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Néron-Severi group preserving lifting of K3 surfaces and applications

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For complex algebraic K3 surfaces, there are criteria in terms of Néron-Severi lattice to be a Kummer surface and to be an Enriques K3 surface. In this paper, using a Néron-Severi group preserving lifting, we prove that the same criteria hold for K3 surfaces of finite height over a field of odd characteristic. We also classify K3 surfaces of Picard number 20 over a field of odd characteristic.

1. Introduction

When X is a complex algebraic K3 surface, the second integral singular cohomology group $H^2(X, \mathbb{Z})$ is an integral lattice that is isomorphic to $U^3 \oplus (-E_8)^2$. Here, U is a hyperbolic unimodular lattice of rank 2 and E_8 is a unimodular root lattice of rank 8. According to Hodge theory, we may consider the global 2-forms of X, $H^0(X, \Omega^2_{X/\mathbb{C}})$ as a direct factor of $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C}$. The Néron-Severi group of X, $NS(X)$ is identified with the intersection of the orthogonal complement of $H^0(X, \Omega^2_{X/\mathbb{C}})$ and $H^2(X, \mathbb{Z})$ inside $H^2(X, \mathbb{C})$. We call the rank of $NS(X)$ the Picard number of X. Thanks to the Torelli theorem for complex K3 surfaces ([19]), we can express many geometric properties of complex K3 surfaces in terms of the Néron-Severi lattice.

Let Y be a complex Enriques surface. The free part of the Néron-Severi group Y admits a lattice structure $\Gamma = U \oplus -E_8$. We say Γ is the Enriques lattice. We use $\Gamma(2)$ to denote the lattice obtained from Γ by multiplying the pairing by 2. Let $f : X \to Y$ be the K3 cover of Y and $\iota : X \to X$ be the associated involution. We say a K3 surface is an Enriques K3 surface if it is the K3 cover of an Enriques surface. A K3 surface is an Enriques K3 surface if and only if it has a fixed point free involution. The covering map induces an embedding of lattices

$$f^* : \Gamma(2) \hookrightarrow NS(X).$$

Then $f^*\Gamma(2)$ is the fixed part of $NS(X)$ for the involution ([13], Proposition 2.3):

$$\iota^* : NS(X) \to NS(X).$$

Let $\Gamma(2)^\perp$ be the orthogonal complement of $f^* : \Gamma(2) \hookrightarrow NS(X)$. Then,

$$\Gamma(2)^\perp = \{x \in NS(X) | \iota^*(x) = -x\}$$

and $\Gamma(2)^\perp$ is an even negative definite lattice. Assume the self intersection of a vector x in $\Gamma(2)^\perp$ is -2. By the Riemann-Roch theorem, either x or $-x$ is effective, but not both. However, x is effective if and only if $\iota^*(x) = -x$ is effective. Therefore $\Gamma(2)^\perp$ does not contain a vector of self intersection -2. Based on the Torelli theorem, the converse is also valid, and we have a lattice criterion for a complex K3 surface to be an Enriques K3 surface.

Theorem. ([13], Theorem 1.14, [6], Theorem 1) *A complex K3 surface X is an Enriques K3 surface if and only if there exists a primitive embedding of lattices $\Gamma(2) \hookrightarrow NS(X)$ such that the orthogonal complement of the embedding does not contain a vector of self intersection -2.*

Assume X is a complex Kummer surface associated with an abelian surface A. X is the blow up of 16 singular points of A_1-type of $A/(-id_A)$. Let $I = \{a_1, \ldots, a_{16}\}$ be the set of two torsion points of A. By the group structure of A, we may consider I as a four-dimensional vector space over $\mathbb{F}_2$, where $\mathbb{F}_2$ is a prime field of characteristic 2. Inside I, there are 30 hyperplanes and each hyperplane has 8 points. Let Q be the subset of the power set of I consisting of 30 hyperplanes, the empty set and I. For each $a_i \in I$, there is a smooth rational curve on X corresponding to a_i. We denote this smooth rational curve using a_i again. The classes of 16 rational curves a_i are linearly independent in $NS(X)$. Let M be the sublattice of $NS(X)$ generated by the 16 classes of a_i. For each $\alpha \in Q$, let $v_\alpha = \frac{1}{2}\sum_{a_i \in \alpha} a_i \in M \otimes \mathbb{Q}$. We set

$$J = M + \sum_{\alpha \in Q} \mathbb{Z} \cdot v_\alpha \subset M \otimes \mathbb{Q}.$$

J is the saturation of M in $NS(X)$ and $J/M = (\mathbb{Z}/2)^5$. We call J the Kummer lattice. The following lattice criterion is also based on the Torelli theorem.

