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Mean value inequalities and conditions to extend Ricci flow

Xiaodong Cao and Hung Tran

This paper concerns conditions related to the first finite singularity time of a Ricci flow solution on a closed manifold. In particular, we provide a systematic approach to the mean value inequality method, suggested by N. Le [13] and F. He [11]. We also display a close connection between this method and time slice analysis as in [23]. As an application, we prove several inequalities for a Ricci flow solution on a manifold with nonnegative isotropic curvature.

1. Introduction

Let M^n be a n-dimensional closed manifold and $g(t)$, $0 \leq t < T$, be a one-parameter family of smooth Riemannian metrics on M satisfying

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}(t),$$

then $(M, g(t))$ is said to be a solution to the Ricci flow, which was first introduced by R. Hamilton in [9]. The Ricci flow has been studied extensively, particularly in the last decade thanks to the seminal contribution by G. Perelman [19]. As a weakly parabolic system, the Ricci flow can develop finite-time singularities. We say that $(M, g(t))$, $t \in [0, T)$, is a maximal solution if it becomes singular at time T. An intriguing but still open problem concerns the precise behavior of the curvature tensor as the flow approaches the finite singular time.

It was first shown by Hamilton that the norm of the Riemannian curvature tensor $|\mathrm{Rm}|$ must blow up approaching the first finite singular time [9]. More recently, by using an application of the non-collapsing result of G. Perelman [19], N. Sesum was able to prove that if the norm of Ricci curvature $|\mathrm{Rc}|$ is bounded then the flow can be extended [21]. Since then the obvious generalized problem of whether the scalar curvature must behave

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similarly has received extensive attention. The question is still open but considerable progress has been made: the Type I case is resolved by J. Ender, R. Muller and P. Topping [8], also independently by Q. Zhang and the first author in [5, 6], while the Kähler case is solved by Z. Zhang [24]. There are various other relevant results such as estimates relating the scalar curvature and the Weyl tensor [5], comparable growth rates of different components of the curvature tensor [23], [22], and integral conditions by N. Le and N. Sesum [14].

It is interesting that elementary but clever analytical techniques proved fruitful to study this problem. Following the mean value inequality trick of Le [13] for the mean curvature flow, F. He developed a logarithmic-improvement condition for the Ricci flow [11]. Our contribution is to provide a more systematic treatment of the mean value inequality method and to find a close connection to the time slice analysis method suggested by B. Wang [23]. Then we apply our analysis to a particular context of Ricci flow with a uniform-growth condition defined below.

Throughout the rest of this paper, M is a closed smooth Riemannian manifold and T is some positive finite time. We will use the following notation:

$$Q(t) = \sup_{M\times\{t\}} |\mathrm{Rm}|,\ P(t) = \sup_{M\times\{t\}} |\mathrm{Rc}|,\ O(t) = \sup_{M\times\{t\}} |\mathrm{R}|.$$

Our first theorem gives a logarithmic-improvement condition relating the Ricci curvature and the Riemannian curvature tensor (in comparison, the logarithmic result in [11] involves a double integral of just the Riemannian curvature).

Theorem 1.1. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution on M. If for some $0 \le p \le 1$, we have*

$$\int_0^T \frac{P(t)}{(\ln(1+Q(t)))^p} dt < \infty,$$

then the solution can be extended past time T.

Since we are interested in the behavior of the scalar curvature at a singular time, this motivates the following definition.

Definition 1.2. A Ricci flow solution on a closed manifold is said to satisfy the **uniform-growth** condition if it develops a singularity in finite time, and any singularity model obtained by parabolic rescaling at the scale of the maximum curvature tensor must has non-flat scalar curvature.

Under the Ricci flow, the uniform-growth condition generalizes both Type I and (non-flat) nonnegative isotropic curvature (NIC) conditions. Combining the above mean value inequality method with the uniform-growth condition yields the following logarithmic-improvement result.

Theorem 1.3. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition on M. If for some $0 \le p \le 1$, we have*

$$\int_0^T \int_M \frac{|\mathrm{R}|^{n/2+1}}{(\ln(1+|\mathrm{R}|))^p} d\mu dt < \infty, \tag{1.1}$$

then the solution can be extended past time T.

The paper is organized as follows. In Section 2, we discuss mean value inequalities and provide the proof of Theorem 1.1. Section 3 displays a close connection to the time-slice analysis and thus gives another proof of the above result as well as some independent estimates. In Section 4 we apply our method to the context of nonnegative isotropic curvature and its generalization.

2. Mean value inequalities

In this section, we describe the method of mean value inequalities to study conditions to extend the Ricci flow. The key idea is to generalize a simple but clever trick from [13] which involves an integral with a carefully chosen weight function. The conclusion is that, regarding the blow-up behavior, the weight function does not really matter.

2.1. A motivation

This section gives a sample of how an elementary technique can be useful in our problem. Precisely, we give an alternative proof to the following result which was proved first by [11] using the Sobolev machinery.

