

Nonabelian Hodge theory in positive characteristic via exponential twisting

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Let k be a perfect field of positive characteristic and X a smooth algebraic variety over k which is W_2 -liftable. We show that the exponent twisting of the classical Cartier descent gives an equivalence of categories between the category of nilpotent Higgs sheaves of exponent $\leq p$ over X/k and the category of nilpotent flat sheaves of exponent $\leq p$ over X/k , by showing that it is equivalent up to sign to the inverse Cartier and Cartier transforms for these nilpotent objects constructed in the nonabelian Hodge theory in positive characteristic by Ogus-Vologodsky [10]. In view of the crucial role that Deligne-Illusie’s lemma has ever played in their algebraic proof of E_1 -degeneration of the Hodge to de Rham spectral sequence and Kodaira vanishing theorem in abelian Hodge theory, it may not be overly surprising that again this lemma plays a significant role via the concept of Higgs-de Rham flow [8] in establishing a p -adic Simpson correspondence in the nonabelian Hodge theory and Langer’s algebraic proof of Bogomolov inequality for semistable Higgs bundles and Miyaoka-Yau inequality [9].

1. Introduction

Let k be a perfect field of positive characteristic and X a smooth algebraic variety over k . We have the following commutative diagram of Frobenii

$$\begin{array}{ccccc}
 X & \xrightarrow{F_{X/k}} & X' & \xrightarrow{\pi} & X \\
 & \searrow & \downarrow & & \downarrow \\
 & & \text{Spec } k & \xrightarrow{\sigma} & \text{Spec } k,
 \end{array}$$

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where the composite of the relative Frobenius $F_{X/k}$ with π is the absolute Frobenius map F_X of X , and σ is the absolute Frobenius of $\text{Spec } k$. A Higgs sheaf over X/k is a pair (E, θ) where E is a coherent sheaf of \mathcal{O}_X -modules and $\theta : E \rightarrow E \otimes \Omega_{X/k}$ an \mathcal{O}_X -linear morphism satisfying the integrability condition $\theta \wedge \theta = 0$. It is said to be nilpotent of exponent $\leq n$ if for all local sections $\partial_1, \dots, \partial_n$ of $T_{X/k}$,

$$\theta(\partial_1) \cdots \theta(\partial_n) = 0.$$

Let $\text{HIG}(X/k)$ be the category of Higgs sheaves over X/k and $\text{HIG}_{\leq n}(X/k)$ the full subcategory of nilpotent Higgs sheaves of exponent $\leq n$. Note that $\text{HIG}_1(X/k)$ is just the category of coherent sheaves of \mathcal{O}_X -modules. On the other hand, we introduce the category $\text{MIC}(X/k)$ of flat sheaves over X/k . A flat sheaf over X/k is a pair (H, ∇) with H a coherent sheaf of \mathcal{O}_X -modules together with an integrable k -connection $\nabla : H \rightarrow H \otimes \Omega_{X/k}$. To each $(H, \nabla) \in \text{MIC}(X/k)$ one associates the p -curvature map $\psi : H \rightarrow H \otimes F_X^* \Omega_{X/k}$, which is \mathcal{O}_X -linear and satisfies $\psi \wedge \psi = 0$ (see §5.0.9 [4]). Following Definition 5.6 [4], (H, ∇) is said to be nilpotent of exponent $\leq n$ if for all local sections $\partial_1, \dots, \partial_n$ of $T_{X/k}$,

$$\psi(\partial_1) \cdots \psi(\partial_n) = 0.$$

Let $\text{MIC}_n(X/k)$ be the full subcategory of nilpotent flat sheaves over X/k of exponent $\leq n$. Note that $\text{MIC}_1(X/k)$ is the category of flat sheaves with vanishing p -curvature. The classical Cartier descent theorem reads as follows:

Theorem 1.1 (Theorem 5.1 [4]). *There is an equivalence of categories between $\text{HIG}_1(X/k)$ and $\text{MIC}_1(X/k)$. Explicitly, one associates $(F_X^* E, \nabla_{can})$ to $(E, 0) \in \text{HIG}_1(X/k)$ and conversely, one associates $\pi_* H^\nabla$ to $(H, \nabla) \in \text{MIC}_1(X/k)$.*

In the above theorem, ∇_{can} means the unique connection on $F_X^* E$ such that the pullback of any local section of E is flat, and the k -subsheaf H^∇ of flat sections is naturally an $\mathcal{O}_{X'}$ -module of the same rank as H . By abuse of notations, we omit π_* by assuming this obvious identification of an object over X' with the corresponding object over X .

In the recent spectacular work [10], Ogus and Vologodsky have established the nonabelian Hodge theory in positive characteristic. Among other important results, they have generalized Theorem 1.1 in a far-reaching way,

and also the fundamental p -curvature formula of the Gauss-Manin connection of Katz [5] and the fundamental decomposition theorem of Deligne-Illusie [1]. A special but essential case of their main construction in loc. cit. is the following

Theorem 1.2 (Theorem 2.8 [10]). *Let $W_2(k)$ be the ring of Witt vectors of length two. Suppose X is $W_2(k)$ -liftable. Then there is an equivalence of categories*

$$\mathrm{HIG}_p(X/k) \xrightleftharpoons[C]{C^{-1}} \mathrm{MIC}_p(X/k).$$

A $W_2 = W_2(k)$ -lifting \tilde{X} of X induces the W_2 -lifting $\tilde{X}' := \tilde{X} \times_{\mathrm{Spec} W_2, \sigma} \mathrm{Spec} W_2$ of X' . Put $(\mathcal{X}, \mathcal{S}) = (X/k, \tilde{X}'/W_2)$. Then the above functor C^{-1} is given by $C_{\mathcal{X}/\mathcal{S}}^{-1} \circ \pi^*$ and C given by $\pi_* C_{\mathcal{X}/\mathcal{S}}$, where $C_{\mathcal{X}/\mathcal{S}}^{-1}$ (resp. $C_{\mathcal{X}/\mathcal{S}}$) is the inverse Cartier transform (resp. Cartier transform) with respect to the pair $(\mathcal{X}, \mathcal{S})$ in [10], restricting to the above subcategories. The full categories in loc. cit. have the merit of being tensor categories and their functors are compatible with tensor product.

Our main result is to show that Theorem 1.2 is nothing but the exponential twisting of the classical Cartier descent Theorem 1.1. More precisely, we have shown the following

Theorem 1.3. *Fix a $W_2(k)$ -lifting \tilde{X} of X/k . Then we construct a functor C_{exp}^{-1} from $\mathrm{HIG}_p(X/k)$ to $\mathrm{MIC}_p(X/k)$ and respectively a functor C_{exp} in the converse direction, which are equivalent to the inverse Cartier transform C^{-1} respectively the Cartier transform C in Theorem 1.2 up to sign.*

As a direct application, we show the Gauss-Manin flat sheaf of a strict p -torsion Fontaine-Faltings module can be reconstructed from its associated Higgs sheaf via the inverse Cartier transform.

