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Boundary regularity of the solution to the complex Monge-Ampère equation on pseudoconvex domains of infinite type

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Let Ω be a C^2-smooth, bounded, pseudoconvex domain in $\mathbb{C}^n$ satisfying the "f-property". The f-property is a consequence of the geometric "type" of the boundary. All pseudoconvex domains of finite type satisfy the f-property as well as many classes of domains of infinite type. In this paper, we prove the existence, uniqueness, and "weak" Hölder-regularity up to the boundary of the solution to the Dirichlet problem for the complex Monge-Ampère equation

$$\begin{cases}\det\left[\dfrac{\partial^2(u)}{\partial z_i\partial\bar z_j}\right]=h\geq 0 & \text{in } \Omega,\\ u=\phi & \text{on } b\Omega.\end{cases}$$

The idea of our proof goes back to Bedford and Taylor [1]. However, the basic geometrical ingredient is based on a recent result by Khanh [12].

1. Introduction

Let Ω be a bounded, weakly pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary $b\Omega$. For given functions $h\geq 0$ defined in Ω and ϕ defined on $b\Omega$, the Dirichlet problem for the complex Monge-Ampère equation consists in finding a continuous, plurisubharmonic function u on Ω such that

$$\begin{cases}\det[u_{ij}]=h & \text{in } \Omega,\\ u=\phi & \text{on } b\Omega,\end{cases}\tag{1.1}$$

where $u_{ij}=\frac{\partial^2 u}{\partial z_i\bar z_j}$ is the $(i,j)^{th}$-entry of $n\times n$-matrix $[u_{ij}]$. When u is not $C^2(\Omega)$, the first Equation in (1.1) means that $(dd^cu)^n=hdV$ in the sense of Bedford-Taylor [1] (where dV is the Lebesgue measure on $\mathbb{C}^n$).

When Ω is a smooth, bounded, strongly pseudoconvex domain in $\mathbb{C}^n$, a great deal of work has been done about the existence, uniqueness, and regularity of the solution to the complex Monge-Ampère problem. The most general related results are those obtained in [1] and [3].

- In [1], Bedford and Taylor establish the classical solvability of the Dirichlet problem (1.1). Via pluripotential theory [15], the right hand side is developed in the sense of positive currents when u is continuous and plurisubharmonic. The authors prove that if Ω is a strongly pseudoconvex, bounded domain in $\mathbb{C}^n$ with C^2 boundary, and if $\phi \in Lip^{2\alpha}(b\Omega)$, $0 \le h^{\frac{1}{n}} \in Lip^{\alpha}(\overline{\Omega})$, where $0 < \alpha \le 1$, then there is a unique solution $u \in Lip^{\alpha}(\overline{\Omega})$ of (1.1). This result is sharp.
- In [3], the smoothness of the solution of (1.1) is also established. In particular, on a bounded strongly pseudoconvex domain with smooth boundary, if $\phi \in C^{\infty}(b\Omega)$, then there exists a unique solution $u \in C^{\infty}(\overline{\Omega})$ when h is smooth and strictly positive on $\overline{\Omega}$. The approach of [3] follows the continuity method applied to the real Monge-Ampère equations [9].

When Ω is not strongly pseudoconvex, there are some known results for the existence and regularity for this problem due to Blocki [2], Coman [7], and Li [18].

- In [2], Blocki also considers the Dirichlet problem (1.1) on a hyperconvex domain. He proves that when the datum $\phi \in C(b\Omega)$ can be continously extended to a plurisubharmonic function on Ω and the right hand is nonnegative and continuous, then the plurisubharmonic solution exists uniquely and is continuous. However, the Hölder continuity for the solution on these domains is still unknown.
- In [7], Coman shows how to connect some geometrical conditions on a domain in $\mathbb{C}^2$ to the existence of a plurisubharmonic upper envelope in Hölder spaces. In particular, the weak pseudoconvexity of finite type m in $\mathbb{C}^2$ and the fact that the Perron-Bremermann function belongs to $Lip^{\frac{\alpha}{m}}$ with corresponding data in Lip^{α} are equivalent. Again, this means that the finite type condition plays a critical role in the Hölder regularity of the solution to the complex Monge-Ampère equation.
- Li [18] studies the problem on a domain admitting a non-smooth, uniformly and strictly plurisubharmonic defining function. In particular, if Ω admits a uniformly and strictly plurisubharmonic defining function in $Lip^{\frac{2}{m}}(\bar{\Omega})$ when $0 < \alpha \le \frac{2}{m}$, and $\phi \in Lip^{m\alpha}(b\Omega)$ and if $h^{\frac{1}{n}} \in Lip^{\alpha}(\bar{\Omega})$,

then the solution $u \in Lip^{\alpha}(\overline{\Omega})$ of (1.1) exists uniquely. Based on results by Catlin [6] and by Fornaess-Sibony [8], there exists a plurisubharmonic defining function in $Lip^{\frac{2}{m}}(\bar{\Omega})$ on pseudoconvex domains of finite type m in $\mathbb{C}^2$ or convex domains of finite type m in $\mathbb{C}^n$.

The main purpose in this paper is to generalize the above results to a pseudoconvex domain, not necessarily of finite type, but admitting an f-property. The f-property assumes the existence of a bounded family of weights in the spirit of [5] and it is sufficient for an f-estimate for the $\bar{\partial}$-Neumann problem [5, 13]. We also notice that when $\lim_{t\to\infty} \frac{f(t)}{\log t} = \infty$ the solution of the $\bar{\partial}$-Neumann problem is regular [14, 16].

