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Scales on Π^2_1 sets

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Assuming $\mathsf{AD}^+ + \theta_0 < \Theta$ we construct scales of optimal complexity on Π^2_1 sets of reals. Namely, the norms of the scale are all ordinal-definable (although the scale itself may not be). This paper extends work of Martin and Woodin from the 1980s as well as more recent work of Jackson. The results of this paper were proved in the author's thesis [13] for more general pointclasses and are presented here for the representative case of the pointclass Π^2_1.

1. Introduction

A central question in descriptive set theory is: Which sets of reals admit scales, and how complex are those scales? The notion of a scale was introduced by Moschovakis as a general way of proving uniformization theorems; that is, proving that if a set in the plane intersects every vertical line, then it contains the graph of a function. Typically, such theorems impose constraints on the complexity of the set and of the function in order to avoid the influence of the Axiom of Choice.

We let $\mathcal{N}$ denote the Baire space, which is the set ω^ω of infinite sequences of natural numbers with the product of the discrete topologies on ω. It is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ with the subspace topology from $\mathbb{R}$, and we will abuse terminology in the usual way by referring to elements of $\mathcal{N}$ as "reals." For any positive integer m the product space $\mathcal{N}^m$ is homeomorphic to $\mathcal{N}$ itself, so we will refer to elements of $\mathcal{N}^m$ as reals also.

We include some basic definitions relating to scales below. For a thorough introduction to the subject see the paper of Kechris and Moschovakis [4] or Moschovakis's book [10].

Definition 1.1. Consider a set of reals $A \subset \mathcal{N}^m$ where m is a positive integer. A function $\varphi : A \to \mathrm{Ord}$, where Ord denotes the class of ordinal numbers, is called a *norm* on A.

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A *semiscale* on A is a sequence $(\varphi_i : i < \omega)$ of norms on A with the property that if $(x_k : k < \omega)$ is a sequence of reals from A converging to some real x, and for all $i < \omega$ the sequence of ordinals $(\varphi_i(x_k) : k < \omega)$ is eventually constant, then x is in A.

A *scale* on A is a semiscale on A with the additional property of *lower semicontinuity*, which says that if $x_k \to x$ as in the definition of a semiscale, then $\varphi_i(x) \leq \lim_{k\to\omega} \varphi_i(x_k)$ for all $i < \omega$.

There is a fundamental connection between scales, semiscales, and trees. For a positive integer m, an ordinal κ, and a tree T on $\omega^m \times \kappa$, we let $[T] \subset \mathcal{N}^m \times \kappa^\omega$ denote the set of branches of T. This set $[T]$ is a closed subset of $\mathcal{N}^m \times \kappa^\omega$ (where κ^ω has the product of the discrete topologies) and conversely every closed subset of $\mathcal{N}^m \times \kappa^\omega$ has the form $[T]$ for some tree T. We let $p[T] \subset \mathcal{N}^m$ denote the projection of this set $[T]$ onto $\mathcal{N}^m$ along the last coordinate.

A set of reals is called κ-*Suslin* if it has the form $p[T]$ for some tree T on $\omega^m \times \kappa$ for some positive integer m, and it is called *Suslin* if it is κ-Suslin for some ordinal κ. For example, the ω-Suslin subsets of $\mathcal{N}^m$ are just the projections of closed subsets of $\mathcal{N}^{m+1}$. These are the $\boldsymbol{\Sigma}^1_1$ (analytic) subsets of $\mathcal{N}^m$.[1] If the Axiom of Choice holds then every set of reals is $2^{\aleph_0}$-Suslin, so it is more interesting to consider Suslin sets under the Axiom of Determinacy (AD), which contradicts AC.

A set of reals is Suslin if and only if it admits a semiscale, which holds in turn if and only if it admits a scale. However, in some cases the construction of a scale from a semiscale can increase the complexity of the norms because the entire sequence of norms of the semiscale is needed to define a single norm of the scale. Because we wish to get scales of optimal complexity, we will construct them more directly.

For the same reason, instead of working with boldface pointclasses we will work with lightface pointclasses. For example, whereas the $\boldsymbol{\Sigma}^1_1$ subsets of $\mathcal{N}^m$ are defined as the projections of closed subsets of $\mathcal{N}^{m+1}$, the Σ^1_1 subsets of $\mathcal{N}^m$ are defined as the projections of *effectively closed* subsets of $\mathcal{N}^{m+1}$, which have the form $[T]$ for T a *computable* tree on ω^{m+1}.