Proposition 1.4. *Let $W(k)$ be the ring of Witt vectors and \mathbf{X} a smooth scheme over $W(k)$. Let $(H, \nabla, \mathrm{Fil}, \Phi)$ be a strict p -torsion Fontaine-Faltings module in $\mathcal{MF}_{[0,n]}^{\nabla}(\mathbf{X}/W(k))$, $n \leq p - 1$ and $(E, \theta) = \mathrm{Gr}_{\mathrm{Fil}}(H, \nabla)$ the associated graded Higgs sheaf over X/k . Then one has an isomorphism*

$$C^{-1}(E, -\theta) \cong (H, \nabla).$$

Remark 1.5. This note grows from our old manuscript [7] and its present form was greatly influenced by remarks in several electronic messages of A. Ogus on our recent works including loc. cit. We intend by no means to

rework the full-fledged theory of Ogus-Vologodsky in [10]. During writing this short note, we have kept the readers whose background are in complex algebraic geometry in mind. Our unclaimed hope is that this work could help more readers nonspecializing this field understand their foundational and beautiful work in char p geometry.

2. Exponential twisting

2.1. Deligne-Illusie's Lemma

Rewrite X by X_0 . Choose and then fix a W_2 -lifting X_1 of X_0 . Then take an affine covering $\mathcal{U} = \{\tilde{U}_\alpha\}_{\alpha \in I}$ of X_1 and for each \tilde{U}_α , take a Frobenius lifting $\tilde{F}_\alpha : \tilde{U}_\alpha \rightarrow \tilde{U}_\alpha$ which mod p is the absolute Frobenius $F_0 : U_\alpha \rightarrow U_\alpha$. Here U_α means the closed fiber of \tilde{U}_α . In the following, we shall always use F_0 for the absolute Frobenius on any variety over k . The induced morphism by \tilde{F}_α on differential forms over \tilde{U}_α is therefore divisible by p . The composite of $\mathcal{O}_{\tilde{U}_\alpha}$ -morphisms

$$\tilde{F}_\alpha^* \Omega_{\tilde{U}_\alpha} \xrightarrow{\tilde{F}_\alpha} p \Omega_{\tilde{U}_\alpha} \xrightarrow{\cong} \Omega_{U_\alpha}$$

induces an \mathcal{O}_{U_α} -morphism

$$\zeta_\alpha := \frac{\tilde{F}_\alpha}{[p]} : F_0^* \Omega_{U_\alpha} \rightarrow \Omega_{U_\alpha}.$$

The basic lemma of Deligne-Illusie in [1] is the following:

Lemma 2.1. *For any $\alpha, \beta, \gamma \in I$, set $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$. There are homomorphisms $h_{\alpha\beta} : F_0^* \Omega_{U_{\alpha\beta}} \rightarrow \mathcal{O}_{U_{\alpha\beta}}$, satisfying the following two properties:*

- (i) *over $F_0^{-1} \Omega_{U_{\alpha\beta}}$ we have $\zeta_\alpha - \zeta_\beta = dh_{\alpha\beta}$;*
- (ii) *the cocycle condition over $U_{\alpha\beta\gamma}$: $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$.*

Proof. Consider the W_2 -morphism $G_\alpha : \tilde{Z}_\alpha \rightarrow \tilde{U}_{\alpha\beta} := \tilde{U}_\alpha \cap \tilde{U}_\beta$ sitting in the following Cartesian diagram:

$$\begin{array}{ccc} \tilde{Z}_\alpha & \xrightarrow{G_\alpha} & \tilde{U}_{\alpha\beta} \\ \tilde{j}_\alpha \downarrow & & \downarrow \tilde{i}_\alpha \\ \tilde{U}_\alpha & \xrightarrow{\tilde{F}_\alpha} & \tilde{U}_\alpha, \end{array}$$

where \tilde{i}_α is the natural inclusion. By reduction modulo p , we obtain the following Cartesian square:

$$\begin{array}{ccc} Z_\alpha & \xrightarrow{G_0} & U_{\alpha\beta} \\ j_\alpha \downarrow & & \downarrow i_\alpha \\ U_\alpha & \xrightarrow{F_0} & U_\alpha. \end{array}$$

Thus we see that Z_α is $U_{\alpha\beta}$ and $G_\alpha : \tilde{Z}_\alpha \rightarrow \tilde{U}_{\alpha\beta}$ is a lifting of the absolute Frobenius F_0 over $U_{\alpha\beta}$. Similarly for $(\tilde{U}_\beta, \tilde{F}_\beta)$, we have $G_\beta : \tilde{Z}_\beta \rightarrow \tilde{U}_{\alpha\beta}$ which is also a lifting of $F_0 : U_{\alpha\beta} \rightarrow U_{\alpha\beta}$. Now we apply Lemma 5.4 [3] to the pair $(G_\alpha : \tilde{Z}_\alpha \rightarrow \tilde{U}_{\alpha\beta}, G_\beta : \tilde{Z}_\beta \rightarrow \tilde{U}_{\alpha\beta})$ of Frobenius liftings of the absolute Frobenius F_0 on $U_{\alpha\beta}$, we get the homomorphisms $h_{\alpha\beta} : F_0^* \Omega_{U_{\alpha\beta}} \rightarrow \mathcal{O}_{U_{\alpha\beta}}$ such that over $F_0^{-1} \Omega_{U_{\alpha\beta}}$ we have $\zeta_\alpha - \zeta_\beta = dh_{\alpha\beta}$ and $h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$. \square

2.2. Inverse Cartier

Given a Higgs sheaf $(E, \theta) \in \text{HIG}_p := \text{HIG}_p(X/k)$, we are going to associate to it a flat sheaf $C_{\text{exp}}^{-1}(E, \theta) \in \text{MIC}_p := \text{MIC}_p(X/k)$.

Description: Over each U_α , we define a local sheaf $H_\alpha := F_0^*(E|_{U_\alpha})$ together with a connection over H_α by the formula

$$\nabla_\alpha = \nabla_{\text{can}} + \zeta_\alpha(F_0^*\theta|_{U_\alpha})$$

where $E|_{U_\alpha}$ is the restriction of E to U_α (similar for the meaning of $\theta|_{U_\alpha}$) and ∇_{can} is the flat connection in the Cartier descent theorem 1.1. Over $U_{\alpha\beta} := U_\alpha \cap U_\beta$, after Lemma 2.1, we define an $\mathcal{O}_{U_{\alpha\beta}}$ -linear morphism

$$h_{\alpha\beta}(F_0^*\theta) : F_0^*E|_{U_{\alpha\beta}} \rightarrow F_0^*E|_{U_{\alpha\beta}}.$$

Because θ is by assumption nilpotent of exponent $\leq p$, we are able to define an element in $G_{\alpha\beta} \in \text{Aut}_{\mathcal{O}_{U_{\alpha\beta}}}(F_0^*E|_{U_{\alpha\beta}})$ by the formula

$$\exp[h_{\alpha\beta}(F_0^*\theta)] := \sum_{i=0}^{p-1} \frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!}.$$