Definition 1.1. For a smooth, monotonic, increasing function $f : [1, +\infty) \to [1, +\infty)$ with $\frac{f(t)}{t^{1/2}}$ decreasing, we say that Ω has an f-property if there exist a neigborhood U of $b\Omega$ and a family of functions $\{\phi_\delta\}$ such that

(i) the functions ϕ_δ are plurisubharmonic, C^2 on U, and satisfy $-1 \leq \phi_\delta \leq 0$, and

(ii) $i\partial\bar{\partial}\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2-defining function of Ω.

Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant. Morever, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

Remark 1.2. For a pseudoconvex domain, the f-property is a consequence of the geometric finite type. In [4, 5], Catlin proves that every smooth, pseudoconvex domain Ω of finite type m in $\mathbb{C}^n$ has the f-property for $f(t) = t^{\epsilon}$ with $\epsilon = m^{-n^2 m^{n^2}}$. Additionally, there are several cases when Ω is known to have the f-property with $f(t) = t^{1/m}$ where m is the type: strongly pseudoconvex, pseudoconvex of finite type in $\mathbb{C}^2$, decoupled or convex in $\mathbb{C}^n$ (cf. [6, 10, 19, 20]).

Remark 1.3. Khanh and Zampieri study the relationship of the general type (both finite and infinite type) and the f-property [10, 14]. They prove that if $P_1, ..., P_n : \mathbb{C} \to \mathbb{R}^+$ are functions such that $\Delta P_j(z_j) \gtrsim \frac{F(|x_j|)}{x_j^2}$ or $\frac{F(|y_j|)}{y_j^2}$ for any $j = 1, ..., n$, then the pseudoconvex ellipsoid

$$C = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} P_j(z_j) \leq 1 \right\}$$

has the f-property for $f(t) = (F^*(t^{-1}))^{-1}$. Here we denote F^* is the inverse function to F.

In this paper, using the f-property we prove the "weak" Hölder regularity for the solution of the Dirichlet problem of complex Monge-Ampère equation. For this purpose we recall the definition of the f-Hölder spaces in [11].

Definition 1.4. Let f be an increasing function such that $\lim_{t\to+\infty} f(t) = +\infty$, $f(t) \lesssim t$. For a subset A of $\mathbb{C}^n$, define the f-Hölder space on A by

$$\Lambda^f(A) = \{u : \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1}) \cdot |u(z) - u(w)| < \infty\}$$

and set

$$\|u\|_{\Lambda^f(A)} = \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1}) \cdot |u(z) - u(w)|.$$

Note that the notion of the f-Hölder space includes the standard Hölder space Λ_α by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha \le 1$. When $1 < \alpha \le 2$, we also define $\Lambda^{t^\alpha}(A) := \Lambda_\alpha(A)$ where

$$\Lambda_\alpha(A) = \left\{u : \|u\|_{\Lambda^{t^\alpha}(A)} := \|Du\|_{\Lambda^{t^{\alpha-1}}(A)} < \infty\right\}.$$

The main result in this paper consists in the following:

Theorem 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain admitting the f-property. Suppose that the function $g : [1,\infty) \to [1,\infty)$ defined by*

$$g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty.$$

If $0 < \alpha \le 2$, $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and $h \ge 0$ on Ω with $h^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\Omega)$, then the Dirichlet problem for the complex Monge-Ampère equation

$$\begin{cases} \det(u_{ij}) = h & \text{in } \Omega, \\ u = \phi & \text{on } b\Omega, \end{cases} \tag{1.2}$$

has a unique plurisubharmonic solution $u \in \Lambda^{g^\alpha}(\overline{\Omega})$.

By Remarks 1.2 and 1.3, we immediately have the following.

Corollary 1.6. *1) Let Ω be a bounded, C^2-boundary, pseudoconvex domain of finite type m in $\mathbb{C}^n$ satisfying at least one of the following conditions: strongly pseudoconvex, convex, in $\mathbb{C}^2$, or decoupled. If $0 < \alpha \le 2$, $\phi \in Lip^{\alpha}(b\Omega)$, and $h \ge 0$ on Ω with $h^{\frac{1}{n}} \in Lip^{\frac{\alpha}{m}}(\Omega)$, then (1.2) has a unique plurisubharmonic solution $u \in Lip^{\frac{\alpha}{m}}(\overline{\Omega})$. If Ω has finite type m, but does not satisfy any one of the above additional conditions, then it is still true that $u \in Lip^{\alpha\epsilon}(\overline{\Omega})$ for $\epsilon = m^{-n^2 m^{n^2}}$.*

2) Let Ω be a complex ellipsoid defined by

$$\Omega = \left\{ z = (z_1, ..., z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \exp\left(-\frac{1}{|z_j|^{s_j}}\right) < e^{-1} \right\}.$$

If $s := \max_{j=1,\dots,n}\{s_j\} < 1$, then under the assumption of ϕ, h and u in Theorem 1.5, we have $u \in \Lambda^{g^\alpha}(\overline{\Omega})$ where $0 < \alpha \le 2$ and $g(t) = \log^{\frac{1}{s}-1} t$.

We organize the paper as follows. In Section 2, we construct a weak Hölder, uniformly and strictly plurisubharmonic defining function via the work of the second author on peak functions [12]. This particular defining function is the crucial point in the establishing the existence of the solution to the complex Monge-Ampère equation. Following the work by Bedford-Taylor [1], we prove Theorem 1.5 in Section 3.

2. The f-property

In this section, under the f-property assumption we construct a uniformly and strictly plurisubharmonic defining function with g^2-Hölder, where g defined in the following theorem.