Theorem. ([14], Theorem 3) *A complex K3 surface X is a Kummer surface if and only if there exists a primitive embedding of the abstract lattice J into $NS(X)$.*

It is also known that every complex Kummer surface is an Enriques K3 surface ([6], Theorem 2).

The Torelli theorem for complex K3 surfaces states that the isomorphism classes of algebraic complex K3 surfaces can be classified by the location of the line of global holomorphic 2 forms in the second complex singular cohomology. Especially for complex K3 surfaces of Picard number 20, we have a more precise classification in terms of the transcendental lattice. The transcendental lattice of a complex K3 surface X is the orthogonal complement of the embedding

$$NS(X) \hookrightarrow H^2(X, \mathbb{Z}).$$

We denote the transcendental lattice of X using $T(X)$. For a complex K3 surface of Picard rank 20, $T(X)$ is an even positive definite lattice of rank 2.

Theorem. ([4], Theorem 4) *The correspondence*

$$X \mapsto T(X)$$

gives a bijection between the set of isomorphism classes of complex K3 surfaces of Picard number 20 and the set of isomorphism classes of even positive definite lattices of rank 2. Every complex K3 surface of Picard number 20 has a model over a number field.

We cannot apply the Torelli theorem for K3 surfaces over a field of positive characteristic. Let k be an algebraically closed field of positive characteristic p. Assume X is a K3 surface over k. The formal Brauer group of X, $\widehat{Br}_X$ is a one-dimensional smooth formal group over k. The height of $\widehat{Br}_X$ is an integer between 1 and 10 or infinite. We define the height of X as the height of $\widehat{Br}_X$.

A K3 surface of infinite height is also called a supersingular K3 surface. If the base characteristic is odd, the Picard number of a supersingular K3 surface is 22 ([2], [9]). Furthermore, if the base characteristic is at least 5, a supersingular K3 surface is unirational ([8]). The discriminant group of the Néron-Severi group of a supersingular K3 surface X, $(NS(X)^*)/NS(X)$ is a σ-dimensional space over a prime field $\mathbb{Z}/p$ for $1 \leq \sigma \leq 10$. We call σ the Artin invariant of X. For supersingular K3 surfaces over a field of odd characteristic, we have the crystalline Torelli theorem ([18]). In a previous work ([5], Theorem 4.1), applying the crystalline Torelli theorem, to a supersingular K3 surface in odd characteristic p, we proved the lattice criterion to be an Enriques K3 surface. By this result and some lattice calculations, we also proved that when $p \geq 23$, a supersingular K3 surface is an Enriques K3 surface if and only if the Artin invariant is less than 6. It is known that a supersingular K3 surface over odd characteristic is a Kummer surface if and only if the Artin invariant is less than 3 ([17], Theorem 7.10). Therefore, if $p \geq 23$, every supersingular Kummer surface is an Enriques K3 surface. In this paper, we use a deformation argument to prove that over a field of odd characteristic, a supersingular K3 surface is an Enriques K3 surface if and only if the Artin invariant is less than 6. (Corollary 2.4)

For a K3 surface of finite height, we do not have a proper replacement of the Torelli theorem. However, over a field of odd characteristic, any K3 surface of finite height has a smooth lifting over the ring of Witt vectors of the base field to which all of the line bundles can extend.

