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On discrete fractional integral operators and related Diophantine equations

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We study discrete versions of fractional integral operators along curves and surfaces. $l^p \to l^q$ estimates are obtained from upper bounds of the number of solutions of associated Diophantine systems. In particular, this relates the discrete fractional integral along the curve $\gamma(m) = (m, m^2, \dots, m^k)$ to Vinogradov's mean value theorem. Sharp $l^p \to l^q$ estimates of the discrete fractional integral along the hyperbolic paraboloid in $\mathbb{Z}^3$ are also obtained except for endpoints.

1. Introduction

The classical Hardy-Littlewood-Sobolev inequality gives the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for the fractional integral operator $f \to |\cdot|^{-d\lambda} * f$ for $0 < \lambda < 1$, $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - (1-\lambda)$. The $l^p(\mathbb{Z}^d) \to l^q(\mathbb{Z}^d)$ bounds for its discrete analogue $f \to \sum_{m \in \mathbb{Z}^d \setminus 0} |m|^{-d\lambda} f(\cdot - m)$ follows from the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds by a simple comparison argument [15]. However, if one considers discrete analogues of recent variants of fractional integrals, where the integration is taken over a sub-manifold, the argument fails and the problem becomes more interesting.

Let us give a few examples. Consider the operators $I_{k,\lambda}$ and $J_{k,\lambda}$ defined by

$$I_{k,\lambda}(f)(n) = \sum_{m=1}^{\infty} \frac{f(n - m^k)}{m^\lambda},$$

$$J_{k,\lambda}(f)(n_1, \dots, n_k) = \sum_{m=1}^{\infty} \frac{f(n_1 - m, n_2 - m^2, \dots, n_k - m^k)}{m^\lambda}.$$

The study of $l^p \to l^q$ estimates of the operator $I_{k,\lambda}$ and $J_{k,\lambda}$ was initiated by Stein and Wainger [15], where they obtained almost sharp bounds for $k = 2$

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and $1/2 < \lambda < 1$ by employing the circle method. Oberlin [11] obtained sharp results for $k = 2$ and $0 < \lambda < 1$ except for endpoints without using the circle method. The endpoint bounds were fully established in a series of papers by Stein and Wainger [15, 16], and Ionescu and Wainger [9], but only in the case $k = 2$. The case $k \geq 3$ seems to be substantially harder. Pierce [13] studied the operator $I_{k,\lambda}$ for $k \geq 3$ using tools from number theory, but it is open for $k \geq 3$ in the full p, q range. See [14] for a generalization of $J_{2,\lambda}$ to quadratic forms in $\mathbb{Z}^d$.

In this paper, we shall consider the operator J_λ^γ acting on (initially) compactly supported functions $f : \mathbb{Z}^d \to \mathbb{C}$ by

$$J_\lambda^\gamma(f)(n) = \sum_{m=1}^{\infty} \frac{f(n - \gamma(m))}{m^\lambda},$$

where $\gamma : \mathbb{N} \to \mathbb{Z}^d$ is an injection. Note that $I_{k,\lambda}$ and $J_{k,\lambda}$ are special cases of J_λ^γ.

It is known [15, Proposition (b)] that the operator J_λ^γ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if $\frac{1}{q} \leq \frac{1}{p} - (1 - \lambda)$ and $1 < p < q < \infty$ for any injection γ. This result is sharp if $\gamma(m) = m$, but not in general. We are interested in obtaining a sharper estimate which is sensitive to γ.

The previous work described above reveals a close connection between $l^p \to l^q$ bounds and the number of solutions of Diophantine equations related to the curves at least in two different ways. One approach [13, 15, 16] is via the Fourier multipliers associated with the operators and Parseval's identity. Another approach [11] is via the combinatorial argument by Christ [4] which originated from the study of an averaging operator along a curve.

The Diophantine equation considered in [13] for the case $\gamma(m) = m^k$ was

$$x_1^k + \cdots + x_s^k = x_{s+1}^k + \cdots + x_{2s}^k \tag{1.1}$$

for $x_i \in [1, P] \cap \mathbb{Z},\ s, P \in \mathbb{N}$. The number of solutions of (1.1) is known as the mean values of Weyl sums, and has applications to Waring's problem.

Following the combinatorial approach [11], for general γ, we shall relate $l^p \to l^q$ bounds of J_λ^γ to the Diophantine system

$$\gamma(x_1) + \cdots + \gamma(x_s) = \gamma(x_{s+1}) + \cdots + \gamma(x_{2s}).$$

Moreover, we shall show that it is desirable to obtain estimates on the number of solutions of Diophantine systems with odd-number of unknowns, which turns out to extend the allowable λ range. See Section 2.1 and 4.

Let us introduce a notation. For fixed $h \in \mathbb{Z}^d$, $r, P \in \mathbb{N}$, let $N_r^\gamma(P;h)$ be the number of r-tuples $(x_1, \dots, x_r)$ of positive integers $x_i \le P$ satisfying the equation

$$\sum_{i=1}^{r} (-1)^{i+1} \gamma(x_i) = h. \tag{1.2}$$

Note that there is a trivial estimate $N_r^\gamma(P;h) \ll P^r$, where $\ll$ denotes the Vinogradov's notation. The following theorem provides $l^p \to l^q$ estimates for J_λ^γ given a non-trivial estimate on $N_r^\gamma(P;h)$. In what follows, we allow implied constants to depend on γ, r, and ϵ. Throughout the paper, we assume $1 \le p, q \le \infty$.