Then we use the set of local isomorphisms $\{G_{\alpha\beta}\}_{\alpha,\beta \in I}$ to glue $\{(H_\alpha, \nabla_\alpha)\}_{\alpha \in I}$, to obtain the claimed flat sheaf $C_{\text{exp}}^{-1}(E, \theta)$ over X/k . The verification details are contained in the following steps:

Step 1: Local sheaves glue. It is to show the cocycle condition:

$$G_{\beta\gamma} \circ G_{\alpha\beta} = G_{\alpha\gamma}.$$

We compute

$$G_{\beta\gamma} \circ G_{\alpha\beta} = \exp[h_{\beta\gamma}(F_0^*\theta)] \exp[h_{\alpha\beta}(F_0^*\theta)].$$

It follows from the integrability of Higgs field that the two morphisms $h_{\alpha\beta}(F_0^*\theta)$ and $h_{\beta\gamma}(F_0^*\theta)$ commute with each other. Thus we compute further that

$$G_{\beta\gamma} \circ G_{\alpha\beta} = \exp[(h_{\beta\gamma} + h_{\alpha\beta})(F_0^*\theta)] = \exp[h_{\alpha\gamma}(F_0^*\theta)] = G_{\alpha\gamma}.$$

The second equality follows from Lemma 2.1 (ii).

Step 2: Local connections glue. It is to show that the local connections $\{\nabla_\alpha\}$ coincide on the overlaps, that is

$$(G_{\alpha\beta} \otimes id) \circ \nabla_\alpha = \nabla_\beta \circ G_{\alpha\beta}.$$

It suffices to show

$$\zeta_\alpha(F_0^*\theta) = G_{\alpha\beta}^{-1} \circ dG_{\alpha\beta} + G_{\alpha\beta}^{-1} \circ \zeta_\beta(F_0^*\theta) \circ G_{\alpha\beta}.$$

We see that

$$G_{\alpha\beta}^{-1} \circ dG_{\alpha\beta} = -dG_{\alpha\beta}^{-1} \circ G_{\alpha\beta} = dh_{\alpha\beta}(F_0^*\theta),$$

and

$$G_{\alpha\beta}^{-1} \circ \zeta_\beta(F_0^*\theta) \circ G_{\alpha\beta} = \zeta_\beta(F_0^*\theta),$$

as $G_{\alpha\beta}$ commutes with $\zeta_\beta(F_0^*\theta)$ due to the integrability of the Higgs field. So

$$G_{\alpha\beta}^{-1} \circ dG_{\alpha\beta} + G_{\alpha\beta}^{-1} \circ \zeta_\beta(F_0^*\theta) \circ G_{\alpha\beta} = dh_{\alpha\beta}(F_0^*\theta) + \zeta_\beta(F_0^*\theta) = \zeta_\alpha(F_0^*\theta).$$

The last equality uses Lemma 2.1 (i).

Step 3: Flatness. This is a local property. First of all, one has

$$F_0^*(\theta) \wedge F_0^*(\theta) = F_0^*(\theta \wedge \theta) = 0,$$

and then

$$\zeta_\alpha(F_0^*\theta) \wedge \zeta_\alpha(F_0^*\theta) = \left(\bigwedge^2 \zeta_\alpha \right) (F_0^*\theta \wedge F_0^*\theta) = 0.$$

It is left to show that $d(\zeta_\alpha(F_0^*\theta)) = 0$. This is done by a local computation: by definition, for $\omega \in \Omega_{U_\alpha}$,

$$\zeta_\alpha(F_0^*\omega) = \frac{1}{[p]}(dF_\alpha(F_\alpha^*\omega')),$$

where $\omega' \in \Omega_{U'_\alpha}$ is any lifting of ω . Then

$$d \circ \zeta_\alpha(F_0^*\omega) = d \circ \frac{1}{[p]}(dF_\alpha(F_\alpha^*\omega')) = \frac{1}{[p]}(d \circ dF_\alpha(F_\alpha^*\omega')).$$

We may write $\omega' = \sum_i f_i dg_i$ for $f_i, g_i \in \mathcal{O}_{U'_\alpha}$. Then

$$d(dF_\alpha(F_\alpha^*\omega')) = \sum_i d(F_\alpha^*f_i) \wedge d(F_\alpha^*g_i) \in p^2\Omega_{U'_\alpha}^2 = 0.$$

Thus $d(\zeta_\alpha(F_0^*\omega)) = 0$. Clearly it follows that $d(\zeta_\alpha(F_0^*\theta)) = 0$.

Step 4: Nilpotency. The p -curvature of the flat sheaf $C_{\text{exp}}^{-1}(E, \theta)$ over U_α takes the form

$$F_0^*\theta_{U_\alpha} : F_0^*E|_{U_\alpha} \rightarrow F_0^*E|_{U_\alpha} \otimes F_0^*\Omega_{U_\alpha}$$

which is clearly nilpotent of exponent $\leq p$.

2.3. Cartier

We shall do the converse process, namely, associate a Higgs sheaf $C_{\text{exp}}(H, \nabla) \in \text{HIG}_p$ to any flat sheaf $(H, \nabla) \in \text{MIC}_p$.

Description: Let $\psi : H \rightarrow H \otimes F_0^*\Omega_{X/k}$ be the p -curvature map of (H, ∇) . Set

$$H_\alpha := H|_{U_\alpha}, \quad \nabla_\alpha := \nabla|_{U_\alpha}, \quad \psi_\alpha := \psi|_{U_\alpha}.$$

Define a new connection ∇'_α on H_α by the formula:

$$\nabla'_\alpha = \nabla_\alpha + \zeta_\alpha(\psi_\alpha).$$

Because of the nilpotency condition on ψ , we can use again Lemma 2.1 to define

$$J_{\alpha\beta} := \exp(h_{\alpha\beta}(\psi)) \in \text{Aut}_{\mathcal{O}_{U_{\alpha\beta}}}(H|_{U_{\alpha\beta}}).$$

Then we use the set $\{J_{\alpha\beta}\}_{\alpha,\beta \in I}$ of local isomorphisms to glue $\{H_\alpha, \nabla'_\alpha\}_{\alpha \in I}$, to obtain a new flat sheaf (H', ∇') whose p -curvature vanishes. The p -curvature map ψ induces an F -Higgs sheaf in a natural way

$$\psi' : H' \rightarrow H' \otimes F_0^* \Omega_{X/k},$$

which is parallel with respect to ∇' . By the Cartier descent Theorem 1.1, the pair (H', ψ') descends to a Higgs sheaf $C_{\text{exp}}(H, \nabla)$. The verification steps are given as follows:

Step 1: Local sheaves glue. This step follows from Lemma 2.1 (ii) in a similar way to Step 1 of inverse Cartier. Also, it is direct to check that

$$\psi_\beta \circ J_{\alpha\beta} = J_{\alpha\beta} \circ \psi_\alpha.$$

Hence ψ induces an F -Higgs sheaf $\psi' : H' \rightarrow H' \otimes F_0^* \Omega_{X/k}$.