Theorem. ([16], p. 505, [7], Corollary 4.2) *Let k be an algebraically closed field of odd characteristic and W be the ring of Witt vectors of k. Assume that X is a K3 surface of finite height defined over k. There exists a smooth lifting of X over W, $\mathbb{X}/W$ such that the reduction map* $\mathrm{Pic}(\mathbb{X}) \to NS(X)$ *is surjective.*

For such a lifting, the reduction map from the Néron-Severi group of the generic fiber to the Néron-Severi group of the special fiber, X, is an isomorphism. We say that a lifting of X that satisfies this condition is a Néron-Severi group preserving lifting of X. For a K3 surface of finite height, using a Néron-Severi group preserving lifting, we may import some of results on complex K3 surfaces expressed in terms of the Néron-Severi groups.

In this paper, we use a Néron-Severi group preserving lifting to prove the lattice criteria to be an Enriques K3 surface and to be a Kummer surface still hold for a K3 surface of finite height in odd characteristic. We also prove that a Kummer surface of finite height is an Enriques K3 surface. Therefore all of these results hold for all of the K3 surfaces over any field of characteristic $p \neq 2$.

Theorem 2.5. *Assume that k is an algebraically closed field of characteristic $p > 2$. A K3 surface X over k is an Enriques K3 surface if and only if there exists a primitive embedding $\Gamma(2) \hookrightarrow NS(X)$ such that the orthogonal complement of the embedding does not have a vector of self intersection -2.*

Theorem 2.6. *Assume that k is an algebraically closed field of characteristic $p > 2$. A K3 surface X over k is a Kummer surface if and only if there exists a primitive embedding of J into $NS(X)$.*

Theorem 2.8. *Assume that k is an algebraically closed field of characteristic $p > 2$. A Kummer surface X over k is an Enriques K3 surface.*

Over a field of odd characteristic, the Picard number of a K3 surface of finite height h ($1 \leq h \leq 10$) is at most $22 - 2h$ ([3], Proposition 5.12). Therefore, if the Picard number of a K3 surface X is 20, the height of X is 1; in other words, X is ordinary. Moreover, in this case, there is a unique Néron-Severi group preserving lifting of X, which is the canonical lifting of X ([15], Definition 1.9). By this fact and the classification of complex K3 surfaces of Picard number 20, we obtain a classification of K3 surfaces of Picard number 20 over a field of odd characteristic p. Let S_p be the set of isomorphic classes of even positive definite lattices of rank 2 such that the discriminant is a non-zero square modulo p. For each $M \in S_p$, there is a unique complex K3 surface of Picard number 20, X_M such that $T(X_M)$ is isomorphic to M. X_M is defined over $\bar{\mathbb{Q}}$. When k is an algebraically closed field of characteristic p, X_M has a good reduction over k. Then the reduction of X_M over k is a K3 surface of Picard number 20. In Section 3, using Néron-Severi group preserving lifting, we prove that this construction classifies K3 surfaces of Picard number 20 over k.

Theorem 3.7. *Let k be an algebraically closed field of characteristic $p > 2$. The isomorphism classes of the K3 surfaces of Picard number 20 are classified by S_p. Every K3 surface of Picard number 20 over k has a model over a finite field.*

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2. Kummer surfaces and Enriques surfaces

Let k be an algebraically closed field of characteristic $p > 2$. Let W be the ring of Witt vectors of k and K be the fraction field of W. Assume that Y is an Enriques surface over k and that $f : X \to Y$ is the K3 cover of Y. Because f is étale,

$$f^*T_{Y/k} = T_{X/k} = \Omega^1_{X/k} \text{ and } f_*\Omega^1_{X/k} = \Omega^1_{Y/k} \oplus T_{Y/k},$$

so

$$H^0(Y, T_{Y/k}) = H^2(T, T_{Y/k}) = 0 \text{ and } \dim H^1(Y, T_{Y/k}) = 10.$$

It follows that the deformation space of Y to k-Artin local algebras with residue field k is formally smooth of 10 dimension over k. Let $\mathcal{S} = \operatorname{Spf} k[[t_1, \dots, t_{10}]]$ be the deformation space. Suppose that (A, m) is an Artin local k-algebra with residue field $A/m \simeq k$ and that $Y_A \in \mathcal{S}(A)$ is a deformation of Y over A. Let $i : Y \hookrightarrow Y_A$ be the canonical embedding.