Theorem 1.1. *Suppose that for fixed $r \in \mathbb{N}$ and $\delta > 0$, we have $N_r^\gamma(P;h) \ll P^{r-\delta+\epsilon}$ for each $\epsilon > 0$ uniformly in $h \in \mathbb{Z}^d$. Let $s \in \mathbb{N}$ be the number such that we have either $r = 2s$ or $r = 2s - 1$. Then J_λ^γ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if $1 - \frac{\delta}{r} < \lambda < 1$ and p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1-\lambda}{\delta}$ *and*

(ii) $\frac{1}{p} > \frac{s}{\delta}(1-\lambda)$, $\frac{1}{q} < 1 - \frac{s}{\delta}(1-\lambda)$.

Several remarks are in order.

1) When $r = 2s$, it is enough to assume that $N_{2s}^\gamma(P;0) \ll P^{2s-\delta+\epsilon}$ since $N_{2s}^\gamma(P;h) \le N_{2s}^\gamma(P;0)$ for all $h \in \mathbb{Z}^d$. This can be seen by writing
$$N_{2s}^\gamma(P;h) = \int_{[0,1]^d} |S^\gamma(\alpha)|^{2s} e(-h \cdot \alpha) d\alpha,$$
where $e(t) \equiv e^{2\pi i t}$ and $S^\gamma(\alpha) = \sum_{m=1}^{P} e(\gamma(m) \cdot \alpha)$.

2) Given the stronger estimate $N_r^\gamma(P;h) \ll P^{r-\delta}$, it is possible to replace $<$ in (i) by $\le$ with a slight modification of the proof of Theorem 1.1. This can be done by applying an abstract analogue ([2, Section 6.2]) of an interpolation argument of Bourgain [1], but we shall not pursue it here.

3) If one considers operators with summation over $m \in \mathbb{Z} \setminus 0$, then the condition $x_i \in [1, P] \cap \mathbb{Z}$ in (1.2) changes to $x_i \in [-P, P] \cap \mathbb{Z}$. See Section 4 for an analogous statement in higher dimensions.

Before we turn to the proof of Theorem 1.1, we shall give applications for the case $\gamma^a(m) = (m^{a_1}, m^{a_2}, \dots, m^{a_d})$ by using Vinogradov's mean value theorem in Section 2. We prove Theorem 1.1 and a sharp (up to endpoints) $l^p \to l^q$ bound for the discrete fractional integral along the hyperbolic paraboloid in $\mathbb{Z}^3$ in Section 3 and Section 4, respectively. In Appendix 5.3, we generalize the Fourier multiplier approach [13] for operators J^a_λ considered in Section 2, giving an alternative proof of Theorem 2.1.

2. The operator J^a_λ

We study the operator $J^a_\lambda \equiv J^{\gamma^a}_\lambda$, where $\gamma^a(m) = (m^{a_1}, \dots, m^{a_d})$ for a d-tuple of strictly increasing natural numbers $a = (a_1, \dots, a_d)$. We define $\|a\| = a_1 + \cdots + a_d$.

Conjecture 1. *Let $0 < \lambda < 1$. J^a_λ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if and only if p and q satisfy*

(i) $\frac{1}{q} \leq \frac{1}{p} - \frac{1}{\|a\|}(1-\lambda)$ *and*

(ii) $\frac{1}{p} > 1 - \lambda, \frac{1}{q} < \lambda$.

See Appendix 5.1 for the necessity of conditions (i) and (ii). One is mainly interested in proving Conjecture 1 for the range of $\frac{\|a\|-1}{2\|a\|-1} < \lambda < 1$, since then one may get the full result by interpolating the result with the trivial $l^1(\mathbb{Z}^d) \to l^\infty(\mathbb{Z}^d)$ bound for $\Re(\lambda) \geq 0$.

In view of Theorem 1.1, the operator J^a_λ is related to the quantity $N^{\gamma^a}_r(P; h)$. For even $r = 2s$, let us denote $N^{\gamma^a}_{2s}(P; 0)$ by $\mathbf{J}^a_s(P)$, i.e. the number of solutions of the Diophantine system

$$x_1^{a_j} + \cdots + x_s^{a_j} = x_{s+1}^{a_j} + \cdots + x_{2s}^{a_j}$$

for $1 \leq j \leq d$ with $x_i \in [1, P] \cap \mathbb{Z}$. By a standard argument (see [6] or [18]), one may show that there is a lower bound

$$P^s + P^{2s-\|a\|} \ll \mathbf{J}^a_s(P). \tag{2.1}$$

It is natural to ask if $P^s + P^{2s-\|a\|}$ is the true order of $\mathbf{J}^a_s(P)$. This question for the case $a = a_k \equiv (1, 2, \dots, k)$ has been of great interest. Non-trivial upper bounds for $\mathbf{J}_{s,k}(P) \equiv \mathbf{J}^{a_k}_s(P)$ are collectively known as Vinogradov's mean value theorem. Recent work by Wooley [19, 20] and Ford and Wooley [7] report the following substantial progress on Vinogradov's mean value theorem.

