Math. Res. Lett.
Volume 22, Number 2, 439–465, 2015

Diagonal solutions to the 2-Toda hierarchy

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Using a combinatorial description of the Bernstein operator and its action on Schur functions, we describe the formal power series solutions to a family of partial differential equations known as the 2-Toda hierarchy. We also characterize diagonal solutions and use this to prove that a special family of formal power series, called content-type series, are solutions to the 2-Toda hierarchy. As examples, we prove that various generating series for permutation factorization problems (including the double Hurwitz problem), correlators for the Schur measure on partitions and the formal character expansion of the HCIZ integral are all solutions to the 2-Toda hierarchy.

1. Introduction

Integrable hierarchies, certain families of partial differential equations, have found a wide range of applications in areas such as random matrices, enumerative combinatorics and stochastic models. Some of the most prominent examples of this are the numerous problems whose generating function is a solution to the 2-Toda hierarchy (and the related KP hierarchy). Curiously, most of these series can be written as (diagonal) formal power series of the form

$$\tau_n = \sum_{\lambda \in \mathcal{P}} g_\lambda(n) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}),$$

where the sum is over all integer partitions, $s_\lambda(\mathbf{p})$ and $s_\lambda(\mathbf{q})$ are Schur polynomials (see Section 2 for details), and the $g_\lambda(n)$ are coefficients independent of $\mathbf{p}$ and $\mathbf{q}$. For example, the unitary matrix model has a formal solution as above with $g_\lambda(n) = 0$ if λ has more than n parts and is 1 otherwise. Additionally, many of the formal series solutions which appear in known examples have a further restricted form in which the $g_\lambda(n)$ are written as products over the cells in λ. This special type of solution appears in various enumerative problems such as those considered in [15] where the solutions were called content-type series. For additional details see Section 3.

The connection between integrable hierarchies and random matrix models has been known for some time and has been used to derive properties of various statistical quantities of interest. Some examples of this include the results of Adler and Moerbeke [1, 2] on the spectrum of random matrices as well as the results of Orlov and Shcherbin [25–28] relating various iterated integrals to the KP and 2-Toda hierarchies. Related to this is the Fredholm determinant approach to random matrix models used, for example, by Tracy and Widom [31–33]. Although the precise relationship between the Fredholm determinant approach and the integrable hierarchies discussed here is not fully known, some progress has been made [30].

In addition to the connection between integrable hierarchies and random matrix models, it is known that many random matrix models satisfy additional linear partial differential equations called the Virasoro constraints [1, 2, 4]. However, not all content-type solutions are known to satisfy Virasoro constraints. For example, the double Hurwitz series gives a content-type solution to the 2-Toda hierarchy [23] (see the examples in Section 4) but currently there are no known Virasoro constraints for this series [5]. One possible approach to the question of Virasoro constraints for content-type series can be found in [3]

One of the more well known applications of integrable hierarchies to enumerative combinatorics is Okounkov's result [23] that the generating series for double Hurwitz numbers satisfies the 2-Toda hierarchy, thus settling a conjecture of Pandharipande [29] concerning Gromov Witten theory of the sphere. This was then used by Kazarian [20] in one of the proofs of Witten's conjecture on the generating function of linear Hodge integrals.

Another application of integrable hierarchies to an enumerative problem is the result of Goulden and Jackson [15] that the number of rooted triangulations (with respect to number of vertices and genus) satisfies a quadratic recurrence. Bender, Gao and Richmond [6] then used this to derive a recurrence for the map asymptotics constant. A related result was obtained by Ercolani [12] using Riemann-Hilbert techniques.

An example of the connection between various stochastic models and integrable hierachies is through the Schur measure introduced by Okounkov [24] and its correlators. The Schur measure is a generalization of the z-measure introduced by Borodin and Olshanksi [8–10] in the context of asymptotic representation theory. In addition, it has been shown [7] that the z-measure encodes a number of stochastic models considered, for example, by Johansson [18].

In this paper we show that the general content-type series mentioned above are formal power series solutions to the 2-Toda hierarchy (for a similar result coming from a different approach see [25–28]). Moreover, we prove a partial converse. That is, we show that if one has a formal power series solution to the 2-Toda hierarchy that also satisfies some constraints on its coefficients then it can be written as a content-type series.

The outline of this paper is as follows. In section 2 we discuss notation and recall some results from previous work[11] concerning the action of the Bernstein operator on Schur polynomials. We then describe the KP hierarchy and the 2-Toda hierarchy, including a characterization of formal power series solutions with respect to their Schur polynomial coefficients.

In section 3 we first specialize the characterization of formal power series solutions of the 2-Toda hierarchy to diagonal solutions. We then introduce the content-type series and, using the specialized characterization, we prove our first main result that the content-type series are solutions to the 2-Toda hierarchy. We finish the section with our second main result, the partial converse.

In the final section we discuss a few different examples of content-type series that appear in the literature. We start by discussing the solution of an enumeration problem which was discussed in [15] concerning certain tuples of permutations. In addition, we recall a specialization of this result which gives an alternate proof of the fact that the double Hurwitz series can be embedded in a solution to the 2-Toda hierarchy. We then discuss the Schur measure on partitions and show that its correlators satisfy the 2-Toda hierarchy. Lastly, we show that the Harish-Chandra Itzykson Zuber integral also satisfies the 2-Toda hierarchy.

2. Background

We begin with some notation for partitions (for additional information see [22]). If $\lambda_1, \dots, \lambda_n$ are integers with $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$ and $\lambda_1 + \cdots + \lambda_n = d$, then $\lambda = (\lambda_1, \dots, \lambda_n)$ is said to be a partition of $|\lambda| := d$ with $\ell(\lambda) := n$ parts. The empty list ϵ of integers is the unique partition with $d = n = 0$. We will use $\mathcal{P}$ to denote the set of all partitions. If λ has f_j parts equal to j for $j = 1, \dots, d$, then we may also write $\lambda = d^{f_d} \cdots 1^{f_1}$ and if $f_j = 1$ for some j we will omit the exponent. Also, Aut λ denotes the set of permutations that fix λ; therefore $|\text{ Aut } \lambda| = \prod_{j \geq 1} f_j!$. We will often identify a partition λ with its diagram, a left justified array of unit squares, called cells, with λ_i cells in the ith row. The conjugate of λ is the partition λ^t whose diagram is the diagram of λ, reflected along the main diagonal. We will use $\square$ to denote a cell and we will write $\square \in \lambda$ to mean that $\square$ is a cell of the partition λ.

Also, for $\Box \in \lambda$ we define the content of the cell $\Box$ by $c(\Box) := j - i$ where j is the column index of $\Box$ and i is the row index.

Now, let $\lambda \in \mathcal{P}$ and $i \geq 1$ be an integer. Let $u_i(\lambda)$ be the unique integer such that

$$\lambda_{u_i(\lambda)} \geq i > \lambda_{u_i(\lambda)+1}.$$

We define

$$\lambda \uparrow i = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{u_i(\lambda)} - 1, i - 1, \lambda_{u_i(\lambda)+1}, \ldots).$$

The operation $\lambda \mapsto \lambda \uparrow i$ can be thought of informally as follows. We add a part of size i to λ such that the result is still a partition and the index of the added part is as large as possible. We then reduce the size of each part not displaced (including the new part) by one. From the definition of $\lambda \uparrow i$, we compute

$$|\lambda \uparrow i| = |\lambda| + i - u_i(\lambda) - 1,$$

and

$$|\lambda| - \ell(\lambda) = |\lambda \uparrow 1| < |\lambda \uparrow 2| < \cdots .$$

Example 2.1. Suppose $\lambda = 754^21$ and that we wish to determine $\lambda \uparrow 4$. The largest index at which 4 can be placed in λ such that the result is still a partition is 5 and so $u_4(\lambda) = 5$. Thus $\lambda \uparrow 4 = 643^31$. Similarly, if we wish to determine $\lambda \uparrow 3$ we have $u_3(\lambda) = 5$ again and so $\lambda \uparrow 3 = 643^221$.