Lemma 2.1. *The canonical map* $i^* : \operatorname{Pic}(Y_A) \to \operatorname{Pic}(Y)$ *is an isomorphism.*

Proof. Let us consider the following exact sequence :

$$0 \to 1 + m\mathcal{O}_{Y_A} \to \mathbb{G}_{m,Y_A} \to \mathbb{G}_{m,Y} \to 0.$$

Since A is a finite algebra over k, $H^2(Y, 1 + m\mathcal{O}_{Y_A})$ is an iterative extension of finite copies of $H^2(Y, \mathcal{O}_Y)$, so $H^2(Y, 1 + m\mathcal{O}_Y) = 0$. By the same reason, $H^1(Y, 1 + m\mathcal{O}_Y) = 0$. Hence, $H^1(Y_A, \mathbb{G}_{m,Y_A}) \to H^1(Y, \mathbb{G}_{m,Y})$ is an isomorphism. □

Let $\mathcal{Y} \to \mathcal{S}$ be the universal family over the deformation space $\mathcal{S}$. According to Lemma 2.1, all of the line bundles of Y extend to $\mathcal{Y}$. In particular, an ample line bundle extends to $\mathcal{Y}$ and $\mathcal{Y} \to \mathcal{S}$ is algebraizable. Let $S = \operatorname{Spec} k[[t_1, \dots, t_{10}]]$ and $Y_S \to S$ be the algebraic model of the formal scheme $\mathcal{Y} \to \mathcal{S}$. If $f_S : X_S \to Y_S$ is the K3 cover of Y_S, then $\pi : X_S \to S$ is a family of Enriques K3 surfaces over S.

Lemma 2.2 (c.f. [12], p. 383). *If X is a supersingular K3 surface of Artin invariant 1 over k, X is an Enriques K3 surface.*

Proof. Let E be a supersingular elliptic curve over k. X is isomorphic to the Kummer surface of the abelian surface $E \times E$ ([17], Corollary 7.14). Assume that a is a non-zero 2-torsion point of E. An involution

$$E \times E \to E \times E, \ (x, y) \mapsto (-x + a, y + a)$$

commutes with $-id_{E\times E}$ and induces a fixed point free involution on X. Hence, X is an Enriques K3 surface. □

Assume that Y is an Enriques surface whose K3 cover X is a supersingular K3 surface of Artin invariant 1. For the family of Enriques K3 surfaces $\pi : X_S \to S$, let us consider a stratification on S :

$$S = M_1 \supset M_2 \supset \cdots \supset M_{10} \supset M_{11} = \Sigma_{10} \supset \Sigma_9 \supset \cdots \supset \Sigma_1. \tag{2.1}$$

Here, M_i is the locus of fibers of height at least i and Σ_i is the locus of supersingular fibers of Artin invariant at most i. Both M_i and Σ_i are closed subsets of S. For the sake of simplicity, we assume that they are all reduced. Because each step in the stratification is defined by one equation, the dimension decreases by one at most for each step ([1], p. 563). Because the central fiber of π is X, the closed point of S is contained in Σ_1. However, a K3 surface of Artin invariant 1 is unique up to isomorphism. Hence, Σ_1 is 0-dimensional and consists of one point. It follows that there are exactly 10 steps of dimension down on (2.1). If an Enriques K3 surface is of finite height, the height is 6 at most and if an Enriques K3 surface is supersingular, the Artin invariant is 5 at most ([5], Proposition 3.5). Therefore, $M_6 = M_{10}$ and $\Sigma_{10} = \Sigma_5$, and if we shorten the stratification 2.1 to

$$S = M_1 \supset M_2 \supset \cdots \supset M_6 \supset \Sigma_5 \supset \Sigma_4 \supset \cdots \supset \Sigma_1,$$

each step is of codimension 1.

