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On a proof of the Bouchard-Sulkowski conjecture

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In this short note, we give a proof of the free energy part of the BKMP conjecture of $\mathbb{C}^3$ proposed by Bouchard and Sulkowski [4]. Combing with the previous results of the open part, it finishes the proof of the full BKMP conjecture of $\mathbb{C}^3$.

1. Introduction

Motivated by B. Eynard and his collaborators' series works on matrix model [7, 9, 10], V. Bourchard, A. Klemm, M. Mariño and S. Pasquetti [1] proposed a new approach (remodeling the B-model) based on the Eynard-Orantin topological recursion to compute the topological string amplitudes of local Calabi-Yau manifolds. Moreover, they conjectured that the remodeling approach is equivalent to the Gromov-Witten theory of corresponding toric Calabi-Yau manifolds [2]. In particular, for the simplest toric Calabi-Yau threefold $\mathbb{C}^3$, V. Bouchard and M. Mariño [3] calculated the correlation functions by Eynard-Orantin topological recursion explicitly in lower genus, it turns out that these correlation functions are equal to the corresponding generating functions of the open Gromov-Witten invariants [13]. Later, L. Chen and J. Zhou [6, 17] gave the proof of this open part BKMP conjecture of $\mathbb{C}^3$ independently(see also [8] for a new proof).

Recently, V. Bouchard and P. Sulkowski [4] proposed the following free energy part of the BKMP conjecture of $\mathbb{C}^3$ (see the Conjecture 2 in [4]).

Conjecture 1.1. *Let Σ_f be the framed curve mirror to $X = \mathbb{C}^3$. Then the free energies obtained through the Eynard-Orantin recursion are given by:*

$$F^{(g)} = \frac{1}{2}(-1)^g \times \frac{|B_{2g}||B_{2g-2}|}{2g(2g-2)(2g-2)!}.$$

In this note, we give a proof of the Conjecture 1.1 based on a Hodge integral identity due to C. Faber and R. Pandharipande [11] and some residue calculations. After completion of this paper, the author contacted with Prof. V. Bouchard and knew that this conjecture was also proved independently by V. Bouchard and his collaborators [5] at the same time.

2. The BKMP conjecture

Let us consider a smooth complex curve given by $\Sigma = \{H(x,y) = 0\}$ in $\mathbb{C}^2$ or $(\mathbb{C}^*)^2$, it is usually called a spectral curve. It defines a non-compact Riemann surface, which also denoted by Σ. $x, y : \Sigma \to \mathbb{C}$ are holomorphic functions on Σ. We should assume the map $x : \Sigma \to \mathbb{C}$ has only simple ramification points. That means the differential dx has only simple zeros. Let us denote by $\{q_1, \ldots, q_a\} \subset \Sigma$ the set of ramifications points. So there exists a small neighborhood U_j of every q_j, $j = 1, \ldots, a$, such that the map x is a double-sheeted covering, Hence we have a deck transformation map $s_j : U_j \to U_j$ which implies $x(z) = x(s_j(z))$ for the local coordinate z on U_j. We mention that the mirror curve Σ_X of a toric Calabi-Yau 3-fold X satisfies these conditions [1, 12].

The Eynard-Orantin recursion process starts with the following ingredients [9].

2.1. Ingredients

First, one needs the meromorphic differential $\omega(p) = \log y(p) dx(p)$ if the curve Σ is in $\mathbb{C}^2$, and $\omega(p) = \log y(p) \frac{dx(p)}{x(p)}$ if Σ is in $(\mathbb{C}^*)^2$.

We define a bilinear meromorphic form called the "Bergman kernel" $B(p_1, p_2)$ which is uniquely defined by the following conditions: (i) It is symmetric; (ii) It has double pole with no residue at $p_1 = p_2$ and no other pole; (iii) It is normalized such that

$$B(p_1, p_2) = \left(\frac{1}{(z_1 - z_2)^2} + \text{regular} \right) dz_1 dz_2,$$

and its period about a basis of A-cycles on Σ vanish. In particular, for the curve Σ with genus 0, the Bergman kernel is given by

$$B(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \ z_i = z(p_i).$$

2.2. BKMP's construction

For a toric Calabi-Yau 3-fold X with mirror curve Σ_X in $(\mathbb{C}^*)^2$, inspired by the work [15], Bouchard, Klemm, Mariño and Pasquetti [1, 2] defined the correlation functions $F^{(g,h)}(p_1,\ldots,p_h)$ on the mirror curve Σ_X based on the topological recursions construced by B. Eynard and N. Orantin [9] as follows.