The following theorem is an important source of scales on sets far beyond the Σ^1_1 sets. Namely, on Σ^2_1 sets, which are sets of the form

$$\{x \in \mathcal{N}^m : \exists A \subset \mathcal{N}\, (V_{\omega+1}; \in, A) \models \varphi[x]\},$$

[1]Projecting closed subsets of $\mathbb{R}^{m+1}$ (rather than $\mathcal{N}^{m+1}$) does not give all analytic subsets of $\mathbb{R}^m$, which is one reason that the Baire space is a more natural setting for descriptive set theory.

where φ is a formula in the language of set theory with a unary predicate symbol for the set A. Such sets are best studied under the axiom AD^+, which is a natural strengthening of AD that holds in all known models of AD.

Theorem 1.2 (Woodin, $\mathsf{ZF} + \mathsf{AD}^+$). *Every Σ^2_1 set of reals $A \subset \mathcal{N}^m$ has a Σ^2_1-scale, and therefore is the projection of a definable tree on $\omega^m \times \boldsymbol{\delta}^2_1$.*

Here $\boldsymbol{\delta}^2_1$ denotes the supremum of the lengths of $\boldsymbol{\Delta}^2_1$ pre-wellorderings of the reals. We will not need to use the precise definition of "Σ^2_1-scale" but we remark that it yields a uniformization by a Σ^2_1 function.

The special case of the theorem for $L(\mathbb{R})$ was proved by Martin and Steel [7] and this was later generalized to $K(\mathbb{R})$ by Steel [11]. The general case remains unpublished, although Steel [12] presents a proof in the case of derived models and remarks that this can be shown to imply the general case.

The next natural targets for scales after the Σ^2_1 sets are their complements, the Π^2_1 sets. In this paper, we give a method for constructing scales on Π^2_1 sets whose norms have the optimal complexity, namely they are ordinal-definable. We will need an extra hypothesis because there are some models of AD^+, such as $L(\mathbb{R})$, where some Π^2_1 sets admit no scales at all. (This observation is due to Kechris and Solovay, according to Martin and Steel [7].)

To state our main theorem we must first recall some standard terminology from the study of AD. We let Θ denote the least ordinal that is not a surjective image of the reals and we let θ_0 denote the least ordinal that is not a surjective image of the reals by an ordinal-definable function. Under AD, the ordinal Θ is the height of the Wadge pre-wellordering of $\wp(\mathbb{R})$ and θ_0 is the supremum of the Wadge ranks of ordinal-definable sets of reals, and the inequality $\theta_0 < \Theta$ is equivalent to the existence of a set of reals that is not ordinal-definable from any real parameter.

Main theorem ($\mathsf{ZF} + \mathsf{AD}^+$). *If $\theta_0 < \Theta$ then every Π^2_1 set of reals A has a scale $\vec{\varphi}$ whose norms φ_i are ordinal-definable.*

Remark 1.3. The method that we will use to prove the main theorem can also be applied in a more general "partial determinacy" setting that can be useful in a core model induction. In that setting, the AD does not hold in V. Instead we have a model $\mathcal{M}$ of $\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 = \Theta$ containing all the reals, and we assume some strong hypothesis in V which takes the place of the hypothesis $\theta_0 < \Theta$.

The optimal-complexity scales on $(\Pi^2_1)^{\mathcal{M}}$ sets will then have norm relations cofinal in the Wadge hierarchy of $\mathcal{M}$, and their construction will be the first step in building a larger determinacy model containing $\mathcal{M}$ and satisfying the stronger determinacy theory $\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \Theta$. For such an application of the method, see the author's thesis [13, Ch. 4].

From the main theorem we can derive a uniformization result in the standard way (see Moschovakis [10].) Namely, let A be a Π^2_1 subset of $\mathcal{N}^2$ that is a total relation, meaning that for every real x the section $A_x = \{y \in \mathcal{N} : (x, y) \in A\}$ is non-empty. Then the sequence

$$(\varphi_0(x, y), y(0), \varphi_1(x, y), y(1), \ldots)$$

attains a lexicographically least value on some $y \in A_x$.