Step 2: Local connections glue. It is to show

$$\zeta_\alpha(\psi) = J_{\alpha\beta}^{-1} \circ dJ_{\alpha\beta} + J_{\alpha\beta}^{-1} \circ \zeta_\beta(\psi) \circ J_{\alpha\beta}.$$

As by Lemma 2.1 (i)

$$J_{\alpha\beta}^{-1} \circ dJ_{\alpha\beta} = -dJ_{\alpha\beta}^{-1} \circ J_{\alpha\beta} = d(\psi(h_{\alpha\beta})) = \zeta_\alpha(\psi) - \zeta_\beta(\psi),$$

and

$$J_{\alpha\beta}^{-1} \circ \zeta_\beta(\psi) \circ J_{\alpha\beta} = \zeta_\beta(\psi),$$

it follows that

$$J_{\alpha\beta}^{-1} \circ dJ_{\alpha\beta} + J_{\alpha\beta}^{-1} \circ \zeta_\beta(\psi) \circ J_{\alpha\beta} = \zeta_\alpha(\psi) - \zeta_\beta(\psi) + \zeta_\beta(\psi) = \zeta_\alpha(\psi).$$

Step 3: p -curvature of the new connection vanishes. This is a local check, which is presumably reduced to check an elementary polynomial identity in char p (see Statement 2.3 below). Instead of showing the identity directly,

we are going to give a proof which uses a trick inspired by Proposition 1.5 [10]. Take a system of étale local coordinates $\{t_1, \dots, t_d\}$ for U_α . Write \mathcal{O} for \mathcal{O}_{U_α} and Ω for Ω_{U_α} and so on. Consider an auxiliary sheaf $\mathcal{O} \oplus F_0^*\Omega$, equipped with the connection ∇_1 defined by the formula:

$$(f, g \otimes \omega) \mapsto (df + g \otimes \zeta(1 \otimes \omega), (1 \otimes \omega) \otimes dg), \quad f, g \in \mathcal{O}, \omega \in \Omega.$$

Its p -curvature ψ_{∇_1} takes a simple form. In fact, it is zero over the factor \mathcal{O} , and for ω ,

$$\begin{aligned} \psi_{\nabla_1}(1 \otimes \partial t_i)(1 \otimes dt_j) &= (\nabla_1(1 \otimes \partial t_i))^p(1 \otimes dt_j) \\ &= \left(\frac{\partial}{\partial t_i}\right)^{p-1}(\partial t_i \lrcorner \zeta_\alpha(1 \otimes dt_j)) = -\delta_{ij}, \end{aligned}$$

where δ_{ij} is the Dirichlet delta-function. So we obtain the formula for ψ_{∇_1} :

$$\psi_{\nabla_1}(f, g \otimes \omega) = -(g, 0) \otimes \omega.$$

Next we extend the connection ∇_1 to the connection ∇_2 on

$$F_0^*(S^{<p}\Omega) := \bigoplus_{i < p} F_0^*S^i\Omega = \mathcal{O} \oplus F_0^*\Omega \oplus F_0^*S^2\Omega \oplus \dots \oplus F_0^*S^{p-1}\Omega$$

by the Leibniz rule. From above, it is easy to see the formula for its p -curvature ψ_{∇_2} :

$$\psi_{\nabla_2}(1 \otimes \partial t_i)(1 \otimes \omega) = -1 \otimes \partial t_i \lrcorner \omega, \quad \omega \in S^{<p}\Omega.$$

Finally, since $F_0^*(S^{<p}T) := \bigoplus_{i < p} F_0^*S^iT$ is naturally dual to $F_0^*(S^{<p}\Omega)$, ∇_2 induces the dual connection ∇_3 on $F_0^*(S^{<p}T)$ given by the formula

$$\partial \langle \alpha, \beta \rangle = \langle \nabla_2(\partial)(\alpha), \beta \rangle + \langle \alpha, \nabla_3(\partial)(\beta) \rangle$$

for $\partial \in T$, $\alpha \in F_0^*(S^{<p}\Omega)$ and $\beta \in F_0^*(S^{<p}T)$, and where $\langle \cdot, \cdot \rangle$ is the natural pairing between $F_0^*(S^{<p}\Omega)$ and $F_0^*(S^{<p}T)$.

Also endow $F_0^*(S^{<p}T)$ (and similarly for $F_0^*(S^{<p}\Omega)$) with its natural ring structure by

$$F_0^*(S^{<p}T) \cong F_0^*(S \cdot T) / I,$$

with I the ideal $F_0^*(S^{\geq p}T) := \bigoplus_{i \geq p} F_0^*S^iT$. Set $f_{ij} := \partial t_j \lrcorner \zeta_\alpha(1 \otimes dt_i)$, 1_Ω the unit of $F_0^*(S^{<p}\Omega)$, 1_T the unit of $F_0^*(S^{<p}T)$. It holds that

$$\langle \nabla_3(\partial t_i)(1_T), 1_\Omega \rangle = -\langle 1_T, \nabla_2(\partial t_i)(1_\Omega) \rangle = 0,$$

and

$$\langle \nabla_3(\partial t_i)(1_T), 1 \otimes dt_j \rangle = -\langle 1_T, \nabla_2(\partial t_i)(1 \otimes dt_j) \rangle = -f_{ji},$$

and for $m \geq 2$

$$\langle \nabla_3(\partial t_i)1_T, 1 \otimes dt_{i_1} \cdots dt_{i_m} \rangle = 0.$$

So we obtain the formula for ∇_3 :

$$(2.1.1) \quad \nabla_3(\partial t_i)(1_T) = \sum_{j=1}^d -f_{ji} \otimes \partial t_j.$$

It follows that for its p -curvature ψ_{∇_3} , one has

$$\begin{aligned} & \langle \psi_{\nabla_3}(1 \otimes \partial t_i)(1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}), 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} \rangle \\ &= \langle [\nabla_3(\partial t_i)]^p(1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}), 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} \rangle \\ &= -\langle 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, [\nabla_2(\partial t_i)]^p(1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d}) \rangle \\ &= -\langle 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, \psi_{\nabla_2}(1 \otimes \partial t_i)(1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d}) \rangle \\ &= \langle 1 \otimes (\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, 1 \otimes \partial t_i \lrcorner ((dt_1)^{j_1} \cdots (dt_d)^{j_d}) \rangle \\ &= \langle 1 \otimes \partial t_i(\partial t_1)^{i_1} \cdots (\partial t_d)^{i_d}, 1 \otimes (dt_1)^{j_1} \cdots (dt_d)^{j_d} \rangle. \end{aligned}$$