$$\begin{aligned}
F^{(g,h)}(p_1,\ldots,p_h) &= \int W^{(g,h)}(p_1,\ldots,p_h),\\
W^{(0,1)}(p) &= \omega(p),\\
W^{(0,2)}(p_1,p_2) &= B(p_1,p_2) - \frac{dp_1 dp_2}{(p_1-p_2)^2},\\
W^{(g,h)}(p_1,\ldots,p_h) &= \tilde{W}^{g,h}(p_1,\ldots,p_h),\ \text{for}\,(g,h)\neq(0,1),\ (0,2).
\end{aligned}$$

where $\tilde{W}^{(g,h)}(p_1,\ldots,p_h)$ is a multilinear meromorphic differential defined by the following topological recursions.

$$\tilde{W}^{(0,1)}(p)=0,\quad \tilde{W}^{(0,2)}(p,q)=B(p,q),$$

$$\begin{aligned}
\tilde{W}^{(g,h+1)}(p,p_1,\ldots,p_h) = &\sum_{q_i} Res_{q=q_i}\frac{dE_{q,\overline{q}}(p)}{\omega(q)-\omega(\overline{q})}\Big(\tilde{W}^{(g-1,h+2)}(q,\overline{q},p_1,\ldots,p_h)\\
&+\sum_{l=0}^{g}\sum_{J\subset H}\tilde{W}^{(g-l,|J|+1)}(q,p_J)\tilde{W}^{(l,|H|-|J|+1)}(\overline{q},p_{H\backslash J})\Big),
\end{aligned}$$

$$H=\{1,\ldots,h\},\quad J=\{i_1,\ldots,i_j\}\subset H,\quad p_J=\{p_{i_1},\ldots,p_{i_j}\},$$

$$dE_{q,\overline{q}}(p)=\frac{1}{2}\int_q^{\overline{q}}B(p,\psi),\ \text{near a ramification point}\ q_i.$$

Moreover, in [2], they defined the free energies $F^{(g)}$ ($g\in\mathbb{Z},\ g\geq 2$) on Σ by

$$F^{(g)}=\frac{(-1)^g}{2-2g}\sum_{q_i} Res_{q=q_i}\theta(q)W^{(g,1)}(q),$$

where $\theta(q)$ is any primitive of $\omega(q)$ given by $d\theta(q)=\omega(q)$. And $F^{(1)}$ is defined separately as

$$F^{(1)}=-\frac{1}{2}\log\tau_B-\frac{1}{24}\log\prod_i\omega^{'}(q_i),$$

where $\omega^{'}(q_i)=\frac{1}{dz_i(p)}d\left(\frac{\log y(x)}{x}\right)|_{p=q_i},\quad z_i(p)=\sqrt{x(p)-x(q_i)}$ and τ_B is the Bergman tau-function [9].

Then the BKMP conjecture for a toric Calabi-Yau 3-fold X can be formulated as follow (Conjecture 1 in [4]).

Conjecture 2.1. *Let Σ_f be the framed mirror curve to a toric Calabi-Yau threefold X.*

1. The free energies $F^{(g)}$ constructed by the Eynard-Orantin recursion are mapped by the mirror map to the genus g generating functions of Gromov-Witten invariants of X.

2. The correlation functions $F^{(g,h)}$ are mapped by the open/closed mirror map to the generating functions of framed open Gromov-Witten invariants.

3. BKMP conjecture for the case of $\mathbb{C}^3$

In this section, we restrict us to consider the special toric Calabi-Yau 3-fold $\mathbb{C}^3$ with the framed mirror curve

$$\Sigma_f = \{H(x,y) := x + y^f + y^{f+1} = 0\} \subset (\mathbb{C}^*)^2.$$

We refer to [4] for the details. It is obvious that Σ_f has only one ramification point (x^*, y^*) with

$$y_* = \frac{-f}{(f+1)}, \quad x_* = \frac{f^f}{(-1-f)^{-1-f}}.$$

By the definition of $\theta(q)$ given in Section 2, we have

$$\theta(y) = \frac{f}{2}(\log y)^2 + \log y \log(1+y) + Li_2(-y). \tag{1}$$

We define the differential form $\Psi_n(y;f)$ for $n \geq 0$ as follow:

(2)

$$\Psi_n(y;f) = -dy\frac{((1+f)y+f)}{y(y+1)}\left(\frac{y(y+1)}{(1+f)y+f}\frac{d}{dy}\right)^{n+1}\frac{1}{(1+f)((1+f)y+f)}.$$

For example, when $n = 0$ and 1:

$$\Psi_0(y;f) = dy\frac{1}{(f+(f+1)y)^2};$$
$$\Psi_1(y;f) = -dy\frac{3(1+f)y(y+1) - (1+2y)(f+(f+1)y)}{(f+(f+1)y)^4}.$$

For convenience, we introduce the notation $\hat{\Psi}_n(y;f)$ given by $\Psi_n(y;f) = -\hat{\Psi}_n(y;f)dy$.