Theorem A. *Suppose that s and $k \geq 3$ are natural numbers and that $1 \leq s \leq \frac{(k+1)^2}{4}$ or $s \geq k^2 - 1$. Then for each $\epsilon > 0$, one has*

$$\mathbf{J}_{s,k}(P) \ll P^{\epsilon}(P^s + P^{2s-\frac{k(k+1)}{2}}). \tag{2.2}$$

Thus, Theorem A answers the question up to ϵ if s is sufficiently larger or smaller than $\|a_k\| = \frac{k(k+1)}{2}$. Let $\tilde{V}(k)$ be the least number $s \geq \|a_k\|$ for which (2.2) holds. A crude standard estimate (see Appendix 5.2) gives

$$\mathbf{J}^a_s(P) \ll P^{2s-\|a\|+\epsilon} \tag{2.3}$$

for $s \geq V(a_d)$. However, we expect that (2.3) holds for a larger range of s in view of the lower bound (2.1). We denote by $V(a)$ the least number s for which (2.3) holds. With $r = 2s = 2V(a)$ and $\delta = \|a\|$, Theorem 1.1 implies the following:

Theorem 2.1. *Let $a_d \geq 3$ and $\lambda_a = 1 - \frac{\|a\|}{2V(a)}$. Then J^a_λ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if $\lambda_a < \lambda < 1$ and p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{\|a\|}(1-\lambda)$ *and*

(ii) $\frac{1}{p} > \frac{V(a)}{\|a\|}(1-\lambda), \frac{1}{q} < 1 - \frac{V(a)}{\|a\|}(1-\lambda)$.

If one has $V(a) = \|a\|$, then Theorem 2.1 would imply the nearly sharp result toward Conjecture 1 for $\frac{1}{2} < \lambda < 1$. In order to extend this to the full range of $0 < \lambda < 1$, we need the stronger estimate

$$N^{\gamma_a}_{2\|a\|-1}(P; h) \ll P^{\|a\|-1+\epsilon} \tag{2.4}$$

uniformly in $h \in \mathbb{Z}^d$. Indeed, Oberlin's result for $J_{2,\lambda}$ is based on the estimate (2.4) which is valid for $a = (1, 2)$. See [11] for details.

We record the results for the special case $J_{k,\lambda} = J^{a_k}_\lambda$, where $a_k = (1, 2, \ldots, k)$ from Theorem A. Theorem 1.1 with $r = 2s = 2(k^2-1)$ and $\delta = \frac{k(k+1)}{2}$ gives

Corollary 2.2. *Let $k \geq 3$ and $\lambda_k = 1 - \frac{k}{4(k-1)}$. Then $J_{k,\lambda}$ extends to a bounded operator from $l^p(\mathbb{Z}^k)$ to $l^q(\mathbb{Z}^k)$ if $\lambda_k < \lambda < 1$ and p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{2}{k(k+1)}(1-\lambda)$ *and*

(ii) $\frac{1}{p} > \frac{2(k-1)}{k}(1-\lambda), \frac{1}{q} < 1 - \frac{2(k-1)}{k}(1-\lambda)$.

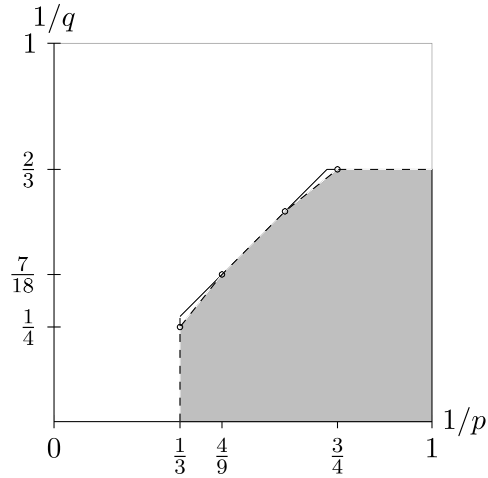

Figure 1: The known region of boundedness of $J_{3,\frac{2}{3}}$

Theorem 1.1 with $r = 2s = 2\delta = 2\lfloor \frac{(k+1)^2}{4} \rfloor$ implies

Corollary 2.3. *Let $k \geq 3$ and $\frac{1}{2} < \lambda < 1$. Then $J_{k,\lambda}$ extends to a bounded operator from $l^p(\mathbb{Z}^k)$ to $l^q(\mathbb{Z}^k)$ if p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{\lfloor \frac{(k+1)^2}{4} \rfloor}(1-\lambda)$ *and*

(ii) $\frac{1}{p} > 1-\lambda, \frac{1}{q} < \lambda$.

Corollary 2.3 complements Corollary 2.2 in the sense that it allows a wider range of applicable λ and that the condition (ii) is optimal at the expense of relaxing condition (i).

Figure 1 illustrates Corollaries 2.2 and 2.3 for the case $k = 3$ and $\lambda = \frac{2}{3}$. It shows the known (shaded) and the conjectured range of exponents $(\frac{1}{p}, \frac{1}{q}) \in [0,1]^2$ where $J_{3,\frac{2}{3}}$ is bounded from $l^p(\mathbb{Z}^3)$ to $l^q(\mathbb{Z}^3)$. Corollaries 2.2 and 2.3 give the vertices $(\frac{4}{9}, \frac{7}{18})$ and $(\frac{1}{3}, \frac{1}{4})$, respectively.

2.1. Relation to Waring's problem

The purpose of this section is twofold; to give a slight improvement to one of the results on the operator $I_{k,\lambda}$ in [13] and to give a correction to an interpolation lemma in [13]. We thank Pierce for encouraging us to clarify the issue here.

We denote by $r_{s,k}(l)$ the number of representations of l as a sum of s positive k-th powers. Then there is an estimate

$$r_{s,k}(l) \ll l^{s/k-1+\epsilon} \tag{2.5}$$

for every sufficiently large s with respect to k, see [18]. Let $\tilde{G}(k)$ to be the least natural number s for which the estimate (2.5) holds. Since the work by Hardy and Littlewood that $\tilde{G}(k) \le (k-2)2^{k-1} + 5$, numerous improvements have been achieved. We refer the reader to [7] and references therein.