Theorem 2.1. *Let $(M, g(t))$, $0 \le t < T < \infty$, be a solution to the Ricci flow. If $F(x) = \int_0^T |Rc|(x,t)dt$ is continuous on M then the solution can be extended past time T.*

Let $H_x(t_1, t_2) = \int_{t_1}^{t_2} |\mathrm{Rc}|(x,t)dt$. H is uniformly continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|t_2 - t_1| < \delta$ then $H_x(t_1, t_2) < \epsilon$, $\forall x \in M$.

Lemma 2.2. *H is uniformly continuous under one of these assumptions:*

a) $\int_0^T P(t)dt < C$;

b) $F(x) = \int_0^T |\mathrm{Rc}|(x,t)dt$ *is continuous on M.*

Proof. **a)** Since $\int_0^T \mathrm{P}(t)dt$ is finite, we can choose η such that $H(t,T) < \epsilon$ for all $\eta \leq t$. If $\eta \leq t_1 \leq t_2 \leq T$ then obviously, $H(t_1,t_2) < \epsilon$. Let $c = \max_{[0,\frac{T+\eta}{2}]} |P(t)|$ and choose $\delta < \min\{\frac{\eta}{2}, \frac{\epsilon}{c}\}$ then the result follows.

b) Let $\mathcal{F}(x,t) = \int_0^t |\mathrm{Rc}|(x,t)dt$. By a standard compactness argument (M is closed and T finite), $\mathcal{F}$ is uniformly continuous on $M \times [0,T]$. The result follows. □

Remark 2.1. Is it possible to replace $\int_0^T \mathrm{P}(t)dt$ by $\int_0^T |\mathrm{Rc}(t)|dt$ at any point in M?

Lemma 2.3. *Let $(M, g(t))$, $0 \leq t < T < \infty$, be a solution to the Ricci flow. If H is uniformly continuous then $g(t)$ is uniformly continuous.*

Proof. For any $x \in M$ and any $V \in T_xM$ we have:

$$\left|\ln \frac{g(x,t_2)(V,V)}{g(x,t_1)(V,V)}\right| = \left|\int_{t_1}^{t_2} \frac{\partial_t g(x,t)(V,V)}{g(x,t)(V,V)}\right| \leq 2\int_{t_1}^{t_2} |\mathrm{Rc}|(x,t) = H_x(t_1,t_2).$$

□

We are now ready to prove the Theorem 2.1.

Proof. The proof is modeled after that of [7, Theorem 6.40].

By Lemmas 2.2 and 2.3, the metric is uniformly continuous. Thus the same argument as in the aforementioned reference would apply if we can show that the singularity model is Ricci flat.

If T is the singular time then by [9, Theorem 14.1], there exist a sequence $t_j \to T$, $Q_j = \max_M |\mathrm{R}(x,t_j)| \to \infty$. We dilate the solution by $g_j(t) = Q_j g(t_j + \frac{t}{Q_j})$. Then $|\mathrm{Rc}|_j(x,t) = \frac{1}{Q_j}|\mathrm{Rc}|(x, t_j + \frac{t}{Q_j})$ and therefore,

$$\int_{-1}^{0} |\mathrm{Rc}_j|(x,t)dt = \int_{t_j - \frac{1}{Q_j}}^{t_j} \frac{|\mathrm{Rc}|(x,s)}{Q_j} Q_j ds = \int_{t_j - \frac{1}{Q_j}}^{t_j} |\mathrm{Rc}|(x,s)ds.$$

Since $Q_j \to \infty$, $t_j - \frac{1}{Q_j} \to T$. As in Lemma 2.2, $\mathcal{F}(x,t) = \int_0^t |\mathrm{Rc}|(x,t)dt$ is uniformly continuous on $M \times [0,T]$. Therefore, the last integral above is approaching zero as $j \to \infty$. By the Cheeger-Gromov-Hamilton convergence theorem and Perelman's non-collapsing result (for more details, see [7, Chapter 8]), that behavior is carried to the singularity model. The result then follows. □

2.2. The main lemma

Here we generalize a trick in [13].

Lemma 2.4. *Let* $f, G : [0, T) \to [0, \infty)$ *be continuous functions and* $\psi : [0, \infty) \to [0, \infty)$ *be a non-decreasing function such that*

$$\int_1^\infty \frac{1}{\psi(s)} ds = \infty. \tag{2.1}$$

If there is a mean value inequality of the form

$$f(t) \leq C_1 \int_0^t \psi(f(s))G(s)ds + C_2 = h(t) \tag{2.2}$$

and $\int_0^T G(t)dt < \infty$*, then* $\limsup_{t\to T} f(t) < \infty$*.*

Proof. For any $T_0 < T$,

$$\begin{aligned}\int_0^{T_0} C_1 G(t)dt &= \int_0^{T_0} \frac{1}{\psi(f(t))} C_1 \psi(f(t))G(t)dt \\ &= \int_{h(0)}^{h(T_0)} \frac{1}{\psi(f(h^{-1}(s)))} ds \text{ (let } s = h(t),\ ds = h'(t)dt) \\ &\geq \int_{h(0)}^{h(T_0)} \frac{1}{\psi(s)} ds.\end{aligned}$$

The last inequality is because of $f(t) \leq h(t)$. If $\int_0^T C_1 G(t)dt < \infty$, then by the choice of ψ, $h(T_0) \leq C < \infty$. Now by the mean value inequality, $f(T_0) \leq h(T_0) \leq C$. Since T_0 is arbitrary, $\sup_{[0,T)} f \leq C < \infty$. □

2.3. Deriving mean value inequalities

In the next several lemmas, we will establish a mean value inequality connecting $Q(t)$ and $P(t)$. We first need the following doubling time estimate.