By using the Eynard-Orantin topological recursions introduced in Section 2, we obtain

$$W^{(0,3)}(y_1,y_2,y_3) = (f(f+1))^2\Psi_0(y_1;f)\Psi_0(y_2;f)\Psi_0(y_3;f);$$

$$W^{(0,4)}(y_1,y_2,y_3,y_4) = (f(f+1))^3\sum_{i=1}^{4}\Psi_1(y_i;f)\prod_{j\neq i}\Psi_0(y_j;f);$$

$$W^{(1,1)}(y) = -\frac{1}{24}\left((1+f+f^2)\Psi_0(y;f) - f(f+1)\Psi_1(y;f)\right).$$

Let $\overline{\mathcal{M}}_{g,h}$ be the moduli space of stable h-pointed genus g complex algebraic curves, λ_i be the i-th Chern class of its Hodge bundle and ψ_i the first Chern class of the line bundle corresponding to the cotangent space of the universal curve at the i-th marked point. We use Witten's notation [16] to denote the following Hodge integrals:

$$\langle \tau_{d_1}\cdots\tau_{d_h}\lambda_1^{k_1}\cdots\lambda_g^{k_g}\rangle_g := \int_{\overline{\mathcal{M}}_{g,h}} \psi_1^{d_1}\cdots\psi_h^{d_h}\lambda_1^{k_1}\cdots\lambda_g^{k_g}.$$

They are nonvanishing only if

$$\sum_{i=1}^{h} d_i + \sum_{j=1}^{g} jk_j = 3g-3+h$$

since the dimension of $\overline{\mathcal{M}}_{g,h}$ is $3g-3+h$.

For the general $g\geq 2$ and $h\geq 1$, L. Chen [6] and J. Zhou [17] have proved the following identity independently (See also [18]).

$$W^{(g,h)}(y_1,\ldots,y_h) = (-1)^g(f(f+1))^{h-1} \cdot \sum_{n_i\geq 0}\left\langle \prod_{i=1}^{h}\tau_{n_i}\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-f-1)\Lambda_g^{\vee}(f)\right\rangle_g \prod_{i=1}^{h}\Psi_{n_i}(y_i;f)$$

where $\Lambda_g^{\vee}(t) = t^g - t^{g-1}\lambda_1 + \cdots + (-1)^g\lambda_g$.

In particularly,

$$W^{(g,1)}(y) = (-1)^g\sum_{n\geq 0}^{3g-2}\langle \tau_n\Lambda_g^{\vee}(1)\Lambda_g^{\vee}(-f-1)\Lambda_g^{\vee}(f)\rangle_g\Psi_n(y;f).$$

Thus the free energy part of the BKMP conjecture of $\mathbb{C}^3$ is given by

$$\begin{aligned} F^{(g)} &= \frac{(-1)^g}{2-2g} Res_{y=\frac{-f}{1+f}} \theta(y) W^{(g,1)}(y) \\ &= \frac{1}{2-2g} \sum_{n\geq 0} \langle \tau_n \Lambda_g^\vee(1)\Lambda_g^\vee(-f-1)\Lambda_g^\vee(f)\rangle_g Res_{y=\frac{-y}{1+y}} \theta(y)\Psi_n(y;f). \end{aligned}$$

Then Conjecture 1.1 is an easy consequence of the following two Lemmas and the Hodge integral identity due to Faber-Pandharipande [11, 14]:

$$\langle \lambda_g\lambda_{g-1}\lambda_{g-2}\rangle_g = \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g}.$$

Lemma 3.1. *The degree $3g-3$ part of $\Lambda_g^\vee(1)\Lambda_g^\vee(-f-1)\Lambda_g^\vee(f)$ is given by*

$$(-1)^{g-1} f(f+1)\lambda_g\lambda_{g-1}\lambda_{g-2}.$$

Proof. By Mumford's relation: $\Lambda_g^\vee(1)\Lambda_g^\vee(-1) = (-1)^g$, we have $\lambda_{g-1}^2 = 2\lambda_g\lambda_{g-2}, \lambda_g^2 = 0$. Then the degree $3g-3$ part of $\Lambda_g^\vee(1)\Lambda_g^\vee(-f-1)\Lambda_g^\vee(f)$ is equal to

$$\begin{aligned} &((-1)^g\lambda_g + (-1)^{g-1}\lambda_{g-1} + (-1)^{g-2}\lambda_{g-2}) \\ &\times (-1)^g(\lambda_g + (f+1)\lambda_{g-1} + (f+1)^2\lambda_{g-2}) \\ &\times ((-1)^g\lambda_g + f(-1)^{g-1}\lambda_{g-1} + f^2(-1)^{g-2}\lambda_{g-2}) \\ = &(-1)^{g-1} f(f+1)\lambda_g\lambda_{g-1}\lambda_{g-2}. \end{aligned}$$