Theorem 2.1. *Let f be as in Definition 1.1 with $g(t)^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$. Assume that Ω is a bounded, pseudoconvex domain admitting the f-property. Then there exists a strictly plurisubharmonic defining function of Ω which belongs to g^2-Hölder space of $\overline{\Omega}$, that means, there is a plurisubharmonic function ρ such that*

(1) *$z \in \Omega$ if and only if $\rho(z) < 0$, $b\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$;*

(2) *$i\partial\bar\partial\rho(X, \bar X) \ge |X|^2$ on Ω in the distribution sense, for any $X \in T^{1,0}\mathbb{C}^n$; and*

(3) *ρ is in the g^2-Hölder space of $\overline{\Omega}$, that is, $|\rho(z) - \rho(z')| \lesssim g(|z - z'|^{-1})^{-2}$ for any $z, z' \in \overline{\Omega}$.*

Remark 2.2. We note that if Ω is strongly pseudoconvex then $f(t) \approx t^{1/2}$ and hence $g(t) \approx t^{1/2}$. In this case, it is easy to choose a defining function satisfying this theorem. So in the following we only consider that Ω is not strongly pseudoconvex, in this case, we can assume that there exists an $\epsilon > 0$ so that $\frac{f(t)}{t^{1/2-\epsilon}}$ is decreasing on $(1, +\infty)$.

The proof of Theorem 2.1 is based on the following result about the existence of a family of plurisubhamonic peak functions which was recently proven by Khanh [12].

Theorem 2.3. *Under the assumptions of Theorem 2.1, for any $\zeta \in b\Omega$, there exists a C^2 plurisuhharmonic function ψ_ζ on Ω which is continuous on $\overline{\Omega}$ and peaks at ζ (that means, $\psi_\zeta(z) < 0$ for all $z \in \overline{\Omega} \setminus \{\zeta\}$ and $\psi_\zeta(\zeta) = 0$). Moreover, there are some positive constants c_1, c_2 and c_3 such that the following hold for any constant $0 < \eta < 1$:*

(1) $|\psi_\zeta(z) - \psi_\zeta(z')| \le c_1|z - z'|^\eta$ *for any* $z, z' \in \overline{\Omega}$*; and*

(2) $g\big((-\psi_\zeta(z))^{-1/\eta}\big) \le c_2|z - \zeta|^{-1}$ *for any* $z \in \overline{\Omega} \setminus \{\zeta\}$.

Before giving the proof of Theorem 2.1, we need the following technical lemma

Lemma 2.4. *Let g and η be in Theorem 2.3. For $\delta \in (0,1)$, let $\omega(\delta) := g(\delta^{-\frac{1}{\eta}})^{-2}$. Then we have*

(i) *ω is increasing function on $(0,1)$ and $\lim_{\delta\to 0^+} \omega(\delta) = 0$;*

(ii) *for a suitable choice of $\eta > 0$, ω is concave downward on $(0,1)$, i.e., $\ddot{\omega}(\delta) \le 0$ for $\delta \in (0,1)$;*

(iii) *the inequality*

$$|\omega(\delta) - \omega(\delta')| \le \omega(|\delta - \delta'|)$$

holds for any $\delta, \delta' \in (0,1)$; and

(iv) *for a constant $c > 0$, there is $c' > 0$ such that $\omega(c\delta) \le c'\omega(\delta)$ for $\delta \in (0,1)$.*

Proof. We first investigate the function g. By the definition of g, i.e., $\frac{1}{g(t)} := \int_t^\infty \frac{da}{af(a)} < \infty$, we have

$$\frac{\dot{g}(t)}{g(t)} = \frac{g(t)}{tf(t)}, \tag{2.1}$$

and

$$\frac{\ddot{g}(t)}{\dot{g}(t)} = \frac{2\dot{g}(t)}{g(t)} - \frac{1}{t} - \frac{\dot{f}(t)}{f(t)}. \tag{2.2}$$

By Remark 2.2, there exists an $\epsilon > 0$ so that $\frac{f(t)}{t^{\frac{1}{2}-\epsilon}}$ is decreasing on $(1, \infty)$, we obtain $\frac{t\dot{f}(t)}{f(t)} \leq \frac{1}{2} - \epsilon$ for any $t \in (0, \infty)$. We also have

$$\frac{f(t)}{g(t)} = f(t)\int_t^\infty \frac{da}{af(a)} = f(t)\int_t^\infty \frac{a^{1/2}}{f(a)} \cdot \frac{da}{a^{3/2}} \geq f(t)\frac{t^{1/2}}{f(t)}\int_t^\infty \frac{da}{a^{3/2}} = 2,$$

for any $t \in (1, \infty)$. Then

$$\frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)} \leq 1 - \epsilon, \quad \text{for any } t \in (1, \infty). \tag{2.3}$$

Now we prove this lemma. The proof of (i) immediately follows by the the first derivative of ω

$$\dot{\omega}(\delta) = \frac{2}{\eta}\delta^{-\frac{1}{\eta}-1}\dot{g}(\delta^{-\frac{1}{\eta}})g^{-3}(\delta^{-\frac{1}{\eta}}) \geq 0.$$