Theorem 2.3. *For each h $(1 \leq h \leq 6)$, there exists an Enriques K3 surface of height h over k. For each σ $(1 \leq \sigma \leq 5)$, there exists a supersingular Enriques K3 surface of Artin invariant σ over k.*

Proof. When h is as above, we choose a point $x \in M_h - M_{h+1}$ or $x \in M_6 - \Sigma_5$ when $h = 6$. The fiber over x, $X_x = X_S \times_S k(x)$ is an Enriques K3 surface of height h over $k(x)$. Let $X_B \to \operatorname{Spec} B$ be an integral model of $X_x \to \operatorname{Spec} k(x)$ with the Enriques involution where B is an integral domain of finite type over k and there is an imbedding $B \hookrightarrow k(x)$ such that $X_B \otimes k(x)$ is isomorphic to X_x. Moreover we may assume that every fiber of $X_B \to \operatorname{Spec} B$ is an Enriques K3 surface of height h considering a stratification on $\operatorname{Spec} B$ as 2.1. Hence a closed fiber of $X_B \to \operatorname{Spec} B$ is an Enriques K3 surface of height h over k. In a similar way, we can construct a supersingular Enriques K3 surface of Artin invariant σ for $1 \leq \sigma \leq 5$. □

In a previous work ([5], Corollary 4.7), we proved that if the characteristic of the base field is $p \geq 23$, then a supersingular K3 surface has an Enriques involution if and only if the Artin invariant is less than 6. If one supersingular K3 surface of Artin invariant σ over k is an Enriques K3 surface, then every supersingular K3 surface of Artin invariant σ over k is supersingular ([5], Remark 4.4). By Theorem 2.3, we obtain the following result in any odd characteristic.

Corollary 2.4. *Let k be an algebraically closed field of odd characteristic. A supersingular K3 surface over k is an Enriques K3 surface if and only if the Artin invariant is less than 6.*

Now we prove that, for a K3 surface of finite height in odd characteristic, the lattice criteria to be an Enriques K3 surface and the lattice criterion to be a Kummer surface holds Recall from Section 1 that Γ is the Enriques lattice and J is the Kummer lattice.

Theorem 2.5. *Assume that k is an algebraically closed field of characteristic $p > 2$. A K3 surface X over k is an Enriques K3 surface if and only if there exists a primitive embedding $\Gamma(2) \hookrightarrow NS(X)$ such that the orthogonal complement of the embedding does not have a vector of self intersection -2.*

Proof. The "only if" part comes from the Riemann-Roch theorem. For the supersingular case, we refer to Theorem 4.1, [5]. Assume that X is of finite height and that there is an embedding $\Gamma(2) \hookrightarrow NS(X)$ satisfying the given condition. We fix a Néron-Severi group preserving lifting of X over W, $\pi : \mathbb{X} \to \operatorname{Spec} W$. Let X_K be the generic fiber of $\mathbb{X}/W$. Since $NS(X) = NS(X_K)$ and K is a field of characteristic 0, $X_K \otimes \bar{K}$ has an Enriques involution g. We assume that g is defined over a finite extension L of K and that V is the ring of integers of L. It follows that g extends to $\mathbb{X} \otimes V$ and that $\bar{g} = g|X$ is an involution on X ([11], p.672, Corollary 1). Because $g^*|H^0(X_K, \Omega^2_{X_K/K}) = -1$, $g^*|\pi_*\Omega^2_{\mathbb{X}\otimes V/V} = -1$ and $\bar{g}^*|H^0(X, \Omega^2_{X/k}) = -1$. Furthermore, $\bar{g}^*$ fixes a sublattice of $NS(X)$ of rank 10. Hence, $\bar{g}$ is an Enriques involution of X. □

Theorem 2.6. *Assume that k is an algebraically closed field of characteristic $p > 2$. A K3 surface X over k is a Kummer surface if and only if there exists a primitive embedding of J into $NS(X)$.*