Lemma 2.5. *[7, Lemma 6.1] If $(M, g(t))$ is a Ricci flow on a closed manifold then for all $t \in [0, \frac{1}{16Q(0)})$,*

$$Q(t) \leq 2Q(0). \tag{2.3}$$

We also need the following definition, for detailed discussion, see [19].

Definition 2.6. A Ricci flow solution $(M, g(t))$, $t \in (0, T)$, is said κ-noncollapsed *(*on all scales*)*, if $\forall g(t)$, every metric ball B of radius r, with $|Rm|(x) \leq r^{-2}$ for every $x \in B$, has volume at least κr^n.

We are now ready to state our first key technical lemma.

Lemma 2.7. *Let $\Sigma(M, \kappa, C_0) = \{(g(t))_{t\in[0,1]}, g(t)$ is κ-noncollapsed, $Q(0) \leq C_0\}$ be a set of Ricci flow solutions on M. Then there exists a constant $C = C(n, \kappa, C_0)$ such that for any $g(t) \in \Sigma$,*

$$\sup_{[0,1]} Q(t) \leq C \int_0^1 Q(t)P(t)dt + 32C_0. \tag{2.4}$$

Proof. The proof is by contradiction. Suppose that the statement is false then there exists a sequence of $g_i(t) \in \Sigma$ and $a_i \to \infty$ such that

$$\sup_{[0,1]} Q_i(t) \geq a_i \int_0^1 Q_i(t)P_i(t)dt + 32C_0.$$

Let $Q_i = \sup_{[0,1]} Q_i(t)$ then we can find (x_i, t_i) such that Q_i is attained. Since $Q_i > 32C_0$ there exists t_{i0} being the first time backward such that $Q_i(t_{i0}) = \frac{1}{2}Q_i$. Consequently, for $t \in [t_{i0}, t_i]$, $32C_0 < Q_i < 2Q_i(t)$, $Q_i(t_{i0}) > 16C_0$ and by Lemma 2.5, $t_{i0} > \frac{1}{16C_0}$.

Claim: There exists a constant $\epsilon_0 = \epsilon_0(n, \kappa)$ such that the following holds: for any $t_0 > 0$, $D \geq \max\{1/t_0, \max_{[0,t_0]} Q\}$, let $t_1 > t_0$ be the first time, if exists, such that $Q(t_1) = D$, and $t_2 > t_1$ be the first time, if exists, such that $|\ln(Q(t_2)/Q(t_1))| = \ln 2$, then

$$\int_{t_1}^{t_2} P(t)dt > \epsilon_0.$$

Proof of claim: This is essentially just a restatement of [23, Lemma 3.2]. If there are no such t_1, t_2, the statement is vacuously true. If they exist then we dilate the solution by $\tilde{g}(t) = Dg(t_1 + t/D)$ then $\tilde{g}(t)$ satisfies the condition of the aforementioned result and the claim follows after rescaling back.

Applying the claim above yields

$$\int_{t_{i0}}^{t_i} P_i(t)dt > \epsilon_0. \tag{2.5}$$

Thus,

$$Q_i \geq 32C_0 + a_i \int_{t_{i0}}^{t_i} Q_i(t)P_i(t)dt \geq 32C_0 + a_i 16C_0\epsilon_0. \tag{2.6}$$

On the other hand,

$$\begin{aligned} Q_i \int_{t_{i0}}^{t_i} P_i(t)dt &\leq 2\int_{t_{i0}}^{t_i} Q_i(t)P_i(t)dt \\ &\leq 2\int_0^1 Q_i(t)P_i(t)dt \\ &\leq 2\frac{Q_i - 32C_0}{a_i}, \end{aligned}$$

hence

$$\int_{t_{i0}}^{t_i} P_i(t)dt \leq \frac{2}{a_i}\frac{Q_i - 32C_0}{Q_i} \to 0,$$

the last limit follows from (2.6) and $a_i \to \infty$. This is in contradiction with (2.5), so the lemma follows. □

We are now in the position to state our mean value inequality.

Proposition 2.8. *Let $(M, g(t))$, $0 \leq t < T$, be a Ricci flow solution on M and $Q(t) = \sup_{M\times\{t\}} |Rm|$. There exist constants*

$$\begin{aligned} C_0 &= C_0(n, \kappa, Q(0)), \\ C_1 &= 32Q(0), \end{aligned}$$

such that

$$\sup_{[0,t]} Q \leq C_0 \int_0^t Q(u)P(u)du + C_1. \tag{2.7}$$

Proof. For $t \in [0, \frac{1}{16Q(0)})$ the statement is true by Lemma 2.5. For any $t \in [\frac{1}{16Q(0)}, T)$ define

$$\tilde{g}(s) = \frac{1}{t} g(ts), \; s \in [0,1],$$
$$\tilde{Q}(s) = tQ(s).$$

Since the non-collapsing constant is a scaling invariant, applying Lemma 2.7 yields

$$\sup_{[0,1]} \tilde{Q} \leq C_0 \int_0^1 \tilde{Q}(s)\tilde{P}(s)ds + 32\tilde{Q}(0),$$
$$\sup_{[0,t]} tQ \leq C_0 t \int_0^t Q(u)P(u)du + 32tQ(0) \; (u = ts),$$
$$\sup_{[0,t]} Q \leq C_0 \int_0^t Q(u)P(u)du + 32Q(0).$$