For (ii), we have

(2.4)

$$\begin{aligned}\ddot{\omega}(\delta) &= -\left(\frac{2}{\eta^2}\delta^{-\frac{1}{\eta}-2}\dot{g}(\delta^{-\frac{1}{\eta}})g^{-3}(\delta^{-\frac{1}{\eta}})\right)\left[\eta + 1 + \frac{\delta^{-\frac{1}{\eta}}\ddot{g}(\delta^{-\frac{1}{\eta}})}{\dot{g}(\delta^{-\frac{1}{\eta}})} - 3\frac{\delta^{-\frac{1}{\eta}}\dot{g}(\delta^{-\frac{1}{\eta}})}{g(\delta^{-\frac{1}{\eta}})}\right] \\ &= -\left(\frac{2}{\eta^2}\delta^{-\frac{1}{\eta}-2}\dot{g}(\delta^{-\frac{1}{\eta}})g^{-3}(\delta^{-\frac{1}{\eta}})\right)\left[\eta - \frac{g(\delta^{-\frac{1}{\eta}})}{f(\delta^{-\frac{1}{\eta}})} - \frac{\delta^{-\frac{1}{\eta}}\dot{f}(\delta^{-\frac{1}{\eta}})}{f(\delta^{-\frac{1}{\eta}})}\right]\end{aligned}$$

where the second equality follows from (2.2). From (2.3) there is a constant $\eta < 1$ such that the bracket term $[\ldots]$ in the last line of (2.4) is positive, particularly we choose $\eta = \frac{1}{2}\left(1 + \sup_{t\in(1,\infty)}\left(\frac{g(t)}{f(t)} + \frac{t\dot{f}(t)}{f(t)}\right)\right) \leq 1 - \frac{\epsilon}{2} < 1$. Therefore, $\ddot{\omega} \leq 0$, i.e., ω is concave downward.

Now we prove that $|\omega(t) - \omega(s)| \leq \omega(|t - s|)$ for any $t, s \in (0, \delta)$. Assume $t \geq s$ for some fixed $s \in [0, \delta)$ and set $k(t) := \omega(t) - \omega(s) - \omega(t - s)$. Since ω is concave downward, $\dot{k}(t) = \dot{\omega}(t) - \dot{\omega}(t - s) \leq 0$. That means k is decreasing, so we obtain $k(t) \leq k(s) = 0$. This completes the proof of (iii).

For the inequality (iv), we notice that if $c \leq 1$ then $\omega(c\delta) \leq \omega(\delta)$ since ω is increasing. Otherwise, if $c > 1$ we use the fact that $\frac{g(t)}{t^{1/2}}$ is decreasing for large t (this is obtained from $\frac{t\dot{g}(t)}{g(t)} = \frac{g(t)}{f(t)} \le \frac{1}{2}$). This implies $\left(\delta^{\frac{1}{2\eta}} g(\delta^{-\frac{1}{\eta}})\right)^{-1}$ is decreasing for small δ , and hence

$$\begin{aligned}\omega(c\delta) = \left(g((c\delta)^{-\frac{1}{\eta}})\right)^{-2} &= (c\delta)^{\frac{1}{\eta}} \left((c\delta)^{\frac{1}{2\eta}} g((c\delta)^{-\frac{1}{\eta}})\right)^{-2} \\ &\le (c\delta)^{\frac{1}{\eta}} \left(\delta^{\frac{1}{2\eta}} g(\delta^{-\frac{1}{\eta}})\right)^{-2} = c^{\frac{1}{\eta}} \omega(\delta).\end{aligned}$$

This completes the proof of Lemma 2.4. □

Now, we will prove the aim of this section. *Proof of Theorem 2.1.* Fix $\zeta \in b\Omega$, we define

$$\rho_\zeta(z) := -2c_2^2 \omega\left(-\psi_\zeta(z)\right) + |z-\zeta|^2,$$

where $\psi_\zeta(.)$ and c_2 are as in Theorem 2.3. We will show that the function $\rho_\zeta(z)$ satisfies the following properties:

(1) $\rho_\zeta(z) < 0$, for $z \in \Omega$, $\rho_\zeta(\zeta) = 0$;

(2) $\rho_\zeta \in C^2(\Omega)$, $i\partial\bar\partial\rho_\zeta(X, \bar X) \ge |X|^2$ on Ω, and $X \in T^{1,0}\mathbb{C}^n$; and

(3) ρ_ζ is in the g^2-Hölder space of $\overline{\Omega}$.

Proof of (1). From the definition of ω and (2) in Theorem 2.3, we have

$$\omega(-\psi_\zeta(z)) = g\left((-\psi_\zeta(z))^{-\frac{1}{\eta}}\right)^{-2} \ge \frac{1}{c_2^2}|z-\zeta|^2. \tag{2.5}$$

Hence,

$$\rho_\zeta(z) = -2c_2^2\omega\left(-\psi_\zeta(z)\right) + |z-\zeta|^2 \le -2|z-\zeta|^2 + |z-\zeta|^2 < 0,$$

where $\zeta \in b\Omega$, and $z \in \Omega$. Moreover, since $\psi_\zeta(\zeta) = 0$ and $\omega(0) = 0$, it follows that $\rho_\zeta(\zeta) = 0$ for any $\zeta \in b\Omega$.

Proof of (2). Fix $\zeta \in b\Omega$, the Levi form of $\omega(-\psi_\zeta)$ on Ω is

$$i\partial\bar\partial\omega(-\psi_\zeta)(X, \bar X) = \dot\omega i\partial\bar\partial\psi_\zeta(X, \bar X) - \ddot\omega|X\psi_\zeta|^2 \ge 0, \tag{2.6}$$

where the inequality follows from Lemma 2.4(i) and (ii).

Proof of (3). From Lemma 2.4(iii), we have

$$\begin{aligned}(2.7)\qquad \big|\omega(-\psi_\zeta(z))-\omega(-\psi_\zeta(z'))\big| &\le \omega\left(|\psi_\zeta(z)-\psi_\zeta(z')|\right)\\ &\le \omega(c|z-z'|^\eta)\\ &\le c'\omega(|z-z'|^\eta)=c'g(|z-z'|^{-1})^{-2}.\end{aligned}$$

Here the inequalities are obtained from Theorem 2.3(1) and Lemma 2.4(iii)–(iv).