Now suppose that $\lambda = \epsilon$. Then for any $i \geq 1$ we have $u_i(\epsilon) = 1$ and so $\epsilon \uparrow i = i - 1$. If $\lambda = 1^k$ then we have $u_1(1^k) = k + 1$ and so $1^k \uparrow 1 = \epsilon$.

We also define the dual operation. Let $\lambda \in \mathcal{P}$ and $j \geq 1$ be an integer. We define

$$\lambda \downarrow j = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{j-1} + 1, \lambda_{j+1}, \ldots).$$

Informally we think of the operation $\lambda \mapsto \lambda \downarrow j$ as removing the part λ_j and then increasing the size of each part not displaced. ¿From the definition of $\lambda \downarrow j$, we compute

$$|\lambda \downarrow j| = |\lambda| + j - \lambda_j - 1,$$

and

$$|\lambda| - \lambda_1 = |\lambda \downarrow 1| < |\lambda \downarrow 2| < \cdots .$$

Example 2.2. Suppose $\lambda = 643^31$. Then we easily see that $\lambda \downarrow 5 = 754^21$. Similarly, if $\lambda = 643^221$ then $\lambda \downarrow 5 = 754^21$.

Now suppose that $\lambda = \epsilon$. Then for any $j \geq 1$ we see that $\epsilon \downarrow j = 1^{j-1}$. If $\lambda = k$ then it is easily seen that $\lambda \downarrow 1 = \epsilon$.

Additionally, from the definition of $\lambda \uparrow i$ and $\lambda \downarrow j$, we have

$$(\lambda \uparrow i) \downarrow (u_i(\lambda) + 1) = \lambda,$$

and

$$(\lambda \downarrow j) \uparrow (\lambda_j + 1) = \lambda.$$

There is another, more combinatorial, description of the operations $\lambda \mapsto \lambda \uparrow i$ and $\lambda \mapsto \lambda \downarrow j$ which was discussed in [11], however, we shall not have need of it here. It does, however, make the following relationship clear,

$$(\lambda^t) \uparrow i = (\lambda \downarrow i)^t.$$

Throughout this paper we use t, a, b, $\mathbf{p} = (p_1, p_2, \ldots)$, $\mathbf{q} = (q_1, q_2, \ldots)$, $\mathbf{w} = (w_1, w_2, \ldots)$, $\mathbf{z} = (z_1, z_2, \ldots)$, and $\mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots)$ to denote algebraically independent indeterminates. We also write $\mathbf{p} + \mathbf{q}$ to mean the sequence $(p_1 + q_1, p_2 + q_2, \ldots)$. For $\lambda \in \mathcal{P}$ we write $p_\lambda = \prod_{i \geq 1} p_{\lambda_i}$.

For $i \geq 0$ define the polynomials $h_i(\mathbf{p})$ by

$$\sum_{i \geq 0} h_i(\mathbf{p}) t^i = \exp\left(\sum_{k \geq 1} \frac{p_k}{k} t^k \right),$$

and for $i < 0$, $h_i(\mathbf{p}) = 0$.

For $\lambda \in \mathcal{P}$ define the polynomials $s_\lambda(\mathbf{p})$ by

$$s_\lambda(\mathbf{p}) = \det\left(h_{\lambda_i - i + j}(\mathbf{p})\right)_{1 \leq i,j \leq n},$$

where $n \geq \ell(\lambda)$. The $s_\lambda(\mathbf{p})$ are called the Schur polynomials and they form a basis for the ring of formal power series in $\mathbf{p}$.

We introduce the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle s_\lambda(\mathbf{p}), s_\mu(\mathbf{p}) \rangle = \delta_{\lambda,\mu}, \ \forall \lambda, \mu \in \mathcal{P}.$$

For any polynomial $f(\mathbf{p})$ we define the adjoint of multiplication by f, written $f^\perp$, by requiring that for all polynomials $g(\mathbf{p})$ and $h(\mathbf{p})$,

$$\langle f^\perp(\mathbf{p}) g(\mathbf{p}), h(\mathbf{p}) \rangle = \langle g(\mathbf{p}), f(\mathbf{p}) h(\mathbf{p}) \rangle.$$

It can be shown (see [22] Chapter I, Section 5, Exercise 3) that $p_i^\perp = i\frac{\partial}{\partial p_i}$ and that for any polynomial $f(\mathbf{p})$, $f^\perp = f(p_1^\perp, p_2^\perp, \ldots)$. In particular, $f^\perp$ is a differential operator.

Note that if we interpret p_i as the ith power sum symmetric function then the Schur polynomials, s_λ, become the Schur symmetric functions and $\langle \cdot, \cdot \rangle$ becomes the Hall inner product[22].

We now define the Bernstein operator,

$$B(\mathbf{p};t) = \exp\left(\sum_{k\geq 1} \frac{t^k}{k} p_k\right) \exp\left(-\sum_{k\geq 1} \frac{t^{-k}}{k} p_k^\perp\right),$$

and the adjoint Bernstein operator

$$B^\perp(\mathbf{p};t) = \exp\left(-\sum_{k\geq 1} \frac{t^k}{k} p_k\right) \exp\left(\sum_{k\geq 1} \frac{t^{-k}}{k} p_k^\perp\right).$$

In [11] we showed that the Bernstein operator and its adjoint act nicely on Schur polynomials. Specifically, we have the following

Theorem 2.3. *Suppose that $a_\lambda, \lambda \in \mathcal{P}$ are scalars. Then*

$$B(\mathbf{p};t) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda(\mathbf{p}) = \sum_{\beta \in \mathcal{P}} s_\beta(\mathbf{p}) \sum_{k \geq 1} (-1)^{k-1} t^{|\beta| - |\beta\downarrow k|} a_{\beta\downarrow k},$$

and

$$B^\perp(\mathbf{p};t) \sum_{\lambda \in \mathcal{P}} a_\lambda s_\lambda(\mathbf{p}) = \sum_{\alpha \in \mathcal{P}} s_\alpha(\mathbf{p}) \sum_{m \geq 1} (-1)^{|\alpha| - |\alpha\uparrow m| + m - 1} t^{|\alpha| - |\alpha\uparrow m|} a_{\alpha\uparrow m}.$$

In addition to showing that the Bernstein operator acts nicely on Schur polynomials, we also showed that the commutator of $B(\mathbf{p};t)$ and a certain translation operator was simple. Specifically, define

$$\Theta(\mathbf{p},\mathbf{q}) = \exp\left(\sum_{k\geq 1} q_k \frac{\partial}{\partial p_k}\right),$$

and

$$\Gamma(\mathbf{q};t) = \exp\left(\sum_{k\geq 1} \frac{t^i}{i} q_i\right).$$

Using multivariate Taylor series, we see that if $f(\mathbf{p})$ is a formal power series then

$$\Theta(\mathbf{p},\mathbf{q})f(\mathbf{p}) = f(\mathbf{p}+\mathbf{q}).$$

It is not difficult to show that the following relations hold between B and Θ. Note that this result essentially arises in the proof of Theorem 5.3 in [11].

Proposition 2.4. *We have*

$$B(\mathbf{p};t)\Theta(\mathbf{p},\mathbf{q}) = \Gamma(\mathbf{q};t)^{-1}\Theta(\mathbf{p},\mathbf{q})B(\mathbf{p};t),$$

and

$$B^{\perp}(\mathbf{p};t)\Theta(\mathbf{p},\mathbf{q}) = \Gamma(\mathbf{q};t)\Theta(\mathbf{p},\mathbf{q})B^{\perp}(\mathbf{p};t).$$

We can use Theorem 2.3 and Proposition 2.4 to describe various families of partial differential equations. For the purposes of this paper we will focus on the 2-Toda hierarchy although other related integrable hierarchies such as the n-KP hierarchy [19] can be treated similarly. It is also likely that similar results can be obtained in the B-type case using the results in [16], however we have not done so here.