So for $\tau \in F_0^*(S^{<p}T)$, $v \in F_0^*T$,

$$(2.1.2) \quad \psi_{\nabla_3}(v)(\tau) = v \cdot \tau.$$

Now let F_0^*T act on $F_0^*(S^{<p}T)$ via ψ_{∇_3} , which extends to an $L := F_0^*(S^<pT)/I$ action on $F_0^*(S^{<p}T)$. By Formula 2.1.2, this action is just the multiplication map. So $F_0^*(S^{<p}T)$ is a rank one free module over L with the basis 1_T . Recall that L acts on H via the p -curvature map ψ . Therefore there is an isomorphism of \mathcal{O} -modules:

$$\lambda : \mathcal{H}om_L(F_0^*(S^{<p}T), H) \cong H, \quad \phi \mapsto \phi(1_T).$$

The connection ∇_3 on $F_0^*(S^{<p}T)$ and ∇ on H induce the connection ∇_4 on $\mathcal{H}om_L(F_0^*(S^{<p}T), H)$, which via the above isomorphism λ induces the connection ∇'' on H . Claim that $\nabla'' = \nabla'$. Note that the claim completes this step, since L acts on $\mathcal{H}om_L(F_0^*(S^{<p}T), H)$ tautologically by zero and therefore the p -curvature map of ∇'' is simply the zero map. So, for any $\phi \in \mathcal{H}om_L(F_0^*(S^{<p}T), H)$ and the corresponding $e := \phi(1_T)$, one calculates

that

$$\begin{aligned}
 \nabla''(\partial t_i)(e) &= (\nabla_4(\partial t_i)\phi)(1_T) \\
 &= \nabla(\partial t_i)(\phi(1_T)) - \phi(\nabla_3(\partial t_i)(1_T)) \\
 &= \nabla(\partial t_i)(e) + \sum_{j=1}^d \phi(f_{ji} \otimes \partial t_j) \\
 &= \nabla(\partial t_i)(e) + \sum_{j=1}^d \phi(\psi_{\nabla_3}(f_{ji} \otimes \partial t_j)(1_T)) \\
 &= \nabla(\partial t_i)(e) + \sum_{j=1}^d \psi_{\nabla_3}(f_{ji} \otimes \partial t_j)(\phi(1_T)) \\
 &= \nabla(\partial t_i)(e) + \partial t_i \lrcorner \zeta(\psi(e)).
 \end{aligned}$$

Thus $\nabla'' = \nabla + \zeta(\psi) = \nabla'$ as claimed.

Remark 2.2. In the following remark, we explain a local computation showing the vanishing of p -curvature for an object in MIC_2 (hence particularly for $p = 2$) and indicate that the same calculation extends over MIC_p by showing an explicit polynomial identity. It suffices to show the case for an affine curve with an étale local coordinate t . The p -curvature ψ of $(H, \nabla) \in \text{MIC}_2$ will be therefore expressed by $K := A \cdot 1 \otimes dt$, with A an endomorphism of H satisfying $A^2 = 0$. The new connection, defined as above, is given by

$$\nabla' = \nabla + \frac{dF^*}{p}(K),$$

where F is any Frobenius lifting. Let f be the element in \mathcal{O} such that $[p]f = \frac{dF}{dt}$. Put $\partial = \partial t$. Then we claim that

$$(\nabla'_{\partial})^p = \sum_{i_1+i_2+i_3+\dots=p} (\nabla_{\partial})^{i_1} \cdot (fA)^{i_2} \cdot (\nabla_{\partial})^{i_3} \dots$$

is actually zero. Notice that if a term in the above expression contains two fA s, then it must be zero. This is because

$$(\nabla_{\partial})^i \cdot (fA) = \sum_{l=0}^i \binom{l}{i} f^{(l)} A \cdot (\nabla_{\partial})^{l-i} = A \cdot B = B \cdot A$$

for some endomorphism B , which follows from the fact that the p -curvature $A = \psi_{1 \otimes \partial}$ commutes with ∇_{∂} , and then $A^2 = 0$ by the assumption. So only

the terms containing at most one fA will survive. Thus we can write

$$(\nabla'_\partial)^p = \sum_{i=0}^{p-1} (\nabla_\partial)^i \cdot (fA) \cdot (\nabla_\partial)^{p-i-1} + (\nabla_\partial)^p,$$

which is further equal to

$$\sum_{i=0}^{p-1} a_i f^{(i)} A \cdot (\nabla_\partial)^{p-i-1} + (\nabla_\partial)^p,$$

with $a_i = \sum_{i \leq j \leq p-1} \binom{j}{i} \pmod p$. But since $\sum_{i=0}^{p-1} a_i T^i = \sum_{j=1}^{p-1} (1+T)^j$, it follows that for $0 \leq i \leq p-2$, $a_i = 0$ and $a_{p-1} = 1$. Thus

$$(\nabla'_\partial)^p = f^{(p-1)} A + A$$

Finally, note that we always have $f^{(p-1)} = -1$. Therefore the claim follows. When the exponent of the nilpotency of the p -curvature becomes larger, the complexity of binomial coefficients in the expression grows. But we find they are neatly expressed by the following identity, which is checked in the appendix:

Statement 2.3. For any $2 \leq i \leq p-1$, set the following polynomial with integer coefficients

$$g_i(T_1, \dots, T_i) = \sum_{0 \leq a_1 + \dots + a_i \leq p-i} \left((1+T_i)^{a_i} (1+T_i+T_{i-1})^{a_{i-1}} \dots \right. \\ \left. \dots (1+T_i+\dots+T_1)^{a_1} \right),$$

where the index a_1, \dots, a_i are natural numbers, and

$$G_i(T_1, \dots, T_i) = \sum_{\sigma \in \mathcal{S}_i} g_i(T_{\sigma(1)}, \dots, T_{\sigma(i)}),$$

where \mathcal{S}_i is the permutation group of i elements. Then $G_i \pmod p$ is zero.

Step 4: Nilpotency. It is clear that the so-obtained Higgs sheaf $C_{\text{exp}}(H, \nabla)$ is nilpotent of exponent $\leq p$.

Remark 2.4. It is not difficult to see that C_{exp}^{-1} and C_{exp} (up to isomorphisms) depend neither on the choices of coverings \mathcal{U} nor on the choices of Frobenius liftings. However, they depend on the choice of W_2 -liftings of

X. In the forthcoming paper [6], we have explicitly determined the effect of different W_2 -liftings on the resulting flat sheaves in one special important case which is in intimate relation to the p -adic Teichmüller theory of S. Mochizuki. Also, it is clear that they respect a G -structure on flat sheaves or Higgs sheaves for any subgroup $G \subset GL$.

3. Equivalence of two constructions

In this section we want to show that our construction coincides with Ougs-Vologodsky's abstract construction up to a minus sign. The notions in [10] will be directly applied in the following.