□

Now we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. First observe that if T is the first singular time then

$$\lim_{t\to T} Q(t) = \infty$$

by [9]. Now applying Lemma 2.4 with the function $\psi(s) = s\ln(1+s)^p$, $0 \leq p \leq 1$ (it is easy to check that it is nondecreasing and $\int_1^\infty \frac{1}{\psi(s)} ds = \infty$) and Proposition 2.8 yields the result. □

Our method is relatively flexible, illustrated by the following result.

Theorem 2.9. *Let $(M, g(t))$, $t \in [0,T)$, be a Ricci flow solution on M and let $f(t) = \int_0^t P(s)ds$. If for some $0 < p \leq 1$, we have*

$$\int_0^T \frac{P(t)}{f(t)[\ln(1+f(t))]^p} dt < \infty,$$

then the solution can be extended past time T.

Proof. Let $\psi(s) = s[\ln(1+s)]^p$, then we have,

$$f(t) = \int_0^t P(u)du = \int_0^t \psi(f(u))\frac{P(u)}{\psi(f(u))}du.$$

If T is a singular time then $\lim_{t\to T} f(t) = \infty$ by [23] or [11]. Therefore, the result follows from Lemma 2.4. □

Remark 2.2. In view of Lemma 3.1, the integral of $P(t)$ dominates the logarithm of $Q(t)$. Therefore, Theorem 2.9 can be seen as a slight improvement of Theorem 1.1.

3. Time slice approach

In the last section, the essential ingredient to obtain the mean value inequality relating $Q(t)$ and $P(t)$ is the estimate in Lemma 2.7. This result points out that, when the curvature doubles, the integral of the maximum of the Ricci tensor norm is bounded below by a universal constant. It turns out that using the time slice analysis, we can deduce similar results in a slightly different manner. To be more precise, the logarithmic quantity and $ln(\int_0^T P(t)dt)$ blow up together at the first singular time. We shall also derive some other results which might be of independent interest.

First let's fix our notation. For a Ricci flow solution developing a finite time singularity, let s_i be the first time such that $Q(s_i) = 2^{i+4}Q(0)$. Then by Lemma 2.5,

$$s_{i+1} \geq s_i + \frac{1}{16Q(s_i)} = s_i + \frac{1}{8Q(s_{i+1})}. \tag{3.1}$$

Lemma 3.1. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M . Then*

$$\sup_{[0,t]} Q(s) \leq 2^{\frac{1}{\epsilon_0}\int_0^t P(s)ds+1} 16Q(0), \tag{3.2}$$

where ϵ_0 is the constant from the claim of Lemma 2.7.

Proof. The result can be deduced directly from [23, Theorem 3.1]. For completeness, we provide a proof here. From the claim in Lemma 2.7, we have

$$\int_{s_i}^{s_{i+1}} P(t)dt \geq \epsilon_0.$$

Let N be the largest interger such that $s_N \leq t$ then

$$N\epsilon_0 \leq \int_{s_0}^{s_N} P(s)ds \leq \int_0^t P(s)ds,$$

hence

$$N \leq \frac{1}{\epsilon_0}\int_0^t P(s)ds.$$

Thus it follows that

$$\sup_{[0,t]} Q(s) \leq 2^{N+1} 16Q(0) \leq 2^{\frac{1}{\epsilon_0}\int_0^t P(s)ds+1} 16Q(0).$$

□

Next we derive a mean value type inequality using the time slice argument.

Theorem 3.2. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Furthermore, let*

$$G(u) = \ln(16Q(0)) + 2\ln 2 + \frac{\ln 2}{\epsilon_0}\int_0^u P(s)ds.$$

Then for $0 \leq p \leq 1$, we have

$$\ln(G(t)) \leq C_1 \int_0^t \frac{P(s)}{(\ln(1+Q(s)))^p} ds + C_2, \tag{3.3}$$

where $C_1 > 0$ only depends on ϵ_0, $C_2 > 0$ depends on ϵ_0 and $Q(0)$.