As an easier example, we begin with a description of the KP hierarchy. Suppose that $\tau(\mathbf{p})$ is a formal power series. Then $\tau(\mathbf{p})$ is a solution to the KP hierarchy if and only if

$$[t^{-1}]\left(B(\mathbf{p};t)\tau(\mathbf{p})\right)\left(B^{\perp}(\mathbf{q};t)\tau(\mathbf{q})\right) = 0.$$

Here we use square brackets to denote the coefficient extraction operator.

Theorem 2.5. *Suppose that $a_\lambda, \lambda \in \mathcal{P}$ are scalars and that*

$$\tau(\mathbf{p}) = \sum_{\lambda\in\mathcal{P}} a_\lambda s_\lambda(\mathbf{p}).$$

The following are equivalent.

(i) *The formal power series $\tau(\mathbf{p})$ is a solution to the KP hierarchy.*

(ii) *The formal power series $\tau(\mathbf{p}+\mathbf{q})$ is a solution to the KP hierarchy in the variables $\mathbf{p}$.*

(iii) *For all* $\alpha, \beta \in \mathcal{P}$,

$$\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} a_{\alpha\uparrow i} a_{\beta\downarrow j} = 0,$$

where the sum is over all integers $i, j \geq 1$ *such that* $|\alpha \uparrow i| + |\beta \downarrow j| = |\alpha| + |\beta| + 1$.

(iv) *For all* $\alpha, \beta \in \mathcal{P}$,

$$\sum_{i,j} (-1)^{|\alpha|-|\alpha\uparrow i|+i+j} (s^{\perp}_{\alpha\uparrow i}(\mathbf{p})\tau(\mathbf{p}))(s^{\perp}_{\beta\downarrow j}(\mathbf{p})\tau(\mathbf{p})) = 0,$$

where the sum is over all integers $i, j \geq 1$ *such that* $|\alpha \uparrow i| + |\beta \downarrow j| = |\alpha| + |\beta| + 1$.

Proof. The fact that (ii) implies (i) follows immediately by setting $q_i = 0$ for all $i \geq 1$. The fact that (i) implies (ii) follows from the definition of the KP hierarchy and Proposition 2.4. That (i) and (iii) are equivalent follows by taking coefficients and using Theorem 2.3. Lastly, that (ii) and (iv) are equivalent follows by coefficient extraction, Theorem 2.3 and the fact that $[s_\lambda(\mathbf{p})]\tau(\mathbf{p}+\mathbf{q}) = s^{\perp}_{\lambda}(\mathbf{p})\tau(\mathbf{p})$ (since the Schur polynomials are orthonormal). □

For the 2-Toda hierarchy we have a similar result, however, it is a little more technical since instead of having a single family of indeterminates, we have two families of indeterminates and a discrete parameter.

A sequence of formal power series $\{\tau_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy if and only if for all $k, m \in \mathbb{Z}$,

$$\begin{aligned}&[t^{k-m}]\left(B(\mathbf{p};t)\tau_m(\mathbf{p},\mathbf{q})\right)\left(B^{\perp}(\mathbf{w};t)\tau_{k+1}(\mathbf{w},\mathbf{z})\right)\\ &= [t^{m-k}]\left(B^{\perp}(\mathbf{q};t)\tau_{m+1}(\mathbf{p},\mathbf{q})\right)\left(B(\mathbf{z};t)\tau_k(\mathbf{w};\mathbf{z})\right).\end{aligned}$$

Theorem 2.6. *Suppose that* $a^{\lambda}_{\mu}(n), \lambda, \mu \in \mathcal{P}$, $n \in \mathbb{Z}$ *are scalars and that for all* $n \in \mathbb{Z}$,

$$\tau_n(\mathbf{p},\mathbf{q}) = \sum_{\lambda,\mu\in\mathcal{P}} a^{\lambda}_{\mu}(n) s_\lambda(\mathbf{p}) s_\mu(\mathbf{q}).$$

The following are equivalent.

(i) *The sequence of formal power series* $\{\tau_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ *is a solution to the 2-Toda hierarchy.*

(ii) *The sequence of formal power series* $\{\tau_n(\mathbf{p}+\mathbf{w},\mathbf{q}+\mathbf{z})\}_{n\in\mathbb{Z}}$ *is a solution to the 2-Toda hierarchy in the variables* $\mathbf{p}$ *and* $\mathbf{q}$.

(iii) *For all* $m,k\in\mathbb{Z}$ *and* $\alpha,\beta,\lambda,\mu\in\mathcal{P}$,

$$\begin{aligned}&\sum_{i,j}(-1)^{|\alpha|-|\alpha\uparrow j|+i+j}a_{\mu}^{\lambda\downarrow i}(m)a_{\beta}^{\alpha\uparrow j}(k+1)\\&=\sum_{s,t}(-1)^{|\mu|+|\mu\uparrow s|+s+t}a_{\mu\uparrow s}^{\lambda}(m+1)a_{\beta\downarrow t}^{\alpha}(k),\end{aligned}$$

where the first sum is over integers $i,j\geq 1$ *such that* $|\lambda\downarrow i|+|\alpha\uparrow j|=|\lambda|+|\alpha|+m-k$ *and the second sum is over integers* $s,t\geq 1$ *such that* $|\mu\uparrow s|+|\beta\downarrow t|=|\mu|+|\beta|+k-m$.

(iv) *for all* $m,k\in\mathbb{Z}$ *and* $\alpha,\beta,\lambda,\mu\in\mathcal{P}$,

$$\begin{aligned}&\sum_{i,j}(-1)^{|\alpha|-|\alpha\uparrow j|+i+j}\left(s_{\lambda\downarrow i}^{\perp}(\mathbf{p})s_{\mu}^{\perp}(\mathbf{q})\tau_m(\mathbf{p},\mathbf{q})\right)\left(s_{\alpha\uparrow j}^{\perp}(\mathbf{p})s_{\beta}^{\perp}(\mathbf{q})\tau_{k+1}(\mathbf{p},\mathbf{q})\right)\\&=\sum_{s,t}(-1)^{|\mu|-|\mu\uparrow s|+s+t}\left(s_{\lambda}^{\perp}(\mathbf{p})s_{\mu\uparrow s}^{\perp}(\mathbf{q})\tau_{m+1}(\mathbf{p},\mathbf{q})\right)\left(s_{\alpha}^{\perp}(\mathbf{p})s_{\beta\downarrow t}^{\perp}(\mathbf{q})\tau_{k}(\mathbf{p},\mathbf{q})\right),\end{aligned}$$

where the first sum is over integers $i,j\geq 1$ *such that* $|\lambda\downarrow i|+|\alpha\uparrow j|=|\lambda|+|\alpha|+m-k$ *and the second sum is over integers* $s,t\geq 1$ *such that* $|\mu\uparrow s|+|\beta\downarrow t|=|\mu|+|\beta|+k-m$.