On the other hand, since Ω is bounded and $g(t)\lesssim t^{\frac{1}{2}}$, we can show that

$$(2.8)\qquad ||z-\zeta|^2-|z'-\zeta|^2|\lesssim|z-z'|\lesssim g\big(|z-z'|^{-1}\big)^{-2}.$$

The inequalities (2.7) and (2.8) verify that $\rho_\zeta(z)\in\Lambda^{g^2}(\overline{\Omega})$ for uniformly in $\zeta\in b\Omega$.

Now, we are ready to prove Theorem 2.1. We define

$$\rho(z)=\sup_{\zeta\in b\Omega}\rho_\zeta(z).$$

The properties of ρ_z imply that the function ρ satisfies (1) of the conclusion and is plurisubharmonic in Ω as a consequence of well-known result by Lelong [17]. Moreover, since $g(0)=0$ and $g:[0,\infty]\to[0,\infty]$, ρ is also g^2-Hölder continuous in $\overline{\Omega}$ – this follows from the theory of modulus of continuity, the superior envelope of g^2-Hölder continuous is g^2-Hölder continuous. Finally, the second property of each ρ_w shows that , in the distribution sense, we have

$$(2.9)\qquad i\partial\bar\partial\rho(X,\bar X)\ge|X|^2,\quad\text{for any } X\in T^{1,0}\mathbb{C}^n.$$

This completes the proof of Theorem 2.1. □

3. Proof of Theorem 1.5

The proof of Theorem 1.5 follows immediately from Theorems 2.1 and 3.1.

Theorem 3.1. *Let Ω be a bounded, pseudoconvex domain. Assume that there is a uniformly and strictly plurisubharmonic defining function ρ of Ω such that $\rho\in\Lambda^{g^2}(\bar\Omega)$. If $0<\alpha\le 2$, $\phi\in\Lambda^{t^\alpha}(b\Omega)$, and $h\ge 0$ on Ω with $h^{\frac{1}{n}}\in\Lambda^{g^\alpha}(\Omega)$, then the Dirichlet problem for the complex Monge-Ampère equation* (1.2) *has a unique plurisubharmonic solution $u\in\Lambda^{g^\alpha}(\overline{\Omega})$.*

Let Ω be a bounded open set in $\mathbb{C}^n$ and $\mathcal{P}(\Omega)$ denote the space of plurisubharmonic functions on Ω. The proof of Theorem 3.1 is adapted from the argument given by Bedford and Taylor [1, Theorem 6.2] for weakly pseudoconvex domains. Based on the approach in [1], we need the following proposition.

Proposition 3.2. *Let Ω be a bounded, pseudoconvex domain. Assume that there is a strictly plurisubharmonic defining function ρ of Ω such that $\rho \in \Lambda^{g^2}(\bar{\Omega})$. Let $0 < \alpha \leq 2$, and $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and let $h \geq 0$ with $h^{1/n} \in \Lambda^{g^\alpha}(\Omega)$. Then, for each $\zeta \in b\Omega$, there exists $v_\zeta \in \Lambda^{g^\alpha}(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ such that*

(i) *$v_\zeta(z) \leq \phi(z)$ for all $z \in b\Omega$, and $v_\zeta(\zeta) = \phi(\zeta)$,*

(ii) *$\|v_\zeta\|_{\Lambda^{g^\alpha}(\overline{\Omega})} \leq C_0$,*

(iii) *$det\ (H(v_\zeta)(z)) \geq h(z)$,*

where C_0 is a positive constant depending only on Ω and $\|\phi\|_{\Lambda^{t^\alpha}(b\Omega)}$.

Proof. For each $\zeta \in b\Omega$, we may choose the family $\{v_\zeta\}$ by two different ways regarding the value of α:

Case 1: if $0 < \alpha \leq 1$ then we choose

$$v_\zeta(z) = \phi(\zeta) - K[-2\rho(z) + |z-\zeta|^2]^{\frac{\alpha}{2}}, \quad z \in \overline{\Omega};$$

Case 2: if $1 < \alpha \leq 2$ then we choose

$$v_\zeta(z) = \phi(\zeta) - \sum_{j=1}^{n} 2\mathrm{Re}\frac{\partial\phi(\zeta)}{\partial\zeta_j}(z_j - \zeta_j) - K[-2\rho(z) + |z-\zeta|^2]^{\frac{\alpha}{2}}, \quad z \in \overline{\Omega};$$

where ρ is defined by Theorem 2.1, and K will be chosen step by step later.

It is easy to see that $v_\zeta(\zeta) = \phi(\zeta)$ for any $\zeta \in b\Omega$ in both cases. Moreover, choosing K such that $K \geq \|\phi\|_{\Lambda^{t^\alpha}}$, for all $z \in b\Omega$ we have in Case 1:

$$v_\zeta(z) \leq \phi(\zeta) - \|\phi\|_{\Lambda^{t^\alpha}}|z-\zeta|^\alpha \leq \phi(z);$$

and in Case 2:

$$\begin{aligned}
v_\zeta(z) &\le \phi(\zeta) - \sum_{j=1}^{n} 2\mathrm{Re}\frac{\partial\phi(\zeta)}{\partial\zeta_j}(z_j-\zeta_j) - \|\phi\|_{\Lambda^{t^\alpha}}|z-\zeta|^\alpha \\
&\le \phi(z) + \sum_{j=1}^{n} 2\mathrm{Re}\frac{\partial\phi(\tau\zeta+(1-\tau)z)}{\partial\zeta_j}(z_j-\zeta_j) \\
&\quad - \sum_{j=1}^{n} 2\mathrm{Re}\frac{\partial\phi(\zeta)}{\partial\zeta_j}(z_j-\zeta_j) - \|\phi\|_{\Lambda^{t^\alpha}}|z-\zeta|^\alpha \\
&\le \phi(z) + \|D\phi\|_{\Lambda^{t^{\alpha-1}}}|z-\zeta|^{\alpha-1}\cdot|z-\zeta| - \|\phi\|_{\Lambda^{t^\alpha}}|z-\zeta|^\alpha \\
&\le \phi(z).
\end{aligned}$$

This proves (i).