Proof. First, without loss of generality, let $Q = \sup_{[0,t]} Q(s) > 2$ and observe that for $0 \leq p \leq 1$,

$$(\ln(1+Q(s)))^p \leq \ln(1+Q(s)) \leq \ln(1+Q).$$

Applying Lemma 3.1,

$$1+Q \leq 2^{\frac{1}{\epsilon_0}\int_0^t P(s)ds+2} 16Q(0),$$

$$\ln(1+Q) \leq \ln(16Q(0)) + 2\ln 2 + \frac{\ln 2}{\epsilon_0}\int_0^t P(s)ds.$$

Since $G(u) = \ln(16Q(0)) + 2\ln 2 + \ln 2 \int_0^u P(s)ds$, we have

$$G'(s) = \frac{\ln 2}{\epsilon_0} P(s) > 0,$$

and

$$G(s) \geq (\ln(1 + Q(s)))^p.$$

Therefore,

$$\frac{\ln 2}{\epsilon_0} \int_0^t \frac{P(s)}{(\ln(1 + Q(s)))^p} ds \geq \int_0^t \frac{G'(s)}{G(s)} ds = \ln G(t) - \ln G(0).$$

The statement now follows immediately. □

Remark 3.1. Theorem 1.1 now follows from Theorem 3.2 and the fact that $\int_0^T P(s)ds$ needs to blow up at the first singular time T *(for example, see [23] or [11])*.

Next we apply the same method to a slightly different setting.

Lemma 3.3. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-non-collapsed Ricci flow solution on M. Then there exists a constant $C = C(Q(0), \kappa)$, such that*

$$Q(s_{i+1}) \leq C \int_{s_i}^{s_{i+1}} \int_M |\mathrm{Rm}|^{\frac{n}{2}+2} d\mu_{g(s)} ds, \tag{3.4}$$

and thus

$$\frac{1}{C} \leq \int_{s_i}^{s_{i+1}} \int_M |\mathrm{Rm}|^{\frac{n}{2}+1} d\mu_{g(s)} ds. \tag{3.5}$$

Proof. Suppose that the statement is false then as $j \to \infty$, there exist $s_{i_j} \to T$ and $a_j \to \infty$, such that

$$a_j \int_{s_{i_j}}^{s_{i_j+1}} \int_M |\mathrm{Rm}|^{n/2+2} d\mu_{g(s)} ds \leq Q(s_{i_j+1}).$$

Therefore, we can choose a blow-up sequence $(x_j, s_{i_j} + 1)$ (in the sense of [7, Theorem 8.4]) and rescale the metric by

$$g_j(t) = Q(s_{i_j+1}) g(s_{i_j+1} + \frac{t}{Q(s_{i_j+1})}).$$

By the Cheeger-Gromov-Hamilton compactness theorem and Perelman's noncollapsing result (for more details, see [7, Chapter 8]), we obtain a singularity model

$$(M_\infty, g_\infty(s), x_\infty)$$

with $|\mathrm{Rm}_\infty(x_\infty, 0)| = 1$.

On the other hand,

$$\begin{aligned}
&\int_{-1/8}^{0}\int_M |\mathrm{Rm}(g_j(t))|^{\frac{n}{2}+2} d\mu_{g_j(t)}dt \\
&= \frac{1}{Q(s_{i_j+1})}\int_{s_{i_j+1}-\frac{1}{8Q(s_{i_j+1})}}^{s_{i_j+1}}\int_M |\mathrm{Rm}(g(s)|^{\frac{n}{2}+2} d\mu_{g(s)}ds \\
&\le \frac{1}{Q(s_{i_j+1})}\int_{s_{i_j}}^{s_{i_j+1}}\int_M |\mathrm{Rm}(g(s)|^{\frac{n}{2}+2} d\mu_{g(s)}ds \\
&\le \frac{1}{a_j} \to 0,
\end{aligned}$$

here (3.1) is used in the first inequality. However, by the dominating convergence theorem, the limit solution is flat, this is a contradiction.

The second statement follows from the first immediately. □

Note that Lemma 3.3 involves a time slice estimate similar in the spirit of the claim in Lemma 2.7 and, thus, applying the same method as before yields the following results. The proofs are omitted as they are almost identical to those of Lemma 3.1 and Theorem 3.2.

Proposition 3.4. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Then*

$$\sup_{[0,t]} Q(s) \le 2^{C\int_0^t\int_M |\mathrm{Rm}|^{\frac{n}{2}+1}d\mu_{g(s)}ds+1}16Q(0). \tag{3.6}$$

Theorem 3.5. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Let*

$$G(u) = \ln(16Q(0)) + 2\ln 2 + C\ln 2\int_0^u\int_M |\mathrm{Rm}|^{\frac{n}{2}+1}d\mu_{g(s)}ds.$$

Then for $0 \le p \le 1$, we have

$$\ln(G(t)) \le C_1\int_0^t\int_M \frac{|\mathrm{Rm}|^{\frac{n}{2}+1}}{(\ln(1+\mathrm{Rm}))^p}d\mu_{g(s)}ds + C_2, \tag{3.7}$$

where $C_1 > 0$ and C_2 only depend on κ and $Q(0)$.

Remark 3.2. It is shown in [22] that the function $G(t)$ must blow up as t approaches the first singular time. Therefore, Theorem 3.5 implies [11, Theorem 1.6].