Proof. The proof is essentially the same as Theorem 2.5. ☐

Example 2.7. In the case of the 2-Toda hierarchy, choose $k=m-1,\alpha=\lambda=\beta=\epsilon$ and $\mu=1$. We compute $\epsilon\uparrow 1=\epsilon$, $\epsilon\uparrow 2=1$, $\epsilon\downarrow 1=\epsilon$, $\epsilon\downarrow 2=1$, $1\uparrow 1=\epsilon$ and $1\uparrow 2=1^2$. The only solutions (i,j) to $|\lambda\downarrow i|+|\alpha\uparrow j|=|\lambda|+|\alpha|+1$ are $(1,2)$ and $(2,1)$. Similarly, the only solution (s,t) to $|\mu\uparrow s|+|\beta\downarrow t|=|\mu|+|\beta|-1$ is $(1,1)$. Theorem 2.6(iii) then gives

$$a_1^{\epsilon}(m)a_{\epsilon}^{1}(m)-a_1^{1}(m)a_{\epsilon}^{\epsilon}(m)=-a_{\epsilon}^{\epsilon}(m+1)a_{\epsilon}^{\epsilon}(m-1)$$

as one of the coefficient constraints. Similarly, Theorem 2.6(iv) gives

$$\begin{aligned}&\left(s_{\epsilon}^{\perp}(\mathbf{p})s_{\epsilon}^{\perp}(\mathbf{q})\tau_{m+1}\right)\left(s_{\epsilon}^{\perp}(\mathbf{p})s_{\epsilon}^{\perp}(\mathbf{q})\tau_{m-1}\right)+\left(s_{1}^{\perp}(\mathbf{p})s_{\epsilon}^{\perp}(\mathbf{q})\tau_{m}\right)\left(s_{\epsilon}^{\perp}(\mathbf{p})s_{1}^{\perp}(\mathbf{q})\tau_{m}\right)\\&=\left(s_{1}^{\perp}(\mathbf{p})s_{1}^{\perp}(\mathbf{q})\tau_{m}\right)\left(s_{\epsilon}^{\perp}(\mathbf{p})s_{\epsilon}^{\perp}(\mathbf{q})\tau_{m}\right)\end{aligned}$$

as one of the partial differential equations in the 2-Toda hierarchy. Using the fact that $s_\epsilon^\perp(\mathbf{p}) = 1$ and $s_1^\perp(\mathbf{p}) = \frac{\partial}{\partial p_1}$, this gives

$$\tau_{m+1}\tau_{m-1} + \left(\frac{\partial}{\partial p_1}\tau_m\right)\left(\frac{\partial}{\partial q_1}\tau_m\right) = \tau_m\left(\frac{\partial^2}{\partial p_1 \partial q_1}\tau_m\right),$$

which can be further simplified to

$$\frac{\partial^2}{\partial p_1 \partial q_1}\log \tau_m = \frac{\tau_{m+1}\tau_{m-1}}{\tau_m^2}.$$

This last equation is called the 2-Toda equation.

We will now take this opportunity to discuss some of the integrable hierarchies that reside within of the 2-Toda hierarchy.

Theorem 2.8. *Suppose the sequence of formal power series $\{\tau_m(\mathbf{p},\mathbf{q})\}_{m\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy. Then for any $m \in \mathbb{Z}$, $r > 0$ and $\lambda, \alpha \in \mathcal{P}$, we have*

$$\sum_{i,j}(-1)^{|\alpha|+|\alpha\uparrow j|+i+j}\left(s^\perp_{\lambda\downarrow i}(\mathbf{p})\tau_m(\mathbf{p},\mathbf{q})\right)\left(s^\perp_{\alpha\uparrow j}(\mathbf{p})\tau_{m-r+1}(\mathbf{p},\mathbf{q})\right) = 0,$$

where the sum is over $i, j \geq 1$ such that $|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + r$. By symmetry, this also implies that

$$\sum_{i,j}(-1)^{|\alpha|+|\alpha\uparrow j|+i+j}\left(s^\perp_{\lambda\downarrow i}(\mathbf{q})\tau_m(\mathbf{p},\mathbf{q})\right)\left(s^\perp_{\alpha\uparrow j}(\mathbf{q})\tau_{m-r+1}(\mathbf{p},\mathbf{q})\right) = 0,$$

where the sum is over $i, j \geq 1$ such that $|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + r$.

Proof. If we set $k = m - r$ and $\mu = \beta = \epsilon$ in Theorem 2.6(iv) then the right hand side of the equation is a sum over integers $s, t \geq 1$ such that

$$|\mu \uparrow s| + |\beta \downarrow t| = |\mu| + |\beta| + k - m = -r < 0,$$

and so there can be no solutions $s, t \geq 1$. The result then follows. □

In particular, if we set $r = 1$ in Theorem 2.8 then the resulting family of partial differential equations are those found in Theorem 2.5(iv). Theorem 2.8 thus implies the well known result that if $\{\tau_m(\mathbf{p},\mathbf{q})\}_{m\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy, then each τ_m is a solution to the KP hierarchy in $\mathbf{p}$ and $\mathbf{q}$ independently.

3. Diagonal solutions

For the remainder of this paper we will assume that the sequence of formal power series $\{\tau_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ is diagonal. That is, instead of viewing each $\tau_n(\mathbf{p},\mathbf{q})$ as a sum over pairs of partitions as in Theorem 2.6, we consider the restricted sums which arise when $\lambda=\mu$. In particular, we require that for each $n\in\mathbb{Z}$,

$$\tau_n(\mathbf{p},\mathbf{q})=\sum_{\lambda\in\mathcal{P}} g_\lambda(n)s_\lambda(\mathbf{p})s_\lambda(\mathbf{q}),$$

where the $g_\lambda(n),\lambda\in\mathcal{P},n\in\mathbb{Z}$ are scalars.

We begin by describing a specialization to diagonal solutions of the characterization of solutions to the 2-Toda hierarchy

Theorem 3.1. *The sequence of formal power series $\{\tau_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy if and only if for all $\lambda,\mu\in\mathcal{P}$, $n,m\in\mathbb{Z}$ and integers $i,j\geq 1$ such that $|\lambda|+|\mu|=|\lambda\uparrow i|+|\mu\downarrow j|+n-m-1$, we have*

$$g_{\lambda\uparrow i}(n)g_{\mu\downarrow j}(m)=g_\lambda(n-1)g_\mu(m+1).$$

Proof. We begin with the characterization of the 2-Toda hierarchy given in Theorem 2.6(iii). Recall that $\{\tau_m(\mathbf{p},\mathbf{q})\}_{m\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy if and only if for all $m,n\in\mathbb{Z}$, $\lambda,\mu,\alpha,\beta\in\mathcal{P}$ we have

$$\begin{aligned}(1)\qquad &\sum_{i,j}(-1)^{|\alpha|-|\alpha\uparrow j|+i+j}a_\mu^{\lambda\downarrow i}(m)a_\beta^{\alpha\uparrow j}(n+1)\\ &=\sum_{s,t}(-1)^{|\mu|+|\mu\uparrow s|+s+t}a^\lambda_{\mu\uparrow s}(m+1)a^\alpha_{\beta\downarrow t}(n),\end{aligned}$$

where the first sum is over $i,j\geq 1$ such that $|\lambda\downarrow i|+|\alpha\uparrow j|=|\lambda|+|\alpha|+m-n$ and the second sum is over $s,t\geq 1$ such that $|\mu\uparrow s|+|\beta\downarrow t|=|\mu|+|\beta|+n-m$.

In order to have a non-trivial sum on the left hand side, it must be true that $\mu=\lambda\downarrow i$, $\beta=\alpha\uparrow j$ for some suitable λ,α,i and j. However, using the fact that

$$(\alpha\uparrow j)\downarrow(u_j(\alpha)+1)=\alpha,$$

and

$$(\lambda\downarrow i)\uparrow(\lambda_i+1)=\lambda,$$

we see that $\lambda=\mu\uparrow(\lambda_i+1)$ and $\alpha=\beta\downarrow(u_j(\alpha)+1)$ and so (1) becomes

$$(-1)^{|\alpha|+|\alpha\uparrow j|+i+j}g_{\lambda\downarrow i}(m)g_{\alpha\uparrow j}(n+1)=(-1)^{|\lambda\downarrow i|-|\lambda|+\lambda_i+u_j(\alpha)}g_\lambda(m+1)g_\alpha(n).$$

Using the fact that

$$|\alpha \uparrow j| = |\alpha| + j + u_j(\alpha) - 1,$$

we have $|\alpha| - |\alpha \uparrow j| + i + j = u_j(\alpha) + i + 1$ and using the fact that

$$|\lambda \downarrow i| = |\lambda| + i - \lambda_i - 1,$$

we have $|\lambda \downarrow i| - |\lambda| + \lambda_i + u_j(\alpha) = u_j(\alpha) + i - 1$.