For the proof of (ii), in both cases we have the following estimates

$$\begin{aligned}
&|v_\zeta(z) - v_\zeta(z')| \\
&\le K\left|[-2\rho(z)+|z-\zeta|^2]^{\frac{\alpha}{2}} - [-2\rho(z')+|z'-\zeta|^2]^{\frac{\alpha}{2}}\right| + K|z-z'| \\
&\le K\left|-2\rho(z)+|z-\zeta|^2+2\rho(z')-|z'-\zeta|^2\right|^{\frac{\alpha}{2}} + K|z-z'| \\
&\le K\left[2|\rho(z)-\rho(z')| + \left||z-\zeta|^2-|z'-\zeta|^2\right|\right]^{\frac{\alpha}{2}} + |z-z'| \\
&\lesssim g^{-\alpha}(|z-z'|^{-1})
\end{aligned} \tag{3.1}$$

Here, the first inequality follows by the fact that $|\delta^{\frac{\alpha}{2}} - \eta^{\frac{\alpha}{2}}| \le |\delta-\eta|^{\frac{\alpha}{2}}$ for all δ,η small and $0<\alpha\le 2$; the last inequality follows by Theorem 2.1,(2.8) and $g(t)\le t^{1/2}\le t^{1/\alpha}$ for large t. This implies $v_\zeta \in \Lambda^{g^\alpha}(\overline{\Omega})$ for all $\zeta\in b\Omega$. Moreover $\|v_\zeta\|_{\Lambda^{g^\alpha}(\overline{\Omega})}$ is independent on ζ.

To establish (iii), we compute $(v_\zeta)_{ij}$ on Ω. In both cases,

$$\begin{aligned}
(v_\zeta(z))_{ij} = K\frac{\alpha}{2}&(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-2} \\
&\cdot\Big[(-2\rho(z)+|z-\zeta|^2)(2\rho(z)_{ij}-\delta_{ij}) \\
&\quad+\left(1-\frac{\alpha}{2}\right)(-2\rho_i+\bar z_i-\bar\zeta_i)\overline{(-2\rho_j+\bar z_j-\bar\zeta_j)}\Big].
\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned} i\partial\bar{\partial}v_\zeta(X,X) &\geq K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-1}(2i\partial\bar{\partial}\rho(X,X)-|X|^2)\\ &\geq K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-1}|X|^2, \end{aligned}$$

for any $X \in T^{1,0}\mathbb{C}^n$. Here the last inequality follows from Theorem 2.1(2). Thus v_ζ is plurisubharmonic and furthermore we obtain

$$\det[(v_\zeta)_{ij}](z) \geq \left[K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\left(\frac{\alpha}{2}-1\right)}\right]^n. \tag{3.3}$$

Now, since $0<\alpha\leq 2$ we choose

$$K \geq \max\left\{\frac{2}{\alpha}\max_{z\in\overline{\Omega},\zeta\in b\Omega}(-2\rho(z)+|z-\zeta|^2)^{1-\frac{\alpha}{2}}\|h^{1/n}\|_{L^\infty(\Omega)}, \|\phi\|_{\Lambda^{t^\alpha}}\right\}.$$

Then

$$\det[(v_\zeta)_{ij}](z) \geq \|h^{1/n}\|^n_{L^\infty(\Omega)} \geq (h^{1/n}(z))^n = h(z), \tag{3.4}$$

for all $z\in\Omega$, and $\zeta\in b\Omega$. This completes the proof of Proposition 3.2. $\square$

Before to give a proof of Theorem 3.1, we recall the existence theorem for the problem (1.1) by Bedford and Taylor [1, Theorem 8.3, page 42].

Theorem 3.3 (Bedford-Taylor [1]). *Let Ω be a bounded open set in $\mathbb{C}^n$. Let $\phi\in C(b\Omega)$ and $0\leq h\in C(\Omega)$. If the Perron-Bremerman family denoted by*

$$\mathcal{B}(\phi,h)=\Big\{v\in\mathcal{P}(\Omega)\cap C(\Omega):\det[(v)_{ij}]\geq h, \\ \limsup_{z\to z_0} v(z)\leq\phi(z_0), \textit{ for all } z_0\in b\Omega\Big\},$$

is non-empty, and its upper envelope

$$u=\sup\{v : v\in\mathcal{B}(\phi,h)\} \tag{3.5}$$

is continuous on $\bar{\Omega}$ with $u=\phi$ on $b\Omega$, then u is a solution to the Dirichlet problem (1.1).

Proof of Theorem 3.1. First, we see that the set $\mathcal{B}(\phi,h)$ is non-empty, in particular, it contains the family of $\{v_\zeta\}_{\zeta\in b\Omega}$ in Proposition 3.2. The proof of this theorem will be completed if the upper envelope defined in (3.5) has the properties

(1) $u(\zeta) = \phi(\zeta)$ for all $\zeta \in b\Omega$;

(2) $u \in \Lambda^{g^\alpha}(\bar{\Omega})$.

We note that the uniqueness of solution follows from the Minimum Principle (cf. [1, Theorem A]).