In [13], (2.5) was applied to obtain estimates on the mean values of the Weyl-sums. Instead, we use it to obtain estimates on $N^{\gamma}_{2s-1}(P;h)$, which is the number of the solutions of the Diophantine equation of $2s-1$ variables

$$x_1^k + \cdots + x_s^k = x_{s+1}^k + \cdots + x_{2s-1}^k + h \tag{2.6}$$

for $x_i \in [1,P] \cap \mathbb{Z}$ and $h \in \mathbb{Z}$. Indeed, one has

$$N^{\gamma}_{2s-1}(P;h) \ll P^{2s-1-k+\epsilon} \tag{2.7}$$

for $s \ge \tilde{G}(k)$ uniformly in $h \in \mathbb{Z}$.

For the convenience of the reader, we record the argument here. One first considers the number of solutions $(x_1, \dots, x_s)$ for each fixed $(x_{s+1}, \dots, x_{2s-1})$. It is $O(P^{s-k+\epsilon})$ uniformly in h by (2.5) since we may assume that the right hand side of (2.6) is $O(P^k)$. We get (2.7) since there are P^{s-1} many choices for $(x_{s+1}, \dots, x_{2s-1})$.

The estimate (2.7) and Theorem 1.1 with $r = 2\tilde{G}(k) - 1$ give the following which slightly improves the allowable range of λ in [13].

Theorem 2.4. *Let $\lambda_k = 1 - \frac{k}{2\tilde{G}(k)-1}$. Then $I_{k,\lambda}$ extends to a bounded operator from $l^p(\mathbb{Z})$ to $l^q(\mathbb{Z})$ if $\lambda_k < \lambda < 1$ and p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{k}(1-\lambda)$ *and*

(ii) $\frac{1}{p} > \frac{\tilde{G}(k)}{k}(1-\lambda), \frac{1}{q} < 1 - \frac{\tilde{G}(k)}{k}(1-\lambda)$.

We note that the optimal condition (ii-c) $1/q < \lambda, 1/p > 1-\lambda$ in the statement of [13, Theorem 4] should be replaced by the weaker condition (ii) in Theorem 2.4. This is due to an error in the interpolation lemma [13, Lemma 2]. The condition (ii) $1/q < \lambda, 1/p > 1-\lambda$ in the lemma should be corrected to a weaker condition (ii) $1/p > \frac{1-\lambda}{2(1-\eta)}, 1/q < 1 - \frac{1-\lambda}{2(1-\eta)}$.

3. Proof of Theorem 1.1

The theorem can be reduced to certain restricted weak type estimates. We decompose the operator J_λ^γ dyadically. Define $J_{\lambda,j}^\gamma$ by

$$J_{\lambda,j}^\gamma(f)(n) = 2^{-\lambda j} \sum_m f(n-\gamma(m))$$

for $j \geq 0$ where $\sum_m$ denotes $\sum_{2^j \leq m < 2^{j+1}}$.

Oberlin [11] used the following lemma contained in the proof of Lemma 1 of [4].

Lemma B (Christ [4]). *Suppose that T is an operator taking characteristic functions χ_E onto measurable functions with $T\chi_E \geq 0$ for any measurable set E. Given $\alpha > 0$ and E with $0 < |E| < \infty$, take $F = \{x : \alpha < T\chi_E(x) < 2\alpha\}$ and $\beta = |E|^{-1}\langle \chi_F, T\chi_E\rangle$. For $k = 0, 1, \dots$ there are positive constants δ_k and ϵ_k (depend only on k) such that the sets E_k and F_k defined by $E_0 = E$, $F_0 = F$,*

$$\begin{aligned} E_{k+1} &= \{x \in E_k : T^*\chi_{F_k}(x) \geq \delta_k \beta\}, \\ F_{k+1} &= \{y \in F_k : T\chi_{E_{k+1}}(y) \geq \epsilon_k \alpha\}, \end{aligned}$$

are nonempty provided that $|F| > 0$.

Let $T = J_{\lambda,j}^\gamma, \alpha > 0, E \subset \mathbb{Z}^d$ and $F \equiv F^j, \beta \equiv \beta_j, E_k \equiv E_k^j, F_k \equiv F_k^j$ be as in Lemma B. We may assume that $|F| > 0$.

Note that $n \in E_k$ implies

$$\sum_m \chi_{F_{k-1}}(n+\gamma(m)) \geq 2^{\lambda j}\delta_{k-1}\beta \tag{3.1}$$

and $n \in F_k$ implies

$$\sum_m \chi_{E_k}(n-\gamma(m)) \geq 2^{\lambda j}\epsilon_{k-1}\alpha \tag{3.2}$$

for $k \geq 1$.

When $r = 2s-1$, we define the sum S_r by

$$S_r = \sum_{m_1}\sum_{m_2}\cdots\sum_{m_r} \chi_E\left(n + \sum_{i=1}^r (-1)^i \gamma(m_i)\right)$$

for a fixed $n \in F_{s-1}$.

Since $n \in F_{s-1}$, there are at least $2^{\lambda j}\epsilon_{s-2}\alpha$ values of $m_1 \in [2^j, 2^{j+1})$ such that $n - \gamma(m_1) \in E_{s-1}$ by (3.2). For each of these m_1, there are at least $2^{\lambda j}\delta_{s-2}\beta$ values of $m_2 \in [2^j, 2^{j+1})$ such that $n - \gamma(m_1) + \gamma(m_2) \in F_{s-2}$ by (3.1). Continuing in this manner, we get the following lower bound of S_r:

$$S_r \gtrsim 2^{(2s-1)\lambda j}\alpha^s\beta^{s-1} \gtrsim 2^{r\lambda j}\alpha^r \frac{|F|^{s-1}}{|E|^{s-1}} \tag{3.3}$$

since $\beta \geq \alpha\frac{|F|}{|E|}$ and $r = 2s-1$.