For $(E, \theta) \in \text{HIG}_p$, let $(E', \theta') := \pi^*(E, \theta)$ be the pull-back Higgs sheaf over X'_0 . Then $C_{(\mathcal{X}, \mathcal{S})}^{-1}(E', \theta')$ is defined as

$$(M, \nabla_M) := \mathcal{B}_{\mathcal{X}/\mathcal{S}} \otimes_{\hat{\Gamma}.T_{X'_0}} \iota^*(E').$$

Now for $(\tilde{U}_\alpha, \tilde{F}_\alpha)$, we set $U'_\alpha := U_\alpha \times_{W_2, \sigma} W_2$ and F'_α the composite of \tilde{F}_α and $\tilde{\pi}^{-1}$. They provide an isomorphism

$$\sigma_\alpha : \mathcal{B}_{\mathcal{X}/\mathcal{S}}|_{U_\alpha} \cong F^*\hat{\Gamma}.T_{U'_\alpha},$$

which induces isomorphisms:

$$\xi_\alpha : M|_{U_\alpha} \cong F^*\hat{\Gamma}.T_{U'_\alpha} \otimes_{\hat{\Gamma}.T_{U'_\alpha}} \iota^*E' \cong F_0^*\hat{\Gamma}.T_{U_\alpha} \otimes_{\hat{\Gamma}.T_{U_\alpha}} \iota^*E,$$

and

$$\varpi_\alpha : F_0^*\hat{\Gamma}.T_{U_\alpha} \otimes_{\hat{\Gamma}.T_{U_\alpha}} \iota^*E \cong F_0^*\iota^*E \cong F_0^*E.$$

Set $\eta_\alpha := \varpi_\alpha \circ \xi_\alpha$. Then under the isomorphism η_α , ∇_M induces a connection ∇_α on F_0^*E . Then for any local section e of E over U_α ,

$$\begin{aligned} \nabla_\alpha(\partial t_k)(1 \otimes e) &= \varpi_\alpha(\nabla_{\partial t_k}(1_T) \otimes e) = \varpi_\alpha \left(\sum_{i=1}^d (f_{ik} \otimes \partial t_i) \otimes e \right) \\ &= - \sum_{i=1}^d \varpi_\alpha(f_{ik} \otimes \theta_{\partial t_i}(e)) = - \sum_{i=1}^d f_{ik} \varpi_\alpha(1 \otimes \theta_{\partial t_i}(e)) \\ &= -\partial t_k \lrcorner \zeta_\alpha(\theta(e)). \end{aligned}$$

Notice that the connection on $F_0^* \hat{\Gamma} T_{U_\alpha}$ in [10] differs from ours by a minus sign. So

$$\nabla_\alpha = \nabla_{can} - \zeta(\theta).$$

Now on the overlap $U_{\alpha\beta}$, $\sigma_\beta \circ \sigma_\alpha^{-1}$ is just the multiplication map by $\exp(h_{\alpha\beta})$, here $h_{\alpha\beta}$ is considered as a local section of $F^* T_{U'_{\alpha\beta}}$. So if we set $\bar{J}_{\alpha\beta} := \eta_\beta \circ \eta_\alpha^{-1}$, then

$$\bar{J}_{\alpha\beta}(1 \otimes e) = \exp(h_{\alpha\beta}) \otimes \pi^*(e) = \exp(-h_{\alpha\beta}(F_0^* \theta(e))).$$

It follows that the inverse Cartier transform $C_{(\mathcal{X}, \mathcal{S})}^{-1}(\pi^*(E, \theta))$ is equivalent to using

$$\{\bar{J}_{\alpha\beta} = \exp(-h_{\alpha\beta}(F_0^* \theta))\}$$

to glue the local models

$$\{(M_\alpha = F_0^* E|_{U_\alpha}, \nabla_\alpha = \nabla_{can} - \zeta_\alpha(\theta))\},$$

which is just $C_{\exp}^{-1}(E, -\theta)$.

On the other hand, given a flat sheaf $(H, \nabla) \in \text{MIC}_p$, its Cartier transform $C_{(\mathcal{X}, \mathcal{S})}(H, \nabla)$ is defined by

$$(E', \theta') = \iota^* \mathcal{H}om_{D_{X_0/\mathcal{S}}}(\mathcal{B}_{\mathcal{X}/\mathcal{S}}, H).$$

Let $D_{X_0/k}$ be the sheaf of PD-differential operators on X_0 . Set $D_\alpha := D_{X_0/k}|_{U_\alpha}$, and $(E, \theta) := \pi_*(E', \theta')$. As the p -curvature map ψ of (H, ∇) is nilpotent of exponent $\leq p - 1$, the above σ_α induces isomorphisms:

$$\mu_\alpha : E|_{U_\alpha} \cong \iota^* \mathcal{H}om_{F_0^* T_{U_\alpha}}(F_0^* T_{U_\alpha}, H)^{D_\alpha},$$

and

$$\kappa_\alpha : \iota^* \mathcal{H}om_{F_0^* T_{U_\alpha}}(F_0^* T_{U_\alpha}, H)^{D_\alpha} \cong H_\alpha^{\nabla'_\alpha}.$$

Here ∇'_α is the connection on H_α induced from that on $\mathcal{H}om_{F_0^* T_{U_\alpha}}(F_0^* T_{U_\alpha}, H_\alpha)$ which is in turn induced by the connection ∇_{T_α} on $F_0^* T_{U_\alpha}$ and ∇ on H . Now the calculation in the Step 3 of our Cartier construction in §2 shows that

$$\nabla'_\alpha = \nabla_H + \zeta_\alpha(\psi).$$

Set $\rho_\alpha := \kappa_\alpha \circ \mu_\alpha$. Then via the isomorphism ρ_α , the Higgs field θ induces a Higgs field θ_α on $H^{\nabla'_\alpha}$. Then for any local section $e = \phi(1_T)$ of H over U_α

annihilated by ∇'_α with ϕ a local section of $\mathcal{H}om_{F_0^*T_{U_\alpha}}(F_0^*T_{U_\alpha}, H)$ over U_α , one has

$$\theta_\alpha(\partial t_k)(e) = \kappa_\alpha(\psi(\partial t_k) \circ \phi) = -\psi(\partial t_k)(\phi(1_T)) = -\psi(\partial t_k)(e).$$

That means $\theta_\alpha = -\psi$. Now on the overlap $U_{\alpha\beta}$, if we set $J_{\alpha\beta} := \rho_\beta \circ \rho_\alpha^{-1}$, then for $e \in H^{\nabla'_\alpha}$, with $e = \phi(1_T)$ for $\phi \in \mathcal{H}om_{F_0^*T_{U_\alpha}}(F_0^*T_{U_\alpha}, H)$, one has

$$\begin{aligned} J_{\alpha\beta}(e) &= \phi(\exp(h_{\beta\alpha})) \\ &= \phi(\psi_T(\exp(h_{\alpha\beta}))(1_T)) \\ &= \exp(\psi(h_{\alpha\beta}))(\phi(1_T)) \\ &= \exp(\psi(h_{\alpha\beta}))(e), \end{aligned}$$

which means

$$J_{\alpha\beta} = \exp(\psi(h_{\alpha\beta})).$$

It follows that the Cartier transform $\pi_*C_{(\mathcal{X}, \mathcal{S})}(H, \nabla)$ is naturally isomorphic to $C_{\exp}(H, -\nabla)$.