4. Nonnegative isotropic curvature condition

The notion of nonnegative isotropic curvature was first introduced by M. Micallef and J. D. Moore in [15]. A Riemannian manifold M of dimension $n \geq 4$ is said to have nonnegative isotropic curvature if for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$, that

$$\mathrm{R}_{1313} + \mathrm{R}_{1414} + \mathrm{R}_{2323} + \mathrm{R}_{2424} - 2\mathrm{R}_{1234} \geq 0.$$

The positive condition is defined similarly by replacing the above with a strict inequality. The isotropic curvature is also related to complex sectional curvatures described as follows. For each $p \in M$, let $T_p^C M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$, then the Riemannian metric g extends naturally to a complex bilinear form

$$g : T_p^C M \times T_p^C M \to \mathbb{C},$$

and so is the Riemannian curvature tensor Rm to a complex multilinear form

$$\mathrm{Rm} : T_p^C M \times T_p^C M \times T_p^C M \times T_p^C M \to \mathbb{C}.$$

Then M has nonnegative isotropic curvature if and only if,

$$\mathrm{Rm}(\theta, \eta, \overline{\theta}, \overline{\eta}) \geq 0$$

for all (complex) vectors θ, η satisfying $g(\theta, \theta) = g(\eta, \eta) = g(\theta, \eta) = 0$ (such a plane spanned by θ and η is called an isotropic plane, for more details, see [3]). Furthermore, this nonnegative isotropic curvature condition is implied by several other commonly used curvature conditions, such as nonnegative curvature operator or point-wise $\frac{1}{4}$-pinched sectional curvature conditions, and it implies nonnegative scalar curvature. For more details, please check, for example, [15] or [3].

Another interesting and relevant fact is that this condition is preserved along the Ricci flow. In dimension 4, it was proved by Hamilton [10]; higher dimension analog was extended by S. Brendle and R. Schoen [4] and also by H. Nguyen [17] independently. Using minimal surface technique, Micallef and

Moore [15] showed that any compact, simply connected manifold with positive isotropic curvature is homeomorphic to S^n. By utilizing the Ricci flow and the aforementioned perseverance, Brendle and Schoen further proved the Differentiable Sphere theorem, which has been a long time conjecture since the (topological) $\frac{1}{4}$-pinched Sphere theorem was proved by M. Berger [1] and W. Klingenberg [12] around 60's. More precisely, Brendle and Schoen showed that any compact Riemannian manifold with point-wise $\frac{1}{4}$-pinched sectional curvature is diffeomorphic to a spherical space form [4].

In this section, we apply our analysis to the context of non-flat manifolds with nonnegative isotropic curvature or, slightly more general, satisfying the uniform-growth assumption as in Definition 1.2. Let's first recall the definition of flag curvature and Berger's Lemma.

Definition 4.1. Given a unit vector e, the flag curvature on the direction e is a symmetric bilinear form on $V_e = e^{\perp}$ (the perpendicular compliment of e in $V = R^n$) given by $\mathrm{R}_e(X, X) = \mathrm{Rm}(e, X, e, X)$ for any $X \in V_e$.

We further define $\rho_e = \sup_{|X|=|Y|=1,<X,Y>=0} (R_e(X, X) - R_e(Y, Y))$ and $\rho = \sup_e \rho_e$.

Remark 4.1. It is clear that ρ is no more than the difference between the maximum and minimum of sectional curvatures at each point.

Lemma 4.2 (Berger [2]). *For orthonormal vectors U, V, X, W in T_pM, we have*

a) $|\mathrm{Rm}(U, V, U, W)| \leq \frac{1}{2}\rho_U$,

b) $|\mathrm{Rm}(U, V, X, W)| \leq \frac{1}{6}\rho_{U+X} + \frac{1}{6}\rho_{U-X} + \frac{1}{6}\rho_{U+W} + \frac{1}{6}\rho_{U-W} \leq \frac{2}{3}\rho$.

The Weitzenböck operator F is defined as

$$\mathrm{F} = \mathrm{Rc} \circ g - 2\mathrm{Rm} = \frac{(n-2)\mathrm{R}}{n(n-1)} g \circ g + \frac{n-4}{n-2} \mathrm{E} \circ g - \mathrm{W}.$$

It is well-known that in dimension four, NIC is equivalent to the nonnegativity of F (see, for example, [15, 16, 18]). Furthermore, the space of bi-vectors Λ^2 can be decomposed into Λ^2_+ and Λ^2_- by the Hodge operator $*$. In particular, the Weyl tensor, considered as an operator on 2-forms, is structurally represented as

$$\mathrm{W} = \mathrm{W}_{+} + \mathrm{W}_{+}.$$

As a consequence, $\mathrm{F} \geq 0$ is equivalent to, for $I_{\pm}$ the identity operators on $\Lambda^2_{\pm}$,

$$\frac{\mathrm{R}}{6} I_{\pm} - W_{\pm} \geq 0.$$

We need the following lemma.

Lemma 4.3. *Let (M, g) be a manifold with NIC then the followings hold true.*

- **a**) *If $n = 4$, $|W| \leq \frac{2}{\sqrt{3}}\mathrm{R}$.*
- **b**) *If $n > 4$, $|\mathrm{Rm}| \leq c(n)\mathrm{R}$.*

Part b) is well-known to experts, for example, see [20] or [3, Prop. 7.3]. We provide a proof here for the sake of completeness.