Next, we define another upper envelope, for each $z \in \overline{\Omega}$,

$$v(z) := \sup_{\zeta \in b\Omega} \{v_\zeta(z)\}.$$

By the first property of $\{v_\zeta\}$ in Proposition 3.2, we have

$$\begin{aligned} v(\zeta) \geq v_\zeta(\zeta) = \phi(\zeta), &\quad \text{for all } \zeta \in b\Omega, \\ v(z) \leq \phi(z), &\quad \text{for all } z \in b\Omega\ , \end{aligned} \tag{3.6}$$

and so $v = \phi$ on $b\Omega$.

From the second property in Proposition 3.2, we have

$$|v_\zeta(z) - v_\zeta(z')| \leq C_0 (g^\alpha(|z-z'|^{-1}))^{-1}, \quad \text{for all } z, z' \in \bar{\Omega}.$$

Notice that C_0 is independent on ζ so taking the supremum in ζ, the theory of the modulus of continuity again implies that

$$|v(z) - v(z')| \leq C_0 (g^\alpha(|z-z'|^{-1}))^{-1}, \quad \text{for all } z, z' \in \bar{\Omega}.$$

By Proposition 2.8 in [1], the following inequality holds

$$\det[(v)_{ij}](z) \geq \inf_{\zeta \in b\Omega} \{\det[(v_\zeta)_{ij}](z)\} \geq h(z), \quad \text{for all } z \in \Omega.$$

Thus, we conclude that $v \in \mathcal{B}(\phi, h) \cap \Lambda^{g^\alpha}(\overline{\Omega})$ and $v(\zeta) = \phi(\zeta)$ for any $\zeta \in b\Omega$.

By a similar construction there exists a plurisuperharmonic function $w \in \Lambda^{g^\alpha}(\bar{\Omega})$ such that $w(\zeta) = \phi(\zeta)$ for any $\zeta \in b\Omega$. Thus, $v(z) \leq u(z) \leq w(z)$ for any $z \in \bar{\Omega}$, and hence $u(\zeta) = \phi(\zeta)$ for any $\zeta \in b\Omega$. We also obtain

$$|u(z) - u(\zeta)| \leq \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}(g^\alpha(|z-\zeta|^{-1})^{-1}, \tag{3.7}$$

for any $z \in \bar{\Omega}, \zeta \in b\Omega$. Here, the inequality follows by the facts that $w, v \in \Lambda^{g^\alpha}(\bar{\Omega})$ and $v(\zeta) = u(\zeta) = w(\zeta) = \phi(\zeta)$ for any $\zeta \in \partial\Omega$.

Finally, using the method by Walsh in [23], we will show that (3.7) also holds for all $\zeta \in \Omega$. For any small vector $\tau \in \mathbb{C}^n$, we define

$$V(z,\tau) = \begin{cases} u(z), & \text{if } z+\tau \notin \Omega,\ z \in \overline{\Omega}, \\ \max\{u(z), V_\tau(z)\}, & \text{if } z, z+\tau \in \Omega, \end{cases}$$

where

$$V_\tau(z) = u(z+\tau) + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1})$$

and here

$$K_1 \geq \max_{k\in\{1,\dots,n\}} \binom{n}{k}^{1/k} \|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}(\overline{\Omega})}, \quad K_2 \geq K_1|z|^2,$$

and

$$K_3 \geq \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|w\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}.$$

We will show that $V(z,\tau) \in \mathcal{B}(\phi, h)$. Observe that $V(z,\tau) \in \mathcal{P}(\Omega)$ for all z, τ. Moreover, for $z \in \partial\Omega$ and $z+\tau \in \Omega$, we have

$$\begin{aligned} V_\tau(z) - u(z) &= u(z+\tau) - u(z) + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) \\ &\leq \max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}, \|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\} g^{-\alpha}(|\tau|^{-1}) \\ &\quad + \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) \\ &\leq 0. \end{aligned} \tag{3.8}$$

Here the first inequality follows by (3.7) and the second follows by the choices of K_2 and K_3. This implies that $\limsup_{z\to\zeta} V(z,\tau) \leq \phi(\zeta)$ for all $\zeta \in b\Omega$. For the proof of $\det[V(z,\tau)_{ij}] \geq h(z)$, we need the following lemma.

Lemma 3.4. *Let $(\alpha_{ij}) \geq 0$ and $\beta \in (0, +\infty)$. Then*

$$det[\alpha_{ij} + \beta I] \geq \sum_{k=0}^{n} \beta^k \det(\alpha_{ij})^{(n-k)/n}.$$

Proof of Lemma 3.4. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of (α_{ij}). We have

$$\begin{aligned} \det[\alpha_{ij} + \beta] = \prod_{j=1}^{n} (\lambda_j + \beta) &\geq \sum_{k=0}^{n} \left(\beta^k \prod_{j=k+1}^{n} \lambda_j\right) \\ &\geq \sum_{k=0}^{n} \left(\beta^k \det[\alpha_{ij}]^{(n-k)/n}\right). \end{aligned} \tag{3.9}$$

Here the last inequality follows by

$$\det[\alpha_{ij}] = \prod_{j=1}^{n} \lambda_j \leq \left(\prod_{j=k+1}^{n} \lambda_j \right)^{n/(n-k)}.$$