Thus, we get

$$g_{\lambda\downarrow i}(m) g_{\alpha\uparrow j}(n+1) = g_\lambda(m+1) g_\alpha(n)$$

where $i, j \geq 1$ are such that

$$|\lambda \downarrow i| + |\alpha \uparrow j| = |\lambda| + |\alpha| + m - n.$$

After shifting $n \mapsto n-1$ and reindexing, the result follows. □

We now describe the content-type series. We begin by defining the shifted content products and discussing some of their properties.

For $m, k \in \mathbb{Z}$ we define

$$Y(m,k) := \begin{cases} \prod_{j=1}^{k} y_{m+1-j}, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0, \\ Y(m-k, -k)^{-1}, & \text{if } k \leq -1. \end{cases}$$

Also, for any $\lambda \in \mathcal{P}$, $n \in \mathbb{Z}$,

$$Y_n(\lambda) := \prod_{i=1}^{\ell(\lambda)} Y(\lambda_i - i + n, \lambda_i).$$

Note that $Y_n(\lambda)$ is the shifted content product in the indeterminates y_i for the partition λ, i.e.,

$$Y_n(\lambda) = \prod_{\square \in \lambda} y_{n+c(\square)}$$

where $c(\square)$ is the content of the cell $\square \in \lambda$.

Lemma 3.2. *Suppose $j, k, s \in \mathbb{Z}$.*

(i)

$$\frac{Y(s,k)}{Y(s,j)} = \frac{1}{Y(s-k,j-k)} = Y(s-j,k-j),$$

(ii)

$$\frac{Y(j,j)}{Y(k,k)} = Y(j,j-k),$$

(iii) *and, if* $j,k \geq 0$,

$$\frac{Y(s+j-k,j)}{Y(s,k)} = Y(s+j-k,j-k).$$

Proof. Each of the identities follows in a straightforward way by first separating into cases and then applying the definition. □

For any $n \in \mathbb{Z}$, define

$$\theta_n := \begin{cases} ab^n y_0^{n/2} \prod_{i=1}^{n-1} Y(i,i), & \text{if } n > 0, \\ a, & \text{if } n = 0, \\ ab^n y_0^{n/2} \prod_{i=1}^{-n-1} Y(-i-1,-i-1)^{-1}, & \text{if } n < 0. \end{cases}$$

We now define a sequence of formal power series which we call content-type series. For $n \in \mathbb{Z}$, define

$$\Phi_n(\mathbf{p},\mathbf{q};a,b,\mathbf{y}) = \sum_{\lambda \in \mathcal{P}} \theta_n Y_n(\lambda) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}).$$

Our aim now is to show that the sequence of formal power series $\{\Phi_n\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy. Note that this result was originally proven by Orlov and Shcherbin in [28], however, their approach is different from ours.

Lemma 3.3. *For any integer* m *we have*

$$\frac{\theta_{m+1}}{\theta_m} = b y_0^{1/2} Y(m,m).$$

Proof. If $m \geq 0$ then

$$\begin{aligned}\frac{\theta_{m+1}}{\theta_m} &= \frac{ab^{m+1}y_0^{(m+1)/2}\prod_{i=1}^{m} Y(i,i)}{ab^m y_0^{m/2}\prod_{i=1}^{m-1} Y(i,i)},\\ &= by_0^{1/2}Y(m,m).\end{aligned}$$

If $m < 0$ we have

$$\begin{aligned}\frac{\theta_{m+1}}{\theta_m} &= \frac{ab^{m+1}y_0^{(m+1)/2}\prod_{i=1}^{-m-2} Y(-i-1,-i-1)^{-1}}{ab^m y_0^{m/2}\prod_{i=1}^{-m-1} Y(-i-1,-i-1)^{-1}},\\ &= by_0^{1/2}Y(m,m).\end{aligned}$$

□

Lemma 3.4. *For integer n, integer $i > 0$ and partition λ,*

$$\frac{Y_n(\lambda \uparrow i)}{Y_{n-1}(\lambda)} = Y(|\lambda \uparrow i| - |\lambda| + n - 1, |\lambda \uparrow i| - |\lambda|).$$

Proof. We have

$$\begin{aligned}\frac{Y_n(\lambda \uparrow i)}{Y_{n-1}(\lambda)} &= Y(i-1-(u_i(\lambda)+1)+n)\left[\frac{\prod_{k=1}^{u_i(\lambda)} Y(\lambda_k - 1 - k + n, \lambda_k - 1)}{\prod_{k=1}^{u_i(\lambda)} Y(\lambda_k - k + n - 1, \lambda_k)}\right]\\ &\quad\times\left[\frac{\prod_{k=u_i(\lambda)+1}^{\ell(\lambda)} Y(\lambda_k - (k+1) + n, \lambda_k)}{\prod_{k=u_i(\lambda)+1}^{\ell(\lambda)} Y(\lambda_k - k + n - 1, \lambda_k)}\right],\\ &= \frac{Y(i - u_i(\lambda) - 1 + n - 1, i - 1)}{\prod_{k=1}^{u_i(\lambda)} Y(n-k, 1)},\end{aligned}$$

where the denominator comes from an application of Lemma 3.2(i). Simplifying and applying Lemma 3.2(iii) gives

$$\begin{aligned}\frac{Y_n(\lambda \uparrow i)}{Y_{n-1}(\lambda)} &= \frac{Y(i - u_i(\lambda) - 1 + n - 1, i - 1)}{Y(n-1, u_i(\lambda))},\\ &= Y(i - u_i(\lambda) - 1 + n - 1, i - u_i(\lambda) - 1),\\ &= Y(|\lambda \uparrow i| - |\lambda| + n - 1, |\lambda \uparrow i| - |\lambda|).\end{aligned}$$

□

Lemma 3.5. *For integer m, integer $j > 0$ and partition μ, we have*

$$\frac{Y_{m+1}(\mu)}{Y_m(\mu \downarrow j)} = Y(|\mu| - |\mu \downarrow j| + m, |\mu| - |\mu \downarrow j|).$$

Proof. We have

$$\begin{aligned}
\frac{Y_{m+1}(\mu)}{Y_m(\mu \downarrow j)} &= Y(\mu_j - j + m + 1, \mu_j) \left[\frac{\prod_{k=1}^{j-1} Y(\mu_k - k + m + 1, \mu_k)}{\prod_{k=1}^{j-1} Y(\mu_k - k + m + 1, \mu_k + 1)} \right] \\
&\quad \times \left[\frac{\prod_{k=j+1}^{\ell(\mu)} Y(\mu_k - k + m + 1, \mu_k)}{\prod_{k=j+1}^{\ell(\mu)} Y(\mu_k - (k-1) + m, \mu_k)} \right], \\
&= \frac{Y(\mu_j - j + m + 1, \mu_j)}{\prod_{k=1}^{j-1} Y(m - k + 1, 1)},
\end{aligned}$$

where the denominator comes from an application of Lemma 3.2(i). After simplifying and applying Lemma 3.2(iii) we have

$$\begin{aligned}
\frac{Y_{m+1}(\mu)}{Y_m(\mu \downarrow j)} &= \frac{Y(\mu_j - j + m + 1, \mu_j)}{Y(m, j-1)}, \\
&= Y(\mu_j - j + m + 1, \mu_j - j + 1), \\
&= Y(|\mu| - |\mu \downarrow j| + m, |\mu| - |\mu \downarrow j|).
\end{aligned}$$