Proof. **a**) Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be eigenvalues of W_+ then

$$|\mathrm{W}^+|^2 = 4\sum_{i=1}^{3} \lambda_i^2,$$
$$0 = \sum_{i=1}^{3} \lambda_i,$$
$$-\frac{\mathrm{R}}{3} \leq \lambda_i \leq \frac{\mathrm{R}}{6},$$

while noticing that, for a $(4, 0)$-tensor, the tensor norm is 4 times the operator norm (sum of squared eigenvalues). We would like to maximize the function $|\mathrm{W}^+|^2 = 4\sum_{i=1}^{3} \lambda_i^2$ on the region identified by the plane $\sum_{i=1}^{3} \lambda_i = 0$ bounded by the the box $-\frac{\mathrm{R}}{3} \leq \lambda_i \leq \frac{\mathrm{R}}{6}$. Since the region is compact, the function attains its maximum.

If $\lambda_1 > -\frac{\mathrm{R}}{3}$ then we can always increase the function by decreasing λ_1 and increasing either λ_2 or λ_3. Thus $|\mathrm{W}_+|$ attains its maximum when $\lambda_1 = -\frac{\mathrm{R}}{3}$ and $\lambda_2 = \lambda_3 = \frac{\mathrm{R}}{6}$. Clearly, the argument holds for W_- and the result follows.

b) If $n > 4$, then we have

$$\begin{aligned} \mathrm{R}_{ikik} + \mathrm{R}_{ilil} + \mathrm{R}_{jkjk} + \mathrm{R}_{jljl} &\geq 0, \\ \mathrm{R}_{ii} + \mathrm{R}_{jj} &\geq 2\mathrm{R}_{ijij}, \\ (n-4)\mathrm{R}_{ii} + \mathrm{R} &\geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{R}_{ii} &\geq -\frac{\mathrm{R}}{n-4}, \\ \mathrm{R}_{ii} &= \mathrm{R} - \Sigma_{j\neq i}\mathrm{R}_{jj} \leq \mathrm{R} + (n-1)\frac{\mathrm{R}}{n-4} = c_1\mathrm{R}, \\ \mathrm{R}_{ijij} &\leq \frac{1}{2}(\mathrm{R}_{ii} + \mathrm{R}_{jj}) \leq c_1\mathrm{R}, \\ \mathrm{R}_{ijij} &\geq -3c_1\mathrm{R}, \end{aligned}$$

Now by Lemma 4.2,

$$|\mathrm{R}_{ijik}| \leq 2c_1\mathrm{R}, \tag{4.1}$$

$$|\mathrm{R}_{ijkl}| \leq \frac{8}{3}c_1\mathrm{R}. \tag{4.2}$$

Thus, there exists a constant $c(n)$ such that

$$|\mathrm{Rm}| \leq c(n)|\mathrm{R}|.$$

□

A direct consequence of the above lemma is the following proposition.

Proposition 4.4. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal Ricci flow solution with NIC, then there exists $c = c(n, g(0))$ such that $|Rm| \leq c\mathrm{R}$ along the flow.*

Proof. If $n > 4$ then the result follows from part b) of Lemma 4.3.
If $n = 4$, then by the pinching estimate of [5],

$$\frac{|\mathring{\mathrm{Rc}}|}{\mathrm{R}} \leq c_1(n, g(0)) + c_2(n) \sup_{M\times[0,T)} \sqrt{\frac{|W|}{R}} \leq c_1 + c_2\sqrt{\frac{2}{\sqrt{3}}}.$$

Furthermore, $|\mathrm{Rm}|^2 = |W|^2 + \frac{\mathrm{R}^2}{6} + 2|\mathring{\mathrm{Rc}}|^2$, the result follows. □

Remark 4.2. One easy consequence is that a non-flat Ricci flow solution on a closed manifold with NIC satisfies the uniform-growth condition as in Definition 1.2.

Our first theorem is similar to[14, Theorems 1.4, 1.5] but we use the new generalized assumption.

Theorem 4.5. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition. If either*

$$\int_M |\mathrm{R}|^\alpha d\mu_{g(t)} < \infty, \quad \textit{for some } \alpha > n/2,$$

or

$$\int_0^T \int_M |\mathrm{R}|^\alpha d\mu_{g(t)} dt < \infty, \quad \textit{for some } \alpha \geq \frac{n}{2} + 1,$$

then the solution can be extended past time T.

Proof. First we observe that, by Holder inequality, for the second condition, it suffices to prove the case $\alpha = \frac{n}{2} + 1$.

The proof is by a contradiction argument. Suppose the flow develops a singularity at time T then we can choose a blow-up sequence (x_i, t_i) (in the sense of [7, Theorem 8.4]), and rescale the metric by $g_i(s) = Q_i g(t_i + \frac{s}{Q_i})$. By the Cheeger-Gromov-Hamilton compactness theorem and Perelman's non-collapsing result (for more details, see [7, Chapter 8]), we obtain a singularity model $(M_\infty, g_\infty(s), x_\infty)$ with

$$|\mathrm{Rm}_\infty(x_\infty, 0)| = 1. \tag{4.3}$$

Recalling the scaling property of R, we calculate,

$$\begin{aligned}\int_M |\mathrm{R}(g_i(.))|^\alpha d\mu_{g_i(.)} &= \int_M Q_i^{-\alpha} |\mathrm{R}(g(.)|^\alpha Q_i^{n/2} d\mu_{g(.)} \\ &= Q_i^{\frac{n}{2}-\alpha} \int_M |\mathrm{R}|^\alpha d\mu_{g(.)} \to 0 \text{ as } i \to \infty.\end{aligned}$$