□

Lemma 3.2.

$$Res_{y=\frac{-f}{1+f}} \theta(y)\Psi_n(y;f) = \begin{cases} \dfrac{1}{f(1+f)}, & n=1, \\ 0, & n\geq 2 \text{ or } n=0. \end{cases}$$

Proof. Let $z = y + \frac{f}{1+f}$, by formula (1), we have

$$\begin{aligned} \theta(z) = &\frac{f}{2}\left(\log\left(z - \frac{f}{1+f}\right)\right)^2 + \log\left(z-\frac{f}{1+f}\right)\log\left(z+\frac{1}{1+f}\right) \\ &+ Li_2\left(-z+\frac{f}{1+f}\right). \end{aligned}$$

Hence,

$$d\theta(z) = \frac{(1+f)z\log\left(z-\frac{f}{1+f}\right)}{(z-\frac{f}{1+f})(z+\frac{1}{1+f})}dz.$$

By formula (2),

$$\hat{\Psi}_n(z;f) = \frac{d}{dz}\left(\hat{\Psi}_{n-1}(z;f)\frac{\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)}{(1+f)z}\right) \text{ for } n \geq 1, \tag{4}$$

and

$$\hat{\Psi}_0(z;f) = -\frac{1}{(1+f)^2 z^2}.$$

By the recursion formula (4), it is easy to show that $\hat{\Psi}_n(z;f)$ takes the following form

$$\hat{\Psi}_n(z;f) = \frac{a_0(f) + a_1(f)z + \cdots + a_{2n}(f)z^{2n}}{((1+f)z)^{2n+2}}$$

where $a_0(f), \ldots, a_{2n}(f)$ are some polynomials of framing f. The following residue identity

$$0 = Res_{z=0} d(f(z)g(z)) = Res_{z=0} g(z)df(z) + Res_{z=0} f(z)dg(z),$$

leads to

$$Res_{z=0} g(z)df(z) = -Res_{z=0} f(z)dg(z). \tag{5}$$

We remark that formula (5) will be used iteratively in the following computations.

For $n = 0$,

$$\begin{aligned}
&Res_{y=\frac{-f}{1+f}}\theta(y)\Psi_0(y;f) \\
&= -Res_{y=-\frac{f}{1+f}}\theta(y)\hat{\Psi}_0(y;f)dy = -Res_{z=0}\theta(z)\hat{\Psi}_0(z;f)dz \\
&= -Res_{z=0}\theta(z)\frac{1}{(1+f)^2}d\left(\frac{1}{z}\right) = Res_{z=0}\frac{1}{(1+f)^2 z}d\theta(z) \\
&= Res_{z=0}\frac{\log\left(z-\frac{f}{1+f}\right)}{(1+f)\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)} = 0.
\end{aligned}$$

For $n = 1$,

$$\begin{aligned}
&Res_{y=\frac{-f}{1+f}}\theta(y)\Psi_1(y;f)\\
=&-Res_{y=-\frac{f}{1+f}}\theta(y)\hat{\Psi}_1(y;f)dy = -Res_{z=0}\theta(z)\hat{\Psi}_1(z;f)dz\\
=&-Res_{z=0}\theta(z)d\left(\hat{\Psi}_0(z;f)\frac{\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)}{(1+f)z}\right)\\
=&Res_{z=0}\hat{\Psi}_0(z;f)\log\left(z-\frac{f}{1+f}\right)dz\\
=&-Res_{z=0}\frac{1}{(1+f)^2z^2}\log\left(z-\frac{f}{1+f}\right)dz\\
=&\frac{1}{f(1+f)}.
\end{aligned}$$