□

Theorem 3.6. *The sequence of formal power series $\{\Phi_n(\mathbf{p}, \mathbf{q}; a, b, \mathbf{y})\}_{n \in \mathbb{Z}}$ is a solution to the 2-Toda hierarchy.*

Proof. By Theorem 3.1 we need to show that for all partitions λ, μ and integers i, j, n, m such that $i, j \geq 1$ and

$$|\lambda| + |\mu| = |\lambda \uparrow i| + |\mu \downarrow j| + n - m - 1,$$

we have

$$g_{\lambda \uparrow i}(n) g_{\mu \downarrow j}(m) = g_\lambda(n-1) g_\mu(m+1).$$

In the case of the content-type series, this becomes

$$\theta_n Y_n(\lambda \uparrow i) \theta_m Y_m(\mu \downarrow j) = \theta_{n-1} Y_{n-1}(\lambda) \theta_{m+1} Y_{m+1}(\mu),$$

or, after rearranging,

$$\frac{Y_n(\lambda\uparrow i)/Y_{n-1}(\lambda)}{Y_{m+1}(\mu)/Y_m(\mu\downarrow j)}=\frac{\theta_{m+1}/\theta_m}{\theta_n/\theta_{n-1}}. \tag{2}$$

So, if we can show that the identity (2) holds then we are done.

Using Lemma 3.4 and Lemma 3.5, the left hand side of (2) becomes

$$\frac{Y(|\lambda\uparrow i|-|\lambda|+n-1,|\lambda\uparrow i|-|\lambda|)}{Y(|\mu|-|\mu\downarrow j|+m,|\mu|-|\mu\downarrow j|)}.$$

Using the fact that $|\lambda|+|\mu|=|\lambda\uparrow i|+|\mu\downarrow j|+n-m-1$, the left hand side of (2) then becomes

$$\frac{Y(|\lambda\uparrow i|-|\lambda|+n-1,|\lambda\uparrow i|-|\lambda|)}{Y(|\lambda\uparrow i|-|\lambda|+n-1,|\lambda\uparrow i|-|\lambda|-m-1+n)}.$$

Applying Lemma 3.2(i) then tells us that the left hand side of (2) is

$$Y(m,m+1-n). \tag{3}$$

Now, using Lemma 3.3, the right hand side of (2) becomes

$$\frac{Y(m,m)}{Y(n-1,n-1)}.$$

Applying Lemma 3.2(ii) then gives (3). Hence (2) is satisfied and the content-type series solves the 2-Toda hierarchy. □

We now give a partial converse to this theorem, showing that many diagonal series that arise as solutions to the 2-Toda hierarchy are in fact content-type series.

Theorem 3.7. *For $n\in\mathbb{Z}$ suppose that*

$$\tau_n=\sum_{\lambda\in\mathcal{P}}g_\lambda(n)s_\lambda(\mathbf{p})s_\lambda(\mathbf{q}),$$

where $g_\lambda(n),\lambda\in\mathcal{P},n\in\mathbb{Z}$ are scalars and that $\{\tau_n(\mathbf{p},\mathbf{q})\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy. Define a sequence $\mathbf{u}=(\ldots,u_{-1},u_0,u_1,\ldots)$ as follows.

For $m \geq 0$,

$$g_{m+1}(0) = u_m g_m(0),$$
$$g_{1^{m+1}}(0) = u_{-m} g_{1^m}(0).$$

If the sequences $\{g_m(0)\}_{m\geq 0}$ and $\{g_{1^m}(0)\}_{m\geq 0}$ can be uniquely recovered from $g_\epsilon(0)$ and the sequence $\mathbf{u}$ and if $g_\epsilon(0), g_1(0) \neq 0$ then for all $n \in \mathbb{Z}$,

$$\tau_n = \Phi_n\left(\mathbf{p}, \mathbf{q}; g_\epsilon(0), \frac{g_\epsilon(1)}{u_0^{1/2} g_\epsilon(0)}, \mathbf{u}\right).$$

Note that the condition on the sequence $\mathbf{u}$ is essentially that if $g_m(0) = 0$ then $g_M(0) = 0$ for all $M \geq m$ and similarly for $g_{1^m}(0)$. Also note that the choice of 0 in the above statement is not special since the conditions on the coefficients of solutions to the 2-Toda hierarchy are invariant under the translation $\tau_n \mapsto \tau_{n+1}$.

Proof. The approach taken will be to show that if the conditions in the theorem are satisfied then each of the coefficients $g_\lambda(n)$ can be constructed from the sequence $\mathbf{u}$ along with $g_\epsilon(0)$ and $g_\epsilon(1)$. The result then follows from the fact that Φ_n also satisfies the conditions in the theorem and that if $\tau_n = \Phi_n$ then for $m \geq 0$,

$$g_{m+1}(0) = \theta_0 Y_0(m+1) = y_m \theta_0 Y_0(m) = y_m g_m(0),$$
$$g_{1^{m+1}}(0) = \theta_0 Y_0(1^{m+1}) = y_{-m} \theta_0 Y_0(1^m) = y_{-m} g_{1^m}(0),$$

and that $g_\epsilon(0) = \theta_0 Y_0(\epsilon) = a$, and $g_\epsilon(1) = \theta_1 Y_1(\epsilon) = aby_0^{1/2}$.

In the following we will use the notation (r, η) where $\eta \in \mathcal{P}$ and $r \geq \eta_1$ to denote the partition $(r, \eta_1, \eta_2, \ldots)$ and we will use the notation $1^s + \eta$ where $\eta \in \mathcal{P}$ and $s \geq \ell(\eta)$ to denote the partition $(\eta_1 + 1, \ldots, \eta_s + 1)$ with the convention that $\eta_i = 0$ if $i > \ell(\eta)$.

Since $\{\tau_n\}_{n\in\mathbb{Z}}$ satisfies the 2-Toda hierarchy and is a diagonal solution, we know that for any partitions $\lambda, \mu \in \mathcal{P}$, $m, n \in \mathbb{Z}$ and integers $i, j \geq 1$ such that

$$|\lambda| + |\mu| = |\lambda \uparrow i| + |\mu \downarrow j| + n - m - 1,$$

the identity

$$g_{\lambda\uparrow i}(n) g_{\mu\downarrow j}(m) = g_\lambda(n-1) g_\mu(m+1) \tag{4}$$

holds.

If we choose $n = m = 0, i = 2, j = 1, \lambda = \mu = \epsilon$ in (4) then we get

$$g_1(0)g_\epsilon(0) = g_\epsilon(-1)g_\epsilon(1).$$

Since $g_1(0), g_\epsilon(0) \neq 0$ we know that $g_\epsilon(1), g_\epsilon(-1) \neq 0$.

If we choose $n = 0, \lambda = \epsilon$ in (4) then we get

$$g_k(0)g_{\mu\downarrow j}(m) = g_\epsilon(-1)g_\mu(m+1),$$

where

$$k = |\mu| - |\mu \downarrow j| + m + 1.$$

In particular, if $\eta \in \mathcal{P}$, $r \geq \eta_1$ and $m \geq -1$, this implies that

$$g_\epsilon(-1)g_{(r,\eta)}(m+1) = g_{r+m+1}(0)g_\eta(m). \tag{5}$$

Since $g_\epsilon(-1) \neq 0$, Equation (5) allows us to construct $g_\eta(n), n > 0$ inductively provided we know $g_\lambda(0)$ for any partition λ.