4. Gauss-Manin connection of a Fontaine-Faltings module

Let $W = W(k)$ be the ring of Witt vectors, and \mathbf{X} a smooth scheme over W . G. Faltings, generalizing the work of Fontaine-Laffaille to a geometric base, introduced the category $\mathcal{MF}_{[0, n]}^\nabla(\mathbf{X}/W)$, $n \leq p - 1$ in [2]. An object in this category shall be called a Fontaine-Faltings module. Under a mild condition, for a smooth proper morphism $f : \mathbf{Y} \rightarrow \mathbf{X}$ over W , the higher direct images of the constant crystal over \mathbf{Y}/W are objects in this category. In that case, let H be the hypercohomology of the relative de Rham complex of f , ∇ the Gauss-Manin connection, Fil the Hodge filtration and Φ the relative Frobenius. Then the four tuple (H, ∇, Fil, Φ) makes a Fontaine-Faltings module coming from geometry. In general, a Fontaine-Faltings module may not come from geometry. We intend to point out a relation of the mod p reduction of (H, ∇) with the above theory. We assume from now on that (H, ∇, Fil, Φ) is a strict p -torsion Fontaine-Faltings module, i.e. $pH = 0$ (one considers otherwise its mod p reduction). Let $(E, \theta) = Gr_{Fil}(H, \nabla)$ be the graded Higgs bundle. We show the following

Proposition 4.1. *The relative Frobenius Φ induces an isomorphism of flat sheaves:*

$$\tilde{\Phi} : C^{-1}(E, -\theta) \cong (H, \nabla).$$

Proof. We use the reinterpretation of C^{-1} via the exponential twisting and write $(H_{\text{exp}}, \nabla_{\text{exp}})$ for $C_{\text{exp}}^{-1}(E, \theta)$. Take a small open affine covering $\{\mathbf{U}_\alpha\}$ of \mathbf{X} , together with a Frobenius lifting $F_{\mathbf{U}_\alpha}$ over $\hat{\mathbf{U}}_\alpha$, the p -adic completion of \mathbf{U}_α . As before, we denote by \tilde{U}_α and \tilde{F}_α their mod p^2 reductions. Recall that over the pair $(\mathbf{U}_\alpha, F_{\mathbf{U}_\alpha})$, the strong p -divisible property of Φ provides an local isomorphism:

$$\tilde{\Phi}_{\tilde{F}_\alpha} := \sum_i \Phi/p^i : H_{\text{exp}}|_{U_\alpha} = F_0^*(E|_{U_\alpha}) \cong H|_{U_\alpha}.$$

Then we shall show that the set of local isomorphisms $\{\tilde{\Phi}_{\tilde{F}_\alpha}\}$ glues into an isomorphism $\tilde{\Phi}$ as claimed.

Step 1: Sheaf isomorphism. For any local section e of E over $U_{\alpha\beta}$, we are going to show that over $U_{\alpha\beta}$,

$$\tilde{\Phi}_{\tilde{F}_\alpha}(F_0^*e) = \tilde{\Phi}_{\tilde{F}_\beta} \circ J_{\alpha\beta}(F_0^*e).$$

We take a system of étale local coordinates $\{t_1, \dots, t_d\}$ of $U_{\alpha\beta}$. For a multi-index $\underline{j} = (j_1, \dots, j_d)$, we put

$$\theta_{\underline{\partial}}^j = (\partial_{t_1} \lrcorner \theta)^{j_1} \dots (\partial_{t_d} \lrcorner \theta)^{j_d}, \quad z_l = \left(\frac{\tilde{F}_\alpha - \tilde{F}_\beta}{[p]} \right) (F_0^*t_l), \quad z^{\underline{j}} = \prod_{l=1}^d z_l^{j_l}.$$

As Φ is horizontal under ∇ , according to the Taylor formula, we have

$$\tilde{\Phi}_{\tilde{F}_\alpha}(F_0^*e) = \tilde{\Phi}_{\tilde{F}_\beta} \circ \left(1 + \sum_{|\underline{j}|=1}^n F_0^*(\theta_{\underline{\partial}}^j) \cdot \frac{z^{\underline{j}}}{\underline{j}!} \right) (F_0^*e).$$

So it suffices to show $J_{\alpha\beta} = 1 + \sum_{|\underline{j}|=1}^n F_0^*(\theta_{\underline{\partial}}^j) \cdot \frac{z^{\underline{j}}}{\underline{j}!}$. As

$$h_{\alpha\beta}(F_0^*\theta) = \sum_{l=1}^d F_0^*(\partial_{t_l} \lrcorner \theta) h_{\alpha\beta}(F_0^*dt_l),$$

and

$$h_{\alpha\beta}(F_0^*dt_l) = \left(\frac{\tilde{F}_\alpha - \tilde{F}_\beta}{[p]} \right) (F_0^*t_l) = z_l,$$

it follows that

$$\frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!} = \frac{(\sum_{l=1}^d F_0^*(\partial_{t_l}\lrcorner\theta)z_l)^i}{i!} = \sum_{|j|=i} \frac{F_0^*(\theta_{\partial}^j)z^j}{j!}.$$

Recall that $J_{\alpha\beta} = \sum_{i=0}^n \frac{(h_{\alpha\beta}(F_0^*\theta))^i}{i!}$, it follows that

$$J_{\alpha\beta} = 1 + \sum_{|j|=1}^n F_0^*(\theta_{\partial}^j) \cdot \frac{z^j}{j!}$$

as wanted.

Step 2: Connection isomorphism. We need to show that under the above isomorphism, the connection ∇_{exp} on H_{exp} is equal to the connection ∇ on H . Take a local section e of E over U_{α} . Set $\theta_l := \partial_{t_l}\lrcorner\theta$. By the horizontal property of $\tilde{\Phi}$, we have

$$\nabla[\tilde{\Phi}_{\tilde{F}_{\alpha}}(F_0^*e)] = \tilde{\Phi}_{\tilde{F}_{\alpha}} \circ \left[\sum_{l=1}^d F_0^*\theta_l \cdot \zeta_{\alpha}(F_0^*dt_l) \right] (F_0^*e).$$

As

$$\nabla_{\text{exp}}(F_0^*e) = \left[\sum_{l=1}^d F_0^*\theta_l \cdot \zeta_{\alpha}(F_0^*dt_l) \right] (F_0^*e),$$

it follows that

$$\tilde{\Phi}_{\tilde{F}_{\alpha}}(\nabla_{\text{exp}}(F_0^*e)) = \left[\sum_{l=1}^d F_0^*\theta_l \cdot \zeta_{\alpha}(F_0^*dt_l) \right] \tilde{\Phi}_{\tilde{F}_{\alpha}}(F_0^*e) = \nabla[\tilde{\Phi}_{\tilde{F}_{\alpha}}(F_0^*e)].$$