Proof. For the "only if" part, we refer to Section 1 of [14]. If X is supersingular, X is a Kummer surface if and only if the Artin invariant of X is 1 or 2 ([17], Theorem 7.10). Assume that the Artin invariant of X is greater than 2. Since $J \otimes \mathbb{Z}_p$ is a unimodular $\mathbb{Z}_p$-lattice of rank 16 with a square discriminant, by Theorem 1.1, [21], there is no embedding of $J \otimes \mathbb{Z}_p$ into $NS(X) \otimes \mathbb{Z}_p$. Therefore, if there is an embedding of J into $NS(X)$, then X is a supersingular Kummer surface. Suppose X is of finite height and there is a primitive embedding of J into $NS(X)$. Let $\mathbb{X}$ be a Néron-Severi group preserving lifting of X over W and X_K be the generic fiber of $\mathbb{X}/W$. Since there is a primitive embedding

$$J \hookrightarrow NS(X) = NS(X_K),$$

X_K is a Kummer surface. Therefore, X_K contains 16 mutually disjoint smooth rational curves $C_1, \dots, C_{16}$ satisfying $\frac{1}{2}\sum C_i \in NS(X_K)$.

Lemma 2.7 (c.f. [7], Lemma 2.3). *A class $v \in NS(X)$ is effective on X if and only if it is effective on X_K. A class $v \in NS(X)$ represents a smooth rational curve if and only if $v \in NS(X_K)$ represents a smooth rational curve.*

Proof. Let us fix a class $h \in NS(X)$ that is ample on both X and X_K. Assume that $v \in NS(X)$ is an effective class on X and D is a curve represented by v. We assume that $D = \sum n_i C_i$ for integral curves C_i. Let w_i be the class in $NS(X)$ which represents C_i. Then $(h, w_i) > 0$ and $(w_i, w_i) \geq -2$ by the adjunction formula. Then by the Riemann-Roch theorem, w_i is an effective class in $NS(X_K)$, so $v = \sum n_i w_i$ is also effective in $NS(K_X)$. The converse follows in the same way. The above argument also asserts that $v \in NS(X)$ is indecomposable and effective if and only if $v \in NS(X_K)$ is indecomposable and effective. A class $v \in NS(X)$ represents a smooth rational curve if and only if $(v, v) = -2$ and v is indecomposable and effective. Then $v \in NS(X_K)$ also represents a smooth rational curve. The converse holds in the same way. □

According to Lemma 2.7, the reduction of each C_i is a smooth rational curve in X. We denote the reduction of C_i using $\bar{C}_i$ again. Let $X' \to X$ be the double cover ramified along the 16 rational curves $\bar{C}_i$. The preimage of each $\bar{C}_i$ in X' is a (-1)-curve. Let A be the surface obtained by blowing down the 16 (-1)-curves of X'. It can be checked that A is an abelian surface and that X is the Kummer surface of A. □

Theorem 2.8. *Assume that k is an algebraically closed field of characteristic $p > 2$. A Kummer surface X over k is an Enriques K3 surface.*

Proof. If X is a supersingular Kummer surface, the Artin invariant of X is 2 at most. It follows that X is an Enriques K3 surface by Corollary 2.4. Assume that X is a Kummer surface of finite height. Let $\pi : \mathbb{X} \to \operatorname{Spec} W$ be a Néron-Severi group preserving lifting of X and X_K be the generic fiber of π. If X is a Kummer surface, there exists a primitive embedding

$$J \hookrightarrow NS(X) = NS(X_K)$$

and X_K is a Kummer surface. Because X_K is defined over a field of characteristic 0, X_K is an Enriques K3 surface and there exists a primitive embedding

$$\Gamma \hookrightarrow NS(X_K) = NS(X)$$

such that the orthogonal complement does not contain a vector of self intersection -2. According to Theorem 2.5, X is an Enriques K3 surface. □

3. Classification of K3 surfaces of Picard number 20

A singular K3 surface is a complex K3 surface of Picard number 20. For a complex K3 surface X, we denote the transcendental lattice of X using $T(X)$. The transcendental lattice of a singular K3 surface is an even positive definite lattice of rank 2. In contrast, for an even positive definite lattice of rank 2, M there exists a unique singular K3 surface X_M up to isomorphism such that $T(X_M)$ is isomorphic to M ([4], Theorem 4). Every singular K3 surface has a model over a number field.