Similarly, if we choose $m = 0, \mu = \epsilon$ in (4) then we get

$$g_{\lambda\uparrow i}(n)g_{1^k}(0) = g_\lambda(n-1)g_\epsilon(1),$$

where

$$k = |\lambda| - |\lambda \uparrow i| - n + 1.$$

In particular, if $\eta \in \mathcal{P}$, $s \geq \ell(\eta)$ and $n \leq 1$ then

$$g_\epsilon(1)g_{1^s+\eta}(n-1) = g_{1^{s-n+1}}(0)g_\eta(n). \tag{6}$$

Since $g_\epsilon(1) \neq 0$, Equation (6) allows us to construct $g_\eta(n), n < 0$ inductively provided we know $g_\lambda(0)$ for any partition λ.

All that is left now is to show that $g_\lambda(0)$ can be constructed provided we know $g_m(0)$, $g_{1^m}(0)$, $g_\epsilon(0)$, $g_\epsilon(1)$ and $g_\epsilon(-1)$.

If we choose $m = -1$ in (5) then we have, for any $\eta \in \mathcal{P}$ and $r \geq \eta_1$,

$$g_\epsilon(-1)g_{(r,\eta)}(0) = g_r(0)g_\eta(-1). \tag{7}$$

If we choose $n = 0$ in (6) then we have, for any $\eta \in \mathcal{P}$ and $s \geq \ell(\eta)$,

$$g_\epsilon(1)g_{1^s+\eta}(-1) = g_{1^{s+1}}(0)g_\eta(0). \tag{8}$$

Note that on the right hand side of both (7) and (8) the partition corresponding to the unknown coefficient is strictly smaller in size than the partition corresponding to the unknown coefficient on the left hand side. Thus, by induction, we can construct $g_\lambda(0)$ for every partition λ.

To see how the last part about $g_\lambda(0)$ works, suppose we wish to determine $g_{42^21^3}(0)$. By repeatedly applying (7) and (8) we get

$$g_\epsilon(-1)g_{42^21^3}(0) = g_4(0)g_{2^21^3}(-1),$$

and

$$g_\epsilon(1)g_{2^21^3}(-1) = g_{1^6}(0)g_{1^2}(0).$$

$\square$

Note that not all diagonal solutions to the 2-Toda hierarchy are content-type series. For example, it is not difficult to show that if we define $\tau_0 = s_1(\mathbf{p})s_1(\mathbf{q}) = p_1q_1$ and $\tau_n = 0$ for $n \neq 0$ then $\{\tau_n\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy.

4. Examples

In this section we briefly discuss a few content-type series that have arisen in the literature. We begin with a family of series that was presented in [15] and which encodes a number of enumerative generating functions.

For $a_1, a_2, \dots \geq 0$ and $\alpha, \beta \in \mathcal{P}$ with $|\alpha| + |\beta| = d$, let $\mathcal{B}_{\alpha,\beta}^{a_1,a_2,\dots}$ be the set of tuples of permutations $(\sigma, \gamma, \pi_1, \pi_2, \dots)$ on $\{1, \dots, d\}$ such that

(i) The permutation σ has cycle type α, γ has cycle type β and $d - \ell(\pi_i) = a_i$ for $i \geq 1$ where $\ell(\pi_i)$ is the number of cycles in the disjoint cycle decomposition of π_i;

(ii) $\sigma\gamma\pi_1\pi_2\cdots = \mathrm{id}$.

Let $b_{\alpha,\beta}^{a_1,a_2,\dots}$ be the number of tuples in $\mathcal{B}_{\alpha,\beta}^{a_1,a_2,\dots}$ (Note that our $b_{\alpha,\beta}^{a_1,a_2,\dots}$ is written $\tilde{b}_{\alpha,\beta}^{a_1,a_2,\dots}$ in [15]). The tuples counted by $b_{\alpha,\beta}^{a_1,a_2,\dots}$ are called constellations in [21].

If we now construct the generating function for these numbers,

$$B := \sum_{\substack{\alpha,\beta\in\mathcal{P},\\ |\alpha|=|\beta|=d\geq 1,\\ a_1,a_2,\dots\geq 0}} \frac{1}{d!} b_{\alpha,\beta}^{a_1,a_2,\dots} p_\alpha p_\beta u_1^{a_1} u_2^{a_2} \cdots ,$$

then using representation theory (see [15] for details) it can be shown that

$$B = \Phi_0\big|_{a=b=1,\ y_j=\prod_{i\geq 1}(1+ju_i),\ j\in\mathbb{Z}},$$

and hence via Theorem 3.6 can be embedded in a solution to the 2-Toda hierarchy.

Various specializations of B give rise to generating functions for a number of different enumerative problems such as map and hypermap enumeration. We choose to focus on only a single specialization here, namely to the double Hurwitz problem, and defer to [15] for others.

Double Hurwitz numbers arise in the enumeration of branched covers of the sphere through an encoding due to Hurwitz [17]. Using the infinite wedge space formalism, Okounkov [23] showed that the generating function for double Hurwitz numbers can be embedded in a solution to the 2-Toda hierarchy. We will see that this also follows from the fact that the generating function for double Hurwitz numbers is a specialization of the generating series B described above. Note that Orlov [25] has also given a proof of this result that is similar to ours although from a different point of view.

For $\alpha, \beta \in \mathcal{P}$, $|\alpha| = |\beta| = d$ and $g \geq 0$ let $r^g_{\alpha,\beta} = \ell(\alpha) + \ell(\beta) + 2g - 2$. The Hurwitz number $H^g_{\alpha,\beta}$ is defined by

$$H^g_{\alpha,\beta} := \frac{1}{d!}|\mathrm{Aut}\alpha||\mathrm{Aut}\beta| b^{a_1,a_2,\ldots}_{\alpha,\beta},$$

where $a_i = 1$ for $1 \leq i \leq r^g_{\alpha,\beta}$ and $a_i = 0$ for $i > r^g_{\alpha,\beta}$. The (disconnected) double Hurwitz series is then

$$H := \sum_{\substack{\alpha,\beta\in\mathcal{P} \\ |\alpha|=|\beta|=d\geq 1, \\ g\geq 0}} \frac{H^g_{\alpha,\beta}}{|\mathrm{Aut}\alpha||\mathrm{Aut}\beta|} p_\alpha q_\beta \frac{t^{r^g_{\alpha,\beta}}}{r^g_{\alpha,\beta}!}.$$

After doing some algebraic manipulations, it can be shown (see [15] for details) that

$$H = B|_{a=b=1,\ e_k(u_1,u_2,\ldots)=\frac{t^k}{k!}\ k\geq 1} = \Phi_0|_{a=b=1,\ y_j=e^{jt},\ j\in\mathbb{Z}},$$

where the e_k are the elementary symmetric functions. Thus, we see that the double Hurwitz series can be embedded in a solution to the 2-Toda hierarchy. This fact was used in [23] to prove a conjecture of Pandharipande[29] concerning the simple Hurwitz numbers.

We now give an example of a content-type series arising in the study of random partitions. We begin by defining a function $\mathcal{M}$ on the set of partitions. For $\lambda \in \mathcal{P}$,

$$\mathcal{M}(\lambda) = \frac{1}{Z} s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q})$$

where $Z = \sum_{\lambda\in\mathcal{P}} s_\lambda(\mathbf{p})s_\lambda(\mathbf{q})$. The function $\mathcal{M}$, considered as a probability measure on partitions, is called the Schur measure and was introduced by Okounkov in [24]. The Schur measure can be thought of as a generalization of the z-measure introduced by Borodin and Olshanski (see [8] for example) and as such contains as specializations or limiting cases a variety of other probability models[7]. One of the results in [24] is that the correlation functions of the Schur measure satisfy the 2-Toda hierarchy in the parameters of the measure. Here we show that this also follows from Theorem 3.1.