More generally, for $n \geq 2$,

$$\begin{aligned}
&Res_{y=\frac{-f}{1+f}}\theta(y)\Psi_n(y;f)\\
=&-Res_{y=-\frac{f}{1+f}}\theta(y)\hat{\Psi}_n(y;f)dy\\
=&-Res_{z=0}\theta(z)\hat{\Psi}_n(z;f)dz\\
=&-Res_{z=0}\theta(z)d\left(\hat{\Psi}_{n-1}(z;f)\frac{\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)}{(1+f)z}\right)\\
=&Res_{z=0}\hat{\Psi}_{n-1}(z;f)\frac{\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)}{(1+f)z}d\theta(z)\\
=&-Res_{z=0}\hat{\Psi}_{n-1}(z;f)\log\left(z-\frac{f}{1+f}\right)dz\\
=&Res_{z=0}d\left(\hat{\Psi}_{n-2}(z;f)\frac{\left(z-\frac{f}{1+f}\right)\left(z+\frac{1}{1+f}\right)}{(1+f)z}\right)\log\left(z-\frac{f}{1+f}\right)\\
=&-Res_{z=0}\hat{\Psi}_{n-2}(z;f)\frac{\left(z+\frac{1}{1+f}\right)}{(1+f)z}\\
=&-Res_{z=0}\frac{a_0(f)+a_1(f)z+\cdots+a_{2n-4}(f)z^{2n-4}}{((1+f)z)^{2n-2}}\left(\frac{1}{(1+f)}+\frac{1}{(1+f)^2z}\right)\\
=&0.
\end{aligned}$$

$\square$

Now, combining the Lemma 3.1 and 3.2, we can finish the proof of the Conjecture 1.1.

Proof.

$$\begin{aligned}
F^{(g)} &= \frac{(-1)^g}{2-2g} Res_{y=-\frac{f}{1+f}} \theta(y) W^{(g,1)}(y) \\
&= \frac{1}{2-2g} \sum_{n=1}^{3g-2} \langle \tau_n \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g Res_{y=-\frac{f}{1+f}} \theta(y) \Psi_n(y;f) \\
&= -\frac{1}{2g-2} \frac{1}{f(1+f)} \langle \tau_1 \Lambda_g^\vee(1) \Lambda_g^\vee(-f-1) \Lambda_g^\vee(f) \rangle_g \\
&= \frac{(-1)^g}{2g-2} \langle \tau_1 \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g \\
&= (-1)^g \langle \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g \\
&= (-1)^g \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}
\end{aligned}$$

where we have used the dilaton equation for Hodge integrals

$$\langle \tau_1 \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g = (2g-2) \langle \lambda_g \lambda_{g-1} \lambda_{g-2} \rangle_g.$$

Thus the Conjecture 1.1 is proved. □

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References

[1] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Remodeling the B-model*. Commun. Math. Phys., **287** (2009), 117–178.

[2] V. Bouchard, A. Klemm, M. Mariño and S. Pasquetti, *Topological open string on orbifolds*. Commun. Math. Phys., **296** (2010), 589–623.

[3] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*. Proc. Symposia Pure Math., **78** (2008), 263–283.

[4] V. Bouchard and P. Sulkowski, *Topological recursion and mirror curves.* Adv. Theor. Math. Phys., **16** (2012), 1443–1483.

[5] V. Bouchard, A. Catuneanuy, O. Marchaly and P. Sulowskiz, *The remodeling conjecture and Faber-Pandharipande formula.* Lett. Math. Phys., **103** (2013), 59–77.

[6] L. Chen, *Bouchard-Klemm-Mariño-Pasquetti conjecture for* $\mathbb{C}^3$. `arXiv:0910.3739v1`.

[7] B. Eynard, *Topological expansion for 1-hermitian matrix model correlation functions.* J. High. Energy. Phys., (2004), no. 11, 031.

[8] B. Eynard, *Intersection numbers of spectral curves.* `arXiv:1104.0176`.

[9] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion.* Commun. Number. Theory. Phys., **1** (2007), no. 2, 347–452.

[10] B. Eynard and N. Orantin, *Algebraic methods in random matrices and enumerative geometry.* `arXiv:0811.3531`.

[11] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory.* Invent. Math., **139** (2000), no. 1, 173–199.

[12] K. Hori and C. Vafa *Mirror symmetry.* `arXiv:hep-th/0002222`.

[13] C.-C. Liu, K. Liu and J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge integrals.* J. Differential Geom., **65** (2003).

[14] C.-C. Liu, K. Liu and J. Zhou, *Mariño-Vafa formula and Hodge integral identities.* J. Algebric Geom., **15** (2006), 379–398.

[15] M. Mariño, *open string amplitudes and large order behavior in topological string theory.* `arXiv:hep-th/0612127`.

[16] E. Witten, *Two dimensional gravity and intersection theory on moduli space.* Surveys Differ. Geom., **1** (1991), 243–310.

[17] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex.* `arXiv:0910.4320`.

[18] S. Zhu, *The Laplace transform of the cut-and-join equation of Mariño-Vafa formula and its applications.* `arXiv:1001.0618`.

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