In this section we assume that k is an algebraically closed field of characteristic $p > 2$ and that its cardinality is equal to or less than the cardinality of $\mathbb{C}$. Let W be the ring of Witt vectors of k and K be the fraction field of W. We fix an isomorphism $\bar{K} \simeq \mathbb{C}$. Let X be a K3 surface of Picard number 20 over k. Then X is an ordinary K3 surface, in particular X is of finite height.

Lemma 3.1. *Assume that X is a K3 surface of Picard number 20 over k. Then $NS(X) \otimes \mathbb{Z}_p$ is an unimodular $\mathbb{Z}_p$-lattice of square discriminant.*

Proof. Because the height of X is 1, the flat cohomology $H^2_{fl}(X, \mathbb{Z}_p(1))$ is a free $\mathbb{Z}_p$-module of rank 20. In the exact sequence ([3], p. 629)

$$0 \to NS(X) \otimes \mathbb{Z}_p \to H^2_{fl}(X, \mathbb{Z}_p(1)) \to T_p(Br_X) \to 0,$$

$T_p(Br_X)$ is a free $\mathbb{Z}_p$-module and the rank of $NS(X) \otimes \mathbb{Z}_p$ is equal to the rank of $H^2_{fl}(X, \mathbb{Z}_p(1))$. Therefore, $T_p(Br_X) = 0$ and $NS(X) \otimes \mathbb{Z}_p \simeq H^2(X, \mathbb{Z}_p(1))$. Because the cup product pairing of $H^2(X, \mathbb{Z}_p(1))$ is unimodular of square discriminant ([18], Remark 1.5), so is $NS(X) \otimes \mathbb{Z}_p$. □

Lemma 3.2. *Let L be a local field of mixed characteristic $(0, p)$ and k be an algebraic closed extension of the residue field of L. Let X_L be a K3 surface defined over L. Assume that X_L has a good reduction over the residue field. Let X be the reduction of X_L over k. The embedding*

$$NS(X_L \otimes \bar{L}) \otimes \mathbb{Z}_l \hookrightarrow NS(X \otimes k) \otimes \mathbb{Z}_l$$

is primitive for any prime number $l \neq p$.

Proof. We may assume that

$$NS(X_L) = NS(X_L \otimes \bar{L}) \text{ and } NS(X) = NS(X \otimes k).$$

For a prime $l \neq p$, the canonical embedding

$$NS(X_L) \otimes \mathbb{Z}_l \hookrightarrow H^2(X_L(\mathbb{C}), \mathbb{Z}_l) \simeq H^2_{\acute{e}t}(X_L \otimes \bar{L}, \mathbb{Z}_l) = H^2_{\acute{e}t}(X \otimes k, \mathbb{Z}_l)$$

factors through

$$NS(X_L) \otimes \mathbb{Z}_l \hookrightarrow NS(X) \otimes \mathbb{Z}_l \hookrightarrow H^2_{\acute{e}t}(X \otimes k, \mathbb{Z}_l).$$

Since the embedding $NS(X) \hookrightarrow H^2(X_L(\mathbb{C}), \mathbb{Z})$ is primitive, $(NS(X)/NS(X_L)) \otimes \mathbb{Z}_l$ has no torsion. □

Remark 3.3. In Lemma 3.2, the quotient group $NS(X)/NS(X_L)$ may have a non-trivial p-torsion. If the ramification index of L is less than $p-1$, then $NS(X)/NS(X_L)$ is torsion-free ([20], Théorème 4.1.2).

Let us fix an embedding

$$\bar{\mathbb{Q}} \hookrightarrow \bar{K} \simeq \mathbb{C}.$$

On each number field F, there exists a unique finite place of residue characteristic p associated with the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{K}$. For any lattice M, we denote the discriminant of M by $d(M)$.