□

5. Appendix

In this appendix, we come back to Statement 2.3 and show that it can actually be deduced by the fact that the p -curvature of the new connection vanishes. Use the notations in Statement 2.3. To this, we shall take $X = \text{Spec } k[t]$, $H = \mathcal{O}^r$, $\nabla_{\partial}(e) = Ne$ with e the standard basis of H and N an $r \times r$ matrix with entries in \mathcal{O} . Suppose its p -curvature is expressed by $\psi_{1 \otimes \partial}(e) = Ae$ for another $r \times r$ matrix A . Take $\text{Spec } W_2[t]$ to be the W_2 -lifting of X

with a Frobenius lifting $F : W_2[t] \rightarrow W_2[t]$ given by $t \mapsto t^p + ph(t)$, so that $f = \frac{dF^*}{p}(dt) = t^{p-1} + h^{(1)}(t)$. Recall that the new connection is expressed by

$$(\nabla'_\partial)^p = \sum_{i_1+i_2+i_3+\dots=p} (\nabla_\partial)^{i_1} \cdot (fA)^{i_2} \cdot (\nabla_\partial)^{i_3} \dots,$$

and we know that it is equal to zero. Now we rewrite the expression into $\sum_{i=0}^{p-1} \mathcal{I}_i$ with \mathcal{I}_i is the summation of all terms containing exactly i As. So

$$\mathcal{I}_i = A^i \left\{ \sum_{d=0}^{p-i} \left[\sum_{s=0}^i h_{d,s}(f, \dots, f) f^{i-s} \right] \nabla_\partial^{p-i-d} \right\},$$

where $h_{d,s}(f, \dots, f)$ is understood as follows: Write $g_i = \sum_{d=0}^{p-i} v_d$ into the sum of its homogenous components with variables $\{T_1, \dots, T_i\}$, and write each v_d further into $v_d = \sum_{s=0}^i h_{d,s}$ with $h_{d,s}$ containing exactly s -variables. Then we write $h_{d,s} = \sum_I a_I \cdot T_{i_1}^{n_1} \dots T_{i_s}^{n_s}$ and require $\sum_j n_j = d$ and $n_j \geq 1$. In terms of this, by $h_{d,s}(f, \dots, f)$ we mean $\sum_I a_I \cdot f^{(n_1)} \dots f^{(n_s)}$.

First we show Statement 2.3 for $i = 2$. By taking any connection matrix N for ∇ with the *constant* coefficients, i.e. in k , satisfying $N^{3p} = 0$ and $N^{3p-1} \neq 0$, we get $A^3 = N^{3p} = 0$, and $A^2 \nabla_\partial^{p-2} = N^{3p-2} \neq 0$. It follows that $\mathcal{I}_i = 0$ for $i \geq 3$. Since we know that $\mathcal{I}_0 + \mathcal{I}_1 = 0$, one must have $\mathcal{I}_2 = 0$. Then

$$\sum_{s=0}^2 h_{d,s}(f, f) f^{2-s} = 0, \quad 0 \leq d \leq p - 2.$$

By adjusting the Frobenius lifting such that the new f is the old one plus any nonzero number, we see that $h_{d,s}$ remain unchanged, and so that

$$h_{d,s}(f, f) = 0.$$

- 1) For $s = 0$, $h_{d,s} = 0$ if $d \neq 0$. As $h_{0,0}$ is a constant, $h_{0,0} = 0$. Thus, $h_{d,0} = 0$ for $0 \leq d \leq p - 2$.
- 2) For $s = 1$, $h_{d,s} = a_1 T_1^d + a_2 T_2^d$, then $a_1 + a_2 = 0$. That is, $h_{d,s}(T_1, T_2) + h_{d,s}(T_2, T_1) = 0$.
- 3) For $s = 2$, $h_{d,s} = \sum_{j=1}^{d-1} b_j T_1^j T_2^{d-j}$. We adjust the Frobenius lifting such that the new f is the old one plus t . Then it implies $b_1 + b_{d-1} = 0$. We can inductively show that $b_j + b_{d-j} = 0$ by adjusting the Frobenius lifting such that the new f is the old one plus t^j . Finally, we get $h_{d,2}(T_1, T_2) + h_{d,2}(T_2, T_1) = 0$.

We finish the proof for $i = 2$. Next we do the induction step. Assuming $G_s = 0$ for $2 \leq s \leq i - 1$, we want to prove $G_i = 0$. We take some connection ∇ , such that its matrix N with its elements in k and $N^{(i+1)p} = 0$, but $N^{(i+1)p-1} \neq 0$. Then $A^{i+1} = 0$, and $A^i \nabla_{\partial}^{p-i} \neq 0$. The induction hypothesis implies that $\mathcal{I}_s = 0$ for $2 \leq s \leq i - 1$. so $\mathcal{I}_i = 0$. By the same argument as $i = 2$, we see that

$$h_{d,s}(f, \dots, f) = 0.$$

We want to prove that

$$\sum_{\sigma \in \mathcal{S}_i} h_{d,s}(T_{\sigma(1)}, \dots, T_{\sigma(i)}) = 0,$$

where \mathcal{S}_i is the permutation group of i elements.

Assume the contrary. For a monomial $a_{j_1 \dots j_i} T_1^{j_1} \dots T_i^{j_i}$, with $\#\{j_m > 0 \mid 1 \leq m \leq i\} = s$, one permutes the variable such that it becomes the form $a_{j_1 \dots j_i} T_1^{i_1} \dots T_s^{i_s}$ with $1 \leq i_1 \leq i_2 \leq \dots \leq i_s$. We shall call (i_1, \dots, i_s) the type of this monomial. Note that the type is invariant under a permutation of variables. We define a partial order on the set of types by the lexicographic order. Now suppose

$$(5.0.1) \quad \sum_{\sigma \in \mathcal{S}_i} h_{d,s}(T_{\sigma(1)}, \dots, T_{\sigma(i)})$$

is nonzero. Let (w_1, \dots, w_s) be the smallest type such that the summation of monomials of this type in the Formula (5.0.1) is nonzero. Then we adjust the Frobenius lifting such that the new f is the old one plus $c \cdot t^{w_1}$ for any $c \in k$. We see that for any monomial h whose type with $i_1 > w_1$, the $h(f, \dots, f)$ remains unchanged. It implies that the summation of monomials whose type with $i_1 = w_1$ in the Formula (5.0.1) is already zero after inserting f . Then we adjust again the Frobenius lifting, such that the new f is the old one plus $c \cdot t^{w_2}$ for any $c \in k$. The same argument as before shows that the summation of terms with type $i_1 = w_1, i_2 = w_2$ in the Formula (5.0.1) is zero after inserting f . Continuing in this way, we show that the summation of terms with type (w_1, \dots, w_s) is zero after inserting f . It implies further that the the summation of monomials of the type (w_1, \dots, w_s) in the Formula (5.0.1) is already zero. A contradiction.

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the equivalence between C^{-1} and C_{exp}^{-1} up to sign should follow from [10, Remark 2.10]. We would like also to thank sincerely the referee for pointing out an error in the original manuscript.

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