Next, we get an upper bound for S_r. Let $E' = n - E$.

$$\begin{aligned} S_r &= \sum_{m_1}\sum_{m_2}\cdots\sum_{m_r}\chi_{E'}\left(\sum_{i=1}^{r}(-1)^{i+1}\gamma(m_i)\right) \\ &= \sum_{l\in E'}\sum_{\substack{\sum_{i=1}^{r}(-1)^{i+1}\gamma(m_i)=l \\ 2^j\leq m_i<2^{j+1}}} 1 \leq \sum_{l\in E'} N_r^{\gamma}(2^{j+1};l) \ll 2^{(r-\delta+\epsilon)j}|E|, \end{aligned} \tag{3.4}$$

where we fix $\epsilon > 0$ so that $\lambda = \frac{r-\delta+2\epsilon}{r}$. (3.3) and (3.4) give

$$\alpha^r|F|^{s-1} \ll 2^{-\epsilon j}|E|^s,$$

or equivalently,

$$\alpha^r|\{J_{\lambda,j}^{\gamma}(\chi_E)(n) > \alpha\}|^{s-1} \ll 2^{-\epsilon j}|E|^s. \tag{3.5}$$

Since $J_\lambda^\gamma(\chi_E)(n) \leq \sum_{j=0}^{\infty} J_{\lambda,j}^{\gamma}(\chi_E)(n)$, (3.5) implies

$$\alpha^r|\{J_\lambda^\gamma(\chi_E)(n) > \alpha\}|^{s-1} \ll |E|^s,$$

which is equivalent to the restricted weak-type $(\frac{2s-1}{s}, \frac{2s-1}{s-1})$ estimate for J_λ^γ.

When $r = 2s$, we take the sum S_r by

$$S_r = \sum_{m_1}\sum_{m_2}\cdots\sum_{m_r}\chi_E\left(n + \sum_{i=1}^{r}(-1)^{i+1}\gamma(m_i)\right)$$

for a fixed $n \in E_s$. After a few similar estimates, we get the restricted weak-type $(\frac{2s}{s+1}, 2)$ estimate for J_λ^γ.

Moreover, the same estimates are valid for a complex-valued λ as long as $\Re(\lambda) > 1 - \frac{\delta}{r}$. For each fixed such λ, we first obtain strong type estimates with bounds uniform in $\Im(\lambda)$ by real interpolation with the $l^1 \to l^\infty$ bound for $\Re(\lambda) \geq 0$. Next, we apply analytic interpolation with the $l^\infty \to l^\infty$ bound for $\Re(\lambda) > 1$. Finally, inclusion property of l^p spaces completes the proof. We refer the reader to [17, Chapter V] for the interpolation theorems used here.

4. Discrete fractional integral along the hyperbolic paraboloid in $\mathbb{Z}^3$

The discrete fractional integrals, where the summation is taken along positive definite quadratic forms in several variables, have been studied by Pierce [14] via studying the Fourier multiplier of the operator. Motivated by [14], we study the discrete fractional integral along the hyperbolic paraboloid in $\mathbb{Z}^3$, defined by

$$\mathcal{P}_\lambda f(n_1, n_2, n_3) = \sum_{m \in \mathbb{Z}^2 \setminus 0} \frac{f(n_1 - m_1, n_2 - m_2, n_3 - (m_1^2 - m_2^2))}{|m|^{2\lambda}},$$

acting on (initially) compactly supported functions $f : \mathbb{Z}^3 \to \mathbb{C}$.

Theorem 4.1. *Let $0 < \lambda < 1$. Then $\mathcal{P}_\lambda$ extends to a bounded operator from $l^p(\mathbb{Z}^3)$ to $l^q(\mathbb{Z}^3)$ if p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{2}(1 - \lambda)$ *and*

(ii) $\frac{1}{p} > 1 - \lambda, \frac{1}{q} < \lambda$.

This result is sharp up to endpoint. One can show the necessity of the condition $\frac{1}{q} \leq \frac{1}{p} - \frac{1}{2}(1 - \lambda)$ by taking $f(n) = |(n_1, n_2)|^{-\alpha} |n_3|^{-\beta}$ for $n_j \geq 1$ and $f(n) = 0$ otherwise, for some appropriate $\alpha, \beta > 0$. The necessity of condition (ii) can be shown as in Appendix 5.1.

To treat the operator $\mathcal{P}_\lambda$, we consider a variant of J_λ^γ defined by

$$\mathcal{J}_\lambda^\gamma(f)(n) = \sum_{m \in \mathbb{Z}^{d_0} \setminus 0} \frac{f(n - \gamma(m))}{|m|^{d_0 \lambda}}$$

acting on (initially) compactly supported functions $f : \mathbb{Z}^d \to \mathbb{C}$, where $\gamma : \mathbb{Z}^{d_0} \to \mathbb{Z}^d$ is an injection.

For fixed $h \in \mathbb{Z}^d$, $r, P \in \mathbb{N}$, let $\mathcal{N}_r^\gamma(P; h)$ denote be the number of solutions of the Diophantine system

$$\sum\nolimits_{i=1}^{r} (-1)^{i+1} \gamma(m_i) = h$$

for $m_i \in B_P$, where $B_P = \{x \in \mathbb{Z}^{d_0} : |x| \leq P\}$ and $|x|^2 = x_1^2 + \cdots + x_{d_0}^2$.

We have the following variant of Theorem 1.1.