In the second case, we have:

$$\begin{aligned}
&\int_{-1}^{0}\int_M |\mathrm{R}(g_i(s))|^{\frac{n}{2}+1} d\mu_{g_i(s)} ds \\
&= \int_{t_i-\frac{1}{Q_i}}^{t_i}\int_M Q_i^{-\frac{n}{2}-1} |\mathrm{R}(g(t)|^{\frac{n}{2}+1} Q_i^{n/2} d\mu_{g(t)} Q_i dt \\
&= \int_{t_i-\frac{1}{Q_i}}^{t_i}\int_M |\mathrm{R}(g(t)|^{\frac{n}{2}+1} d\mu_{g(t)} dt \to 0 \text{ as } i \to \infty.
\end{aligned}$$

By the dominating convergence theorem, the singularity model $(M_\infty, g_\infty(s), x_\infty)$ is scalar flat, which is a contradiction to our uniform-growth condition. □

Applying Lemma 2.4 in this context, we obtain the following lemma.

Lemma 4.6. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition. Suppose $\psi : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that*

$$\int_1^\infty \frac{1}{\psi(s)} ds = \infty. \tag{4.4}$$

If there is a mean value inequality of the form

$$O(t) \leq \int_0^t C_1 \psi(O(s)) G(s) ds + C_2 = h(t), \tag{4.5}$$

and $\int_0^T G(t)dt < \infty$, then the solution can be extended past time T.

Proof. First observe that if T is a first singular time then

$$\lim_{t\to T} Q(t) = \infty$$

by [9]. The uniform-growth condition implies that the curvature tensor and the scalar curvature blow up together. Applying Lemma 2.4 we obtain a contradiction, hence the result holds. □

We are ready to state a mean value inequality.

Lemma 4.7. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal Ricci flow solution satisfying the uniform-growth condition. Then the following mean value inequality holds: there exists $C_1 = C_1(n, g(0))$ and C_0 such that,*

$$\sup_{[0,t]} O(t) \leq C_0 \int_0^t \int_M |\mathrm{R}(g(t))|^{n/2+2} d\mu_{g(t)} dt + C_1 \tag{4.6}$$

for all $t < T$.

Proof. First we observe that there is a constant $c_0(n)$ such that

$$|\mathrm{R}|(x,t) \leq c_0 |\mathrm{Rm}|(x,t).$$

Also by Lemma 2.5, if $t \leq \frac{1}{16Q_0}$ then

$$O(t) \leq c_0 Q(t) \leq 2c_0 Q(0). \tag{4.7}$$

Let

$$C_1 = 2c_0 Q(0).$$

Now suppose the statement is false then there exist sequences $t_i \to T$ and $a_i \to \infty$ such that

$$a_i \int_0^{t_i} \int_M |\mathrm{R}|^{n/2+2} d\mu_{g(s)} ds + 2c_0 Q(0) \leq \sup_{[0,t_i]} O(t) \leq c_0 \sup_{[0,t_i]} Q(t).$$

Let $Q_i = \sup_{[0,t_i]} Q(t)$ then there exist x_i, $\tilde{t}_i \to T$ such that $Q_i = |\mathrm{Rm}(x_i, \tilde{t}_i)|$. Now we can invoke a blow-up argument as in Theorem 4.5 around these points to obtain a singularity model $(M_\infty, g_\infty(t), x_\infty)$, $t \in [-\infty, 0]$, with $|\mathrm{Rm}_\infty(x_\infty, 0)| = 1$.

On the other hand, we have

$$\begin{aligned}\int_{-1}^0 \int_M |\mathrm{R}(g_i(s))|^{n/2+2} d\mu_{g_i(s)} ds &= \frac{1}{Q_i} \int_{\tilde{t}_i - \frac{1}{Q_i}}^{\tilde{t}_i} \int_M |\mathrm{R}(g(t)|^{n/2+2} d\mu_{g(t)} dt \\ &\leq \frac{c_0 Q_i - 2c_0 Q(0)}{a_i Q_i} \to 0.\end{aligned}$$

Thus, by the dominating convergence theorem, the limit solution is scalar flat, which is a contradiction to the uniform-growth condition. ☐

Proof of Theorem 1.3. Applying Lemma 4.6 with the function $\psi(s) = s\ln(1+s)^p$, $0 \leq p \leq 1$ (it is easy to check that it is nondecreasing and $\int_1^\infty \frac{1}{\psi(s)} ds = \infty$) and Lemma 4.7 yields the result. ☐

Remark 4.3. If $p = 0$ we recover the first half of Theorem 4.5.

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Department of Mathematics, Cornell University
Ithaca, NY 14853-4201, USA
E-mail address: cao@math.cornell.edu

Department of Mathematics, UCI
Irvine, CA 92697, USA
E-mail address: hungtran@math.cornell.edu

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