Lemma 3.4. *Let F be a number field and X_F be a singular K3 surface defined over F. Let v be the place of F associated with the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{K}$. X_F has a potentially good reduction at v. Let k be an algebraically closed extension of the residue field of v and X be a good reduction of X_F over k. We suppose $d(NS(X_F \otimes \bar{F}))$ is a non-zero square modulo p. Then X is ordinary and $NS(X) = NS(X_F \otimes \bar{F})$.*

Proof. According to Corollary 0.5, [10], X has a potentially good reduction at v. Since there is an embedding

$$NS(X_F \otimes \bar{F}) \hookrightarrow NS(X),$$

following the assumption, X is supersingular of Artin invariant 1 or ordinary. Because $d(NS(X_F \otimes \bar{F}))$ is a non-zero square modulo p, there is no embedding of $NS(X_F \otimes \bar{F}) \otimes \mathbb{Z}_p$ into the Néron-Severi group of a supersingular K3 surface ([21], Theorem 1.1). Hence, X is ordinary. Since X is ordinary, $NS(X) = NS(X_F \otimes \bar{F})$ by Lemma 3.2. □

Corollary 3.5 ([8], Theorem 2.6). *Every K3 surface X over k of Picard number 20 has a model over a finite field.*

Proof. Assume that $\mathbb{X}$ is a Néron-Severi group preserving lifting of X. It follows that $X_{\bar{K}} = \mathbb{X} \otimes \bar{K}$ is a singular K3 surface and has a model over a number field. We assume that X'_F is a K3 surface defined over a number field F such that $X'_F \otimes \bar{K}$ is isomorphic to $X_{\bar{K}}$. Let v be the place of F corresponding with the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{K}$. We may assume X'_F has a good reduction X'_v at v by Lemma 3.4. Then X is isomorphic to $X'_v \otimes k$. □

Proposition 3.6. *Let X_F and X'_F be two K3 surfaces of Picard number 20 defined over a number field F. Let v be a place of F whose residue characteristic is $p > 2$ and let k be an algebraically closed extension of the residue field k_v. Assume that $d(NS(X_F \otimes \bar{F}))$ and $d(NS(X'_F \otimes \bar{F}))$ are not divisible by p. If the reduction of X_F over k is isomorphic to the reduction of X'_F over k and both are ordinary, then $X_F \otimes \bar{F}$ is isomorphic to $X'_F \otimes \bar{F}$.*

Proof. It is enough to show that $X_F \otimes \bar{K}$ is isomorphic to $X'_F \otimes \bar{K}$. We set $L = KF_v \subset \bar{K}$ and V is the ring of integers of L. Assume that $\mathbb{X}$ and $\mathbb{X}'$ are smooth integral models of X_F and X'_F over V respectively. Let X be the common special fiber of $\mathbb{X}$ and $\mathbb{X}'$. Following the assumption, X is ordinary and

$$NS(X) = NS(X_F) = NS(X'_F).$$

Since the Picard number of X is 20, the deformation space of X together with $NS(X)$ is formally smooth of 0-dimensional over W ([7], Proposition 4.1). Therefore, $\mathbb{X}$ is isomorphic to $\mathbb{X}'$ and $X_F \otimes \bar{K}$ is isomorphic to $X'_F \otimes \bar{K}$. □

We summarize all of the preceding results to classify the K3 surfaces of Picard number 20 over k. Let S_p be the set of isomorphic classes of positive definite even $\mathbb{Z}$-lattices of rank 2 such that the discriminant is a non-zero square modulo p. For each $M \in S_p$, there is a unique singular K3 surface X_M over $\bar{\mathbb{Q}}$ such that $T(X_M)$ is isomorphic to M via the given embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{K} \simeq \mathbb{C}$. Let $X_{k,M}$ be the reduction of X_M over k given in the proof of Lemma 3.4. The correspondence $M \mapsto X_{k,M}$ is a bijection from S_p to the set of isomorphic classes of the K3 surfaces of Picard number 20 over k.

Theorem 3.7. *Let k be an algebraically closed field of characteristic $p >$ 2. The isomorphism classes of the K3 surfaces of Picard number 20 are classified by S_p. Every K3 surface of Picard number 20 over k has a model over a finite field.*

Remark 3.8. In [8], in a similar argument, C. Liedtke proves that every K3 surface of Picard number 20 over a field of odd characteristic has a model over a finite field and has a Shioda-Inose type "sandwich."

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