Theorem 4.2. *Suppose that for each $\epsilon > 0$ we have $\mathcal{N}_r^\gamma(P;h) \ll P^{d_0(r-\delta)+\epsilon}$ for a fixed $r \in \mathbb{N}$ and $\delta > 0$ uniformly in $h \in \mathbb{Z}^d$. Let $s \in \mathbb{N}$ be the number such that we have either $r = 2s$ or $r = 2s-1$. Then $\mathcal{J}_\lambda^\gamma$ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if $1 - \frac{\delta}{r} < \lambda < 1$ and p, q satisfy*

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1-\lambda}{\delta}$ *and*

(ii) $\frac{1}{p} > \frac{s}{\delta}(1-\lambda)$, $\frac{1}{q} < 1 - \frac{s}{\delta}(1-\lambda)$.

The proof of Theorem 4.2 is a straightforward modification of the proof of Theorem 1.1 and will be omitted.

Proof of Theorem 4.1. By Theorem 4.2, it is enough to show that

$$\mathcal{N}_3^\gamma(P;h) \ll P^{2+\epsilon} \tag{4.1}$$

uniformly in $h \in \mathbb{Z}^3$.

Here, $d_0 = 2$, $m = (m_1, m_2) \in \mathbb{Z}^2$ and $\gamma(m) = (m, \tau(m))$, where $\tau(m) = m_1^2 - m_2^2$. Recall that $\mathcal{N}_3^\gamma(P;h)$ is the number of solutions of the Diophantine system

$$\begin{aligned} x + y &= z + v \\ \tau(x) + \tau(y) &= \tau(z) + t \end{aligned}$$

for $x, y, z \in B_P$ and $h = (v, t) \in \mathbb{Z}^{2+1}$.

Since there are $O(P^2)$ many z in B_P, it is enough to show that the number of solutions of

$$\begin{aligned} x + y &= v' \\ \tau(x) + \tau(y) &= t' \end{aligned} \tag{4.2}$$

for $x, y \in B_P$ is $O(P^\epsilon)$ uniformly in $h' = (v', t') \in \mathbb{Z}^{2+1}$.

Suppose that (x, y) is a solution of (4.2). Then $|v'| \le 2P$ and $|t'| \le 2P^2$. We make a change of variables $X = 2x - v'$ and $Y = 2y - v'$. Then (X, Y) is a solution of the Diophantine system

$$\begin{aligned} X + Y &= 0 \\ \tau(v' + X) + \tau(v' + Y) &= 4t', \end{aligned}$$

which is equivalent to the Diophantine equation

$$(X_1 + X_2)(X_1 - X_2) = N := 2t' - ({v'_1}^2 - {v'_2}^2), \tag{4.3}$$

where $X = (X_1, X_2)$ and $v' = (v'_1, v'_2)$.

Therefore, the number of solutions (X_1, X_2) of (4.3) is $O(d(N))$, where $d(N)$ is the number of divisors of N. The fact that $d(N) = O(|N|^\epsilon)$ (see Chapter 18 of [8]) and $|N| = O(P^2)$ implies that the number of solutions (X_1, X_2) of (4.3) is $O(P^\epsilon)$ for any $\epsilon > 0$, which in turn implies that the number of solutions of (4.2) is $O(P^\epsilon)$.

Theorem 4.2 with $r = 3$ and $\delta = 2$ implies Theorem 4.1 for $\frac{1}{3} < \lambda < 1$. Interpolating the result with the trivial $l^1(\mathbb{Z}^3) \to l^\infty(\mathbb{Z}^3)$ bound for $\Re(\lambda) \geq 0$ finishes the proof. $\square$

5. Appendix

5.1. Necessity of conditions in Conjecture 1.

For the necessity of the second condition, we take the example in [15]; $f(0) = 1$, $f(n) = 0$ for $n \neq 0$. Then $f \in l^p(\mathbb{Z}^d)$ for all p, and

$$J_\lambda^a(f)(n) = \begin{cases} m^{-\lambda} & \text{if } n = \gamma^a(m) \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\|J_\lambda^a(f)\|_{l^q(\mathbb{Z}^d)}^q = \sum_{m \geq 1} m^{-\lambda q},$$

where the sum converges only if $1/q < \lambda$. Duality gives $1/p > 1 - \lambda$.

For the necessity of the first condition, we take $f : \mathbb{Z}^d \to \mathbb{C}$ by

$$f(n) = \begin{cases} \prod_{j=1}^d |n_j|^{-\alpha} & \text{if } n_j \neq 0 \text{ for all } 1 \leq j \leq d \\ 0 & \text{otherwise} \end{cases}$$

for a fixed constant $\alpha > 1/p$ so that $f \in l^p(\mathbb{Z}^d)$.

For $n_1 > 1$ and $n_j \geq n_1^{a_j/a_1}$ for $2 \leq j \leq d$,

$$J_\lambda^a(f)(n) \geq \sum_{1 \leq m^{a_1} < n_1} m^{-\lambda} \prod_{j=1}^d (n_j - m^{a_j})^{-\alpha} \gtrsim n_1^{(1-\lambda)/a_1} \prod_{j=1}^d n_j^{-\alpha}.$$

Thus,

$$\begin{aligned}\|J_\lambda^a(f)\|_{l^q(\mathbb{Z}^d)}^q &\gtrsim \sum_{n_1>1} n_1^{q(1-\lambda)/a_1-q\alpha} \sum_{\substack{n_1^{a_j}\le n_j^{a_1}\le 2n_1^{a_j}\\ 2\le j\le d}} \prod_{j=2}^d n_j^{-q\alpha}\\ &\gtrsim \sum_{n_1>1} n_1^{q(1-\lambda)/a_1-q\alpha} \prod_{j=2}^d n_1^{a_j(1-q\alpha)/a_1}\\ &\gtrsim \sum_{n_1>1} n_1^{q(1-\lambda)/a_1+\|a\|(1-q\alpha)/a_1-1},\end{aligned}$$

where the last sum converges only if $\frac{1}{q} < \alpha - \frac{1}{\|a\|}(1-\lambda)$. We get the first necessary condition since we may decrease α to $1/p$ as close as we want.