For any partition λ, let $\mathfrak{S}(\lambda) = \{\lambda_i - i\}_{i\geq 1}$. This can be thought of as the sequence of contents of the rightmost cell in each row of λ where we append a countable number of parts of size 0 to λ. Note that λ is uniquely recoverable from $\mathfrak{S}(\lambda)$. For any $X \subseteq \mathbb{Z}$, we define

$$\rho(X) = \sum_{\substack{\lambda\in\mathcal{P}\\ X\subseteq\mathfrak{S}(\lambda)}} \mathcal{M}(\lambda).$$

The $\rho(X)$ are the correlators of the Schur measure. If $X = \{x_1, x_2, \ldots\}$ then for $n \in \mathbb{Z}$ we use $X - n$ to denote the set $\{x_1 - n, x_2 - n, \ldots\}$.

Proposition 4.1. *For any $X \subseteq \mathbb{Z}$, the sequence $\{Z\rho(X - n)\}_{n\in\mathbb{Z}}$ is a solution to the 2-Toda hierarchy.*

Proof. First, notice that

$$\tau_n = Z\rho(X - n) = \sum_{\substack{\lambda\in\mathcal{P}\\ X-n\subseteq\mathfrak{S}(\lambda)}} s_\lambda(\mathbf{p})s_\lambda(\mathbf{q}).$$

Suppose $\lambda, \mu \in \mathcal{P}$ and $n, m \in \mathbb{Z}$ are such that

$$X - n + 1 \subseteq \mathfrak{S}(\lambda),$$

and

$$X - m - 1 \subseteq \mathfrak{S}(\mu).$$

Further, suppose that $i, j \in \mathbb{Z}$ are such that $|\lambda| + |\mu| = |\lambda \uparrow i| + |\mu \downarrow j| + n - m - 1$. We will now show that

$$X - n \subseteq \mathfrak{S}(\lambda \uparrow i)$$

and

$$X - m \subseteq \mathfrak{S}(\mu \downarrow j)$$

which, by Theorem 3.1, will prove the proposition.

First, recall that $\lambda \uparrow i = (\lambda_1 - 1, \ldots, \lambda_{u_i(\lambda)} - 1, i - 1, \lambda_{u_i(\lambda)+1}, \ldots)$ where $u_i(\lambda)$ is the unique integer such that $\lambda_{u_i(\lambda)} \geq i > \lambda_{u_i(\lambda)+1}$ and that $\mu \downarrow j = (\mu_1 + 1, \ldots, \mu_{j-1} + 1, \mu_{j+1}, \ldots)$. Thus, we have $|\lambda| - |\lambda \uparrow i| = u_i(\lambda) + 1 - i$ and $|\mu| - |\mu \downarrow j| = \mu_j - j + 1$ so that the constraint $|\lambda| + |\mu| = |\lambda \uparrow i| + |\mu \downarrow j| + n - m - 1$ becomes $u_i(\lambda) - i + \mu_j - j + 2 = n - m - 1$.

Now, for any $x \in X$, we know that

$$x - n + 1 = \lambda_t - t$$

for some positive integer t (recall that $\lambda_t = 0$ if $t > \ell(\lambda)$). If $t \leq u_i(\lambda)$ then

$$x - n = \lambda_t - 1 - t = (\lambda \uparrow i)_t - t.$$

Similarly, if $t > u_i(\lambda)$ then

$$x - n = \lambda_t - (t + 1) = (\lambda \uparrow i)_{t+1} - (t + 1).$$

In particular, notice that this implies there is no $x \in X$ such that

$$x - n = (\lambda \uparrow i)_{u_i(\lambda)+1} - (u_i(\lambda) + 1) = i - u_i(\lambda) - 2.$$

So far we have shown that $X - n \subseteq \mathfrak{S}(\lambda \uparrow i)$ and so now we must show that $X - m \subseteq \mathfrak{S}(\mu \downarrow j)$.

For any $x \in X$, we know that

$$x - m - 1 = \mu_t - t$$

for some positive integer t. If $t < j$ then

$$x - m = (\mu_t + 1) - t = (\mu \downarrow j)_t - t.$$

Similarly, if $t > j$ then

$$x - m = \mu_t - (t - 1) = (\mu \downarrow j)_{t-1} - (t - 1).$$

If $t = j$ then since $x - m - 1 = \mu_j - j$ and $u_i(\lambda) - i + \mu_j - j + 2 = n - m - 1$ we have

$$x - n = m + 1 - n + \mu_j - j = i - u_i(\lambda) - 2,$$

a contradiction and hence $X - m \subseteq \mathfrak{S}(\mu \downarrow j)$. $\square$

For the last example we look at a matrix integral that arises in mathematical physics and, more recently, in a combinatorial context[13, 14]. Consider the integral

$$I_n := \int_{U(n)} e^{\mathrm{Tr}(XUYU^*)} dU,$$

where $U(n)$ is the group of n by n unitary matrices, dU is the Haar measure on $U(n)$ and X and Y are diagonal matrices.

The integral I_n appeared recently in the context of a combinatorial problem involving enumeration of certan restricted factorizations in the symmetric group. This problem is called the monotone double Hurwitz problem and is related to the double Hurwitz problem discussed earlier. We refer to [13] for an analytic treatment of I_n and [14] for a solution to the corresponding enumeration problem.

If we let $p_k = \mathrm{Tr}(X^k)$ and $q_k = \mathrm{Tr}(Y^k)$ then using the character expansion method [25] we have

$$I_n = \sum_{\substack{\lambda \in \mathcal{P}, \\ \ell(\lambda) \leq n}} \frac{s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q})}{\prod_{\square \in \lambda}(n + c(\square))}.$$

Given a partition λ, the cell with the smallest content is in the first column and the last row and the content of this cell is $1 - \ell(\lambda)$. ¿From this it is easy to see that $\lambda \in \mathcal{P}$ is such that $\ell(\lambda) \leq n$ if and only if for all $\square \in \lambda$, $c(\square) > -n$. Now, consider the series $\tilde{I_n} = \Phi_n$ where we set $a = 1, b = y_0^{-1/2}$ and $y_i = \frac{1}{i}$ if $i > 0$ and $y_i = 0$ if $i \leq 0$. Note that we first make the substitution $b = y_0^{-1/2}$ and each of the coefficients in the resulting series is a monomial in the y_i and so we may set $y_0 = 0$. We have

$$\tilde{I_n} = \tilde{\theta_n} \sum_{\lambda \in \mathcal{P}} \tilde{Y_n}(\lambda) s_\lambda(\mathbf{p}) s_\lambda(\mathbf{q}),$$

where

$$\tilde{\theta_n} = \left(\prod_{i=1}^{n-1} i! \right)^{-1}.$$

Also,

$$\tilde{Y_n}(\lambda) = \prod_{\square \in \lambda} y_{n+c(\square)},$$

and so $\tilde{Y_n}(\lambda) = 0$ unless $\forall \Box \in \lambda$, $n + c(\Box) > 0$ or $c(\Box) > -n$. If $\tilde{Y_n}(\lambda) \neq 0$ then

$$\tilde{Y_n}(\lambda) = \prod_{\Box \in \lambda} y_{n+c(\Box)} = \prod_{\Box \in \lambda} \frac{1}{n + c(\Box)}.$$

Thus,

$$\tilde{I_n} = \left(\prod_{i=1}^{n-1} i! \right)^{-1} I_n,$$

and hence, via Theorem 3.6, $\left\{ \left(\prod_{i=1}^{n-1} i! \right)^{-1} I_n \right\}_{n \in \mathbb{Z}}$ is a solution to the 2-Toda hierarchy where $I_n = 0$ if $n < 0$. Note that a similar result appears in [13] with a slightly different proof. Also, this result appears in [25] from a different perspective and can be thought of as a generalization of a result of Zinn-Justin [34].

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Received September 10, 2011