a and b.

For all $v \in S\tilde{M}$ and $u \in H^+(v)$, we will write $d^+(u,v)$ the Hamenstädt-Hersonsky-Paulin distance (see [Rob03], [H-P]) on the horosphere $H^+(v)$, and $B^+(v,r)$ the ball with center v and radius r for this distance. We recall that, by definition,

$$d^+(u,v) = \lim_{t\to+\infty} e^{-t+\frac{1}{2}d(\pi g^t v, \pi g^t u)}.$$

where d is the Riemannian distance. Triangle inequality implies $d(\pi g^t v, \pi u) - t \geq d(\pi v, \pi g^{-t} v) - d(\pi u, \pi g^{-t} v)$, therefore $\mathcal{B}_{v^+}(\pi u, \pi v) \geq 0$ as $t \to \infty$. It also implies $d(\pi g^t v, \pi u) - t \geq d(\pi g^t v, \pi g^t u) - 2t$, which gives after taking limit and exponentiating,

$$e^{\mathcal{B}_{v^+}(\pi v, \pi u)} \leq \min\left(1, \frac{1}{d^+(v,u)^2}\right). \tag{4.3}$$

Moreover, balls for d^+ are preserved by the geodesic flow:

$$B^+(v,r) = g^{\log r} B^+(g^{-\log r} v, 1).$$

We recall that on negatively curved manifold, the geodesic flow is uniformly hyperbolic (cf. [Klin], Chap. 9): for all $v \in S\tilde{M}$, all ξ tangent to $H^+(v)$ and all $t \geq 0$,

$$C_1{}^{-1} \|\xi\| e^{-bt} \leq \left\|Dg^{-t}.\xi\right\| \leq C_1 \|\xi\| e^{-at},$$

where we write Dg^t for the differential of the geodesic flow at the origin of $T_v S\tilde{M}$. Therefore, for all $r \geq 1$,

$$V_{H^+(v)}\big(B^+(v,r)\big) \leq C_1{}^n r^{nb} V_{H^+(g^{-\log r} v)}\big(B^+(g^{-\log r} v, 1)\big). \tag{4.4}$$

The same argument shows that $V_{H^+(v)}\big(B^+(v,1)\big) \leq C_2 V_{H^+(g^t v)}\big(g^t B^+(v, 1)\big)$ for $|t| \leq \frac{1}{2}$. Now, a standard comparison with the hyperbolic space with constant curvatures $-a^2$ shows that on $\tilde{M}$, $\frac{2}{a}\sinh\frac{a}{2}d(\pi u, \pi v) \leq d^+(u,v)$ (equality occurs when the curvatures of $\tilde{M}$ are constant $-a^2$). Together with the previous inequalities, this gives

$$\begin{aligned} V_{H^+(v)}\big(B^+(v,1)\big) &\leq C_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} V_{H^+(g^t v)}\big(g^t B^+(v,1)\big) dt \\ &\leq C_2 V\big(B(\pi v, C_3)\big) \\ &\leq C_4, \end{aligned}$$

where the last inequality comes again from a volume comparison (cf. [G-H-L] p. 169) since $K \leq -a^2$.

Gathering this inequality with (4.4), we get that for all $r \geq 1$,

$$V_{H^+(v)}\big(B^+(v,r)\big) \leq C_5 r^{nb}. \tag{4.5}$$

From (4.3) and (4.5), an integration by part gives

$$\int_{H^+(v)} e^{\delta \mathcal{B}_{v^+}(\pi v, \pi u)} dV_{H^+(v)}(u) \leq C_6 \int_1^\infty r^{nb-2\delta-1} dr,$$

which concludes the proof of our proposition. □

Proof of Theorem 4.1. Let M be a Riemannian manifold whose sectional curvatures satisfy $-b^2 \leq K \leq -a^2$ and whose Bowen-Margulis measure is finite. Let δ_Γ be the critical exponent of its fundamental group Γ, and μ be a conformal density of dimension δ_Γ invariant by Γ, and $\phi_\mu : \tilde{x} \mapsto \|\mu_{\tilde{x}}\|$. By Proposition 4.2, ϕ_μ is a positive map on M, satisfying

$$\delta_\Gamma(na - \delta_\Gamma)\phi_\mu \leq \Delta\phi_\mu \leq \delta_\Gamma(nb - \delta_\Gamma)\phi_\mu.$$

Proposition 3.2, together with Brooks estimate $\lambda_0^{ess}(M) \leq \frac{(nb)^2}{4}$ (cf. [Bro]), then gives (1).

The proof of (2) goes as in the proof of Theorem 3.1.