5.2. Proof of (2.3)

Let $a = (a_1, \ldots, a_d)$ be given. Let $l = a_d - d$ and $\{b_i\}_{i=1}^l$ be the increasing sequence of natural numbers such that $\{b_1, \ldots, b_l\} = ([1, a_d] \cap \mathbb{Z}) \setminus \{a_1, \ldots, a_d\}$. Considering the underlying Diophantine systems, we have

$$\begin{aligned}\text{(5.1)}\qquad \mathbf{J}_s^a(P) &= \sum_{|h_{b_1}|\le sP^{b_1}} \cdots \sum_{|h_{b_l}|\le sP^{b_l}} \int_{[0,1]^{a_d}} |S(\alpha)|^{2s} \prod_{i=1}^l e(-h_{b_i}\alpha_{b_i})d\alpha\\ &\ll P^{\sum_{i=1}^l b_i}\mathbf{J}_{s,a_d}(P) = P^{\frac{a_d(a_d+1)}{2}-\|a\|}\mathbf{J}_{s,a_d}(P),\end{aligned}$$

where $S(\alpha) = \sum_{m=1}^P e(\alpha_1 m + \alpha_2 m^2 + \cdots + \alpha_{a_d} m^{a_d})$.

Combining (2.2) and (5.1), we have

$$\text{(5.2)}\qquad \mathbf{J}_s^a(P) \ll P^{2s-\|a\|+\epsilon}$$

for $s \ge \tilde{V}(a_d)$.

5.3. Fourier multiplier approach for J_λ^a

Let $\hat{f}(\alpha) = \sum_{n\in\mathbb{Z}^d} f(n)e(-n\cdot\alpha)$ be the Fourier transform of $f \in l^1(\mathbb{Z}^d)$, where $e(t) \equiv e^{2\pi it}$. We shall study the Fourier multiplier m_λ^a of the operator J_λ^a

$$m_\lambda^a(\alpha) = \sum_{n=1}^\infty \frac{e(-\gamma^a(n)\cdot\alpha)}{n^\lambda}$$

for $\alpha \in [0,1]^d$, given by the relation $\widehat{J_\lambda^a(f)}(\alpha) = m_\lambda^a(\alpha)\hat{f}(\alpha)$.

By closely following the argument in [13, 15], we generalize the Weyl sum approach on the Fourier multipliers (Proposition 5 of [13]) as follows:

Proposition 5.1. *Let $s \in \mathbb{N}$, $0 < \delta \leq 2s$, and $\lambda \in \mathbb{C}$. $\mathbf{J}_s^a(P) \ll P^{2s-\delta+\epsilon}$ for each $\epsilon > 0$ if and only if $m_\lambda^a \in L^{2s}([0,1]^d)$ for all $\Re(\lambda) > 1 - \frac{\delta}{2s}$.*

The "folk" lemma (Lemma 2 of [15]) implies the following result.

Corollary 5.2. *Suppose that one has $\mathbf{J}_s^a(P) \ll P^{2s-\delta+\epsilon}$ for some $0 < \delta \leq 2s$. Then the operator J_λ^a extends to a bounded operator from $l^{\frac{2s}{s+1}}(\mathbb{Z}^d)$ to $l^2(\mathbb{Z}^d)$ for $\Re(\lambda) > 1 - \frac{\delta}{2s}$.*

Note that Theorem 2.1 follows from Corollary 5.2 and a complex interpolation.

Proof of Proposition 5.1. For $l \in \mathbb{Z}^d$, let $r_s^a(l)$ be the number of solutions of the Diophantine system

$$x_1^{a_j} + \cdots + x_s^{a_j} = l_j, \tag{5.3}$$

where $l = (l', l_d) = (l_1, \ldots, l_d)$ and $x_i \geq 1$ for $1 \leq j \leq d$. In addition, we define $r_s^a(l; P)$ to be the number of solutions of (5.3) for $1 \leq x_i \leq P$.

Let $\delta > 0$ be given. We observe that $\mathbf{J}_s^a(P) \ll P^{2s-\delta+\epsilon}$ is equivalent to

$$\sum_{l_d=1}^{P} R_s^a(l_d) \ll P^{(2s-\delta+\epsilon)/a_d}, \qquad \text{where } R_s^a(l_d) = \sum_{l' \in \mathbb{Z}^{d-1}} \left(r_s^a(l', l_d)\right)^2. \tag{5.4}$$

This follows from Parseval's identity applied to $\mathbf{J}_s^a(P) = \int_{[0,1]^d} |S^{\gamma^a}(\alpha)|^{2s} d\alpha$, where $S^{\gamma^a}(\alpha) = \sum_{m=1}^{P} e(\gamma^a(m) \cdot \alpha)$. Indeed, one has

$$\mathbf{J}_s^a(P) = \sum_{l \in \mathbb{Z}^d} (r_s^a(l; P))^2 = \sum_{1 \leq l_d \leq sP^{a_d}} \sum_{l' \in \mathbb{Z}^{d-1}} (r_s^a(l', l_d))^2 = \sum_{1 \leq l_d \leq sP^{a_d}} R_s^a(l_d).$$

Therefore, it is enough to show that (5.4) is equivalent to

$$m_\lambda^a \in L^{2s}([0,1]^d), \qquad \text{for every } \Re(\lambda) > 1 - \frac{\delta}{2s}. \tag{5.5}$$