□

Continuing the proof of Theorem 1.5, for any $z, z+\tau \in \Omega$ we have

$$\begin{aligned}
(3.10) \qquad & \det[(V_\tau(z))_{ij}] \\
&= \det[u_{ij}(z+\tau) + K_1 g^{-\alpha}(|\tau|^{-1}) I] \\
&\geq \det[u_{ij}(z+\tau)] + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k}. \det[u_{ij}(z+\tau)]^{\frac{n-k}{n}} \\
&\geq h(z+\tau) + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k}.(h(z+\tau))^{\frac{n-k}{n}},
\end{aligned}$$

where the first inequality is derived by Lemma 3.4. Since $h^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\Omega)$, we obtain

$$h^{\frac{1}{n}}(z) - h^{\frac{1}{n}}(z+\tau) \leq g^{-\alpha}(|\tau|^{-1}) \|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}}, \quad \text{for any } z, z+\tau \in \Omega,$$

and hence

$$(3.11) \quad h(z) \leq h(z+\tau) + \sum_{k=1}^{n} \binom{n}{k} h(z+\tau)^{(n-k)/n} \left(g^{-\alpha}(|\tau|^{-1}) \|h^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}} \right)^k.$$

Combining (3.10), (3.11) with the choice of K_1, we get

$$\det[(V_\tau)_{ij}](z) \geq h(z), \quad \text{for any} \quad z, z+\tau \in \Omega.$$

We conclude that $V(z,\tau) \in \mathcal{B}(\phi, h)$. It follows that for all $z \in \Omega$, $V(z,\tau) \leq u(z)$. If $z, z+\tau \in \Omega$, this yields

$$\begin{aligned}
(3.12) \quad u(z+\tau) - u(z) &\leq V(\tau, z) - \left(K_1|z|^2 - K_2 - K_3\right) g^{-\alpha}(|\tau|^{-1}) - u(z) \\
&\leq \left(-K_1|z|^2 + K_2 + K_3\right) g^{-\alpha}(|\tau|^{-1}) \\
&\leq (K_2 + K_3)\, g^{-\alpha}(|\tau|^{-1}).
\end{aligned}$$

By reversing the role of z and $z+\tau$, we assert that $u \in \Lambda^{g^\alpha}(\overline{\Omega})$. This completes the proof. □

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References

[1] E. Bedford and B. A. Taylor, *The Dirichlet Problem for a complex Monge-Ampère Equation.* Inventiones Math., **37** (1976), 1–44.

[2] Z. Blocki, *The complex Monge-Ampère operator in hyperconvex domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **23** (1996), 721–747.

[3] L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations II. Complex Monge-Ampère equations and uniformly elliptic equations.* Comm. Pure Appl. Math., **38** (1985), 209–252.

[4] D. Catlin, *Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem.* Ann. of Math. (2), **117**(1) (1983), 147–171.

[5] D. Catlin, *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains.* Ann. of Math. (2), **126**(1) (1987), 131–191.

[6] D. Catlin, *Estimates of invariant metrics on pseudoconvex domains of dimension two.* Math. Z., **200**(3) (1989), 429–466.

[7] D. Coman, *Domains of finite type and Hölder continuity of the Perron-Bremermann function.* Proc. Amer. Math. Soc., **125**(12) (1997), 3569–3574.

[8] J. E. Fornaess and N. Sibony, *Construction of P.S.H. functions on weakly pseudoconvex domains.* Duke Math. J., **58**(3) (1989), 633–655.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order.* Second edition, Springer-Verlag, Berlin, 1983.

[10] T. V. Khanh, *A general method of weights in the $\bar{\partial}$-Neumann problem.* `arxiv:1001.5093v1`, 2010, Ph.D. Thesis.

[11] T. V. Khanh, *Supnorm and f-Hölder estimates for $\bar{\partial}$ on convex domains of general type in $\mathbb{C}^2$.* J. Math. Anal. Appl., **403** (2013), 522–531.

[12] T. V. Khanh, *Lower bounds on the Kobayashi metric near a point of infinite type.* J. Geom. Anal., in press, `DOI:10.1007/s12220-015-9565-y`.

[13] T. V. Khanh and G. Zampieri, *Regularity of the $\overline{\partial}$-Neumann problem at point of infinite type.* J. Funct. Anal., **259**(11) (2010), 2760–2775.

[14] T. V. Khanh and G. Zampieri, *Necessary geometric and analytic conditions for general estimates in the $\bar{\partial}$-Neumann problem.* Invent. Math., **188**(3) (2012), 729–750.

[15] M. Klimek, *Pluripotential theory.* London. Math. Soc. Monos., New series 6, 1991.

[16] J. J. Kohn, *Superlogarithmic estimates on pseudoconvex domains and CR manifolds.* Ann. of Math. (2), **156**(1) (2002), 213–248.

[17] P. Lelong, *Plurisubharmonic functions and positive differential forms.* New York: Gordon and Breach, 1969.

[18] S. Y. Li, *On the existence and regularity of Dirichlet problem for complex Monge-Ampere equations on weakly pseudoconvex domains.* Calc. Var., **20** (2004), 119-132.

[19] J. D. McNeal, *Local geometry of decoupled pseudoconvex domains.* In "Complex analysis" (Wuppertal, 1991), Aspects Math., E17, pages 223–230, Vieweg, Braunschweig (1991).

[20] J. D. McNeal, *Convex domains of finite type.* J. Funct. Anal., **108**(2) (1992), 361–373.

[21] D. H. Phong and J. Sturm, *The Dirichlet problem for degenerate complex Monge-Ampère equations.* Comm. Anal. Geom., **18**(1) (2010), 145–170.

[22] D. H. Phong, J. Song and J. Sturm, *Complex Monge-Ampère equations.* Surv. Differ. Geom., **17** (2012), 327–410.

[23] J. B. Walsh, *Continuity of envelopes of plurisubharmonic functions.* J. Math. Mech., **18** (1968), 143–148.

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