Assume now that $\delta_\Gamma > \frac{nb}{2}$. Then by Proposition 4.3 together with Proposition 3.5, the map ϕ_μ is in $L^2(M)$. Therefore,

$$\lambda_0(M) \leq \delta_\Gamma(nb - \delta_\Gamma),$$

which concludes the proof of (3). □

Theorem 4.1 shows in particular that the Laplacian has isolated eigenvalues as soon as the critical exponent of the fundamental group of the manifold is big enough.

Corollary 4.4. *Let (M,g) be a geometrically finite $(n+1)$-manifold, whose sectional curvatures satisfies $-b^2 \leq K_g \leq -a^2$ and whose Bowen-Margulis measure is finite. Assume the critical exponent of its fundamental group Γ satisfies*

$$\delta_\Gamma > \frac{nb}{2}\left(1 + \sqrt{1 - \frac{a^2}{b^2}}\right).$$

Then $\lambda_0(M)$ is an isolated eigenvalue for the Laplacian.

Proof. Assume that (M, g) satisfies the above hypotheses. Since M has finite Bowen-Margulis measure, Theorem 4.1 implies that $\lambda_0(M) < \frac{(na)^2}{4}$. Therefore, we just have to show that on geometrically finite manifolds whose sectional curvatures satisfy $-b^2 \leq K_g \leq -a^2$, the bottom of the essential spectrum is at least $\frac{(na)^2}{4}$. This last fact has been shown in [Ham04] when $n \geq 2$. However, the proof of Lemmas 2.2 and 2.3 of [Ham04] is still valid when $n = 1$, i.e. for surfaces. Since the ends of a negatively curved geometrically finite surface can only be finite volume cusps and convex-cocompact funnels, the conclusion $\lambda_0^{ess}(M) \geq \frac{(na)^2}{4}$ is still valid in this case. □

This criterion is of course not optimal. In particular, as follows from the proof, if one modifies the metric inside a compact domain, without modifying the *lower bound* on the curvatures, it does not affect this criterion which only relies on b and on the essential spectrum of the Laplacian. This provides the following refinement of Corollary 4.4.

Corollary 4.5. *Let (M, g) be a geometrically finite $(n+1)$-manifold, whose sectional curvatures satisfies $-b^2 \leq K_g \leq -a^2$ and whose Bowen-Margulis measure is finite.*

Let $a_\infty \in [a, b]$ be the supremum over all compact subsets $K \subset M$ of the $\alpha \geq a$ such that on $M \backslash K$, one has $K_g \leq -\alpha^2$.

Assume the the critical exponent of its fundamental group Γ satisfies

$$\delta_\Gamma > \frac{nb}{2}\left(1 + \sqrt{1 - \frac{a_\infty^2}{b^2}}\right).$$

Then $\lambda_0(M)$ is an isolated eigenvalue for the Laplacian.

Conformally compact manifolds with negative sectional curvatures, as defined for instance in [Maz88], form a special class of geometrically finite manifolds without cusps, i.e. convex-cocompact manifolds. On such a manifold M, the sectional curvatures tend to a scalar function $K^\infty : \partial M \to [-b^2, -a^2]$ on the boundary at infinity. We get another refinement of Corollary 4.4 in this conformally compact case.

Corollary 4.6. *Let M be a conformally compact Riemannian manifold, whose sectional curvatures satisfies $-b^2 \leq K \leq -a^2$. Assume the the critical* *exponent of its fundamental group* Γ *satisfies*

$$\delta_\Gamma > \frac{nb}{2}\left(1+\sqrt{1-\frac{K_{max}^\infty}{b^2}}\right),$$

where K_{max}^∞ *is the supremum of the sectional curvatures on the boundary at infinity. Then* $\lambda_0(M)$ *is an isolated eigenvalue for the Laplacian.*

Proof. Let M be a conformally compact Riemannian $(n+1)$-manifold, whose sectional curvatures satisfies $-b^2 \leq K \leq -a^2$. A conformally compact manifold is convex-cocompact, therefore its Bowen-Margulis measure is finite. Theorem 4.1 implies then that

$$\lambda_0(M) \leq \delta_\Gamma(nb-\delta_\Gamma) < \frac{n^2 K_{max}^\infty}{4}.$$

Moreover, it follows from Theorem 1.3 of [Maz88] that the bottom of the essential spectrum of M is exactly $\lambda_0^{ess}(M) = \frac{n^2 K_{max}^\infty}{4}$, where K_{max}^∞ is the supremum of the sectional curvatures at infinity. Therefore $\lambda_0(M)$ is an isolated eigenvalue. □

Such a dynamical criterion for the existence of discrete eigenvalue of the Laplacian on conformally compact manifolds answers a question of R. Mazzeo.

Let us conclude by the following question, in the light of Corollary 3.7 for constant curvature manifolds. Does there exist a complete Riemannian manifold (M,g) with pinched negative curvatures, *infinite volume and infinite Bowen-Margulis measure*, such that $\lambda_0(M) < \lambda_0^{ess}(M)$?

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