As in [13, 15], one has

$$m_\lambda^a(\alpha) = C_{a,\lambda} \int_0^1 S_y^a(\alpha) y^{\lambda/a_d} \frac{dy}{y} + O(1),$$

where $C_{a,\lambda} = \Gamma(\lambda/a_d)$ and $S_y^a(\alpha) = \sum_{n\geq 1} e^{-n^{a_d}y} e(-\gamma^a(n)\cdot\alpha)$ which is well-defined for each $y > 0$. This follows from the observation

$$\int_0^\infty e^{-n^{a_d}y} y^{\lambda/a_d} \frac{dy}{y} = n^{-\lambda}\Gamma(\lambda/a_d).$$

Thus,

$$\|m_\lambda^a\|_{L^{2s}([0,1]^d)} \leq C_{a,\lambda} \int_0^1 \|S_y^a\|_{L^{2s}([0,1]^d)} y^{\Re(\lambda)/a_d} \frac{dy}{y} + O(1). \tag{5.6}$$

By Parseval's identity and summation by parts, (5.4) implies

$$\|S_y^a\|_{L^{2s}([0,1]^d)}^{2s} = \sum_{l_d\geq 1} e^{-2l_d y} R_s^a(l_d) \ll y^{-(2s-\delta+\epsilon)/a_d}. \tag{5.7}$$

Thus we get (5.5) by (5.6) and (5.7).

For the converse, it is enough to assume (5.5) only for $\lambda \in \mathbb{R}$. Then $m_\lambda^a(\alpha)^s = \sum_{l\in\mathbb{Z}^d} a_l e(-l\cdot\alpha) \in L^2$, where

$$a_l = \sum_{n_1,\ldots,n_s\geq 1} \sum_{\gamma^a(n_1)+\cdots+\gamma^a(n_s)=l} \frac{1}{n_1^\lambda \cdots n_s^\lambda}.$$

Parseval's identity implies that $\sum_{l\in\mathbb{Z}^d} |a_l|^2$ is finite. Since $n_i^{a_d} \leq l_d$ in the sum, we have $a_l \geq r_s^a(l) l_d^{-s\lambda/a_d}$ if $l_d \geq 1$. Thus,

$$\infty > \sum_{l\in\mathbb{Z}^d} |a_l|^2 \geq \sum_{l_d\geq 1} \sum_{l'\in\mathbb{Z}^{d-1}} (r_s^a(l))^2 l_d^{-2s\lambda/a_d} = \sum_{l_d\geq 1} R_s^a(l_d) l_d^{-2s\lambda/a_d}$$

for every $\lambda > 1 - \frac{\delta}{2s}$, which is equivalent to (5.4). □

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References

[1] J. Bourgain, *Estimations de certaines fonctions maximales.* C. R. Acad. Sci. Paris Sér. I Math., **301** (1985), no. 10, 499–502.

[2] A. Carbery, A. Seeger, S. Wainger and J. Wright, *Classes of singular integral operators along variable lines.* J. Geom. Anal., **9** (1999), no. 4, 583–605.

[3] M. Christ, *Endpoint bounds for singular fractional integral operators.* Unpublished manuscript, 1988.

[4] M. Christ, *Convolution, curvature, and combinatorics: a case study.* Internat. Math. Res. Notices, (1998), no. 19, 1033–1048.

[5] H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities.* Cambridge Mathematical Library, Cambridge University Press, Cambridge, second edition (2005).

[6] K. B. Ford, *New estimates for mean values of Weyl sums.* Internat. Math. Res. Notices, (1995), no. 3, 155–171.

[7] K. B. Ford and T. D. Wooley, *On Vinogradov's mean value theorem: Strongly diagonal behaviour via efficient congruencing.* Acta Math., **213** (2014), no. 2, 199–236.

[8] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers.* Oxford University Press, Oxford, sixth edition (2008).

[9] A. D. Ionescu and S. Wainger, *L^p boundedness of discrete singular Radon transforms.* J. Amer. Math. Soc., **19** (2006), no. 2, 357–383.

[10] J. Kim, *On discrete fractional integral operators and related Diophantine equations.* Master's thesis, Pohang University of Science and Technology (2011).

[11] D. M. Oberlin, *Two discrete fractional integrals.* Math. Res. Lett., **8** (2001), no. 1-2, 1–6.

[12] L. B. Pierce, *Discrete analogues in harmonic analysis.* Ph.D. Thesis, Princeton University (2009).

[13] L. B. Pierce, *On discrete fractional integral operators and mean values of Weyl sums.* Bull. Lond. Math. Soc., **43** (2011), no. 3, 597–612.

[14] L. B. Pierce, *Discrete fractional Radon transforms and quadratic forms.* Duke Math. J., **161** (2012), no. 1, 69–106.

[15] E. M. Stein and S. Wainger, *Discrete analogues in harmonic analysis. II. Fractional integration.* J. Anal. Math., **80** (2000), 335–355.

[16] E. M. Stein and S. Wainger, *Two discrete fractional integral operators revisited.* J. Anal. Math., **87** (2002), 451–479.

[17] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J. (1971).

[18] R. C. Vaughan, *The Hardy-Littlewood method.* Cambridge Tracts in Mathematics, Vol. 125, Cambridge University Press, Cambridge, second edition (1997).

[19] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing.* Ann. of Math. (2), **175** (2012), no. 3, 1575–1627.

[20] T. D. Wooley, *Vinogradov's mean value theorem via efficient congruencing, II*, Duke Math. J., **162** (2013), no. 4, 673–730.

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