This is because for every $k < \omega$ we can take a real $y_k \in A_x$ making the finite sequence $(\varphi_0(x, y_k), y_k(0), \ldots, \varphi_{k-1}(x, y_k), y_k(k-1))$ lexicographically least, and let $y = \lim_{k\to\omega} y_k$. Then we have $y \in A_x$ by the semiscale property, and the desired lexicographic minimality property of y follows from the lower semicontinuity property of scales.

If $f(x)$ denotes the unique real $y \in A_x$ attaining this least value, then the function given by $x \mapsto f(x) \restriction i$ is definable from the norms $\varphi_0, \ldots, \varphi_{i-1}$, giving the following result.

Corollary 1.4 ($\mathsf{ZF} + \mathsf{AD}^+$). *If $\theta_0 < \Theta$ then every Π^2_1 total relation $A \subset \mathcal{N}^2$ is uniformized by a function $f : \mathcal{N} \to \mathcal{N}$ such that for every $i < \omega$ the function $\mathcal{N} \to \omega^i$ given by $x \mapsto f(x) \restriction i$ is ordinal-definable.*

Remark 1.5. Under AD every well-ordered set of reals is countable, so for every real x there is a real y that is not ordinal-definable from x. In other words, the relation $A = \{(x, y) \in \mathcal{N}^2 : y \notin \mathrm{OD}_x\}$ is total. Under $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$[2] this relation is Π^2_1 by Woodin's Σ_1 reflection theorem (see Steel [12]). Therefore by Corollary 1.4 there is a function $f : \mathcal{N} \to \mathcal{N}$ such that $f(x)$ is not in OD_x for any $x \in \mathcal{N}$, but for every $i < \omega$ the initial segments $f(x) \restriction i$ are OD_x uniformly in x.

This remark demonstrates the necessity of the hypothesis $\theta_0 < \Theta$ in the main theorem and in Corollary 1.4: any uniformization of the set $\{(x, y) \in$

[2]Without the assumption $V = L(\wp(\mathbb{R}))$ we would have to replace "$y \in \mathrm{OD}_x$" with "y is ordinal-definable from x in the model $L(B, \mathbb{R})$ for some set of reals B" but our argument would be otherwise unchanged.

$\mathcal{N}^2 : y \notin \mathrm{OD}_x\}$ must have Wadge rank at least θ_0 because no such uniformization can be ordinal-definable from a real. Indeed, if $g(x) \notin \mathrm{OD}_x$ for every real x and the function g is OD_{x_0}, then we get a contradiction from $g(x_0) \in \mathrm{OD}_{x_0}$.

There are several existing results approximating the main theorem, which will be briefly discussed below. Our starting point is a method of constructing weak homogeneity systems due to Martin and Woodin [9], which can be used to show that Π^2_1 sets are Suslin (and therefore admit scales) under the additional hypothesis of $\mathsf{AD}_{\mathbb{R}}$. Woodin [14] subsequently weakened the hypothesis to that of the main theorem.

(The result of Martin and Steel [8] can also be used to show under the hypothesis of the main theorem that Π^2_1 sets are Suslin, but their method is less relevant to the present paper.)

By the result of Woodin stated as Theorem 1.2 above, every Π^2_1 set $A \subset \mathcal{N}^m$ has the form $\mathcal{N}^m \setminus p[T]$ for some definable tree T on $\omega^m \times \boldsymbol{\delta}^2_1$. A weak homogeneity system $\vec{\mu}$ for T can be obtained by the methods [9, 14] mentioned above. Then the method of Martin and Solovay [6] can be applied to T and $\vec{\mu}$ to construct a semiscale on the set $A = \mathcal{N}^m \setminus p[T]$ with the property that each of its norms, given by $(*)$ below, is definable from a measure in $\vec{\mu}$.

Under AD^+ (or just $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) every measure on an ordinal $\kappa < \Theta$ is ordinal-definable by a theorem of Kunen (see Steel [12, Theorem 8.6] for a proof) and this generalizes trivially to measures on $\kappa^{<\omega}$, so the norms of the Martin–Solovay semiscale are ordinal-definable. However, obtaining a *scale* with ordinal-definable norms is more difficult because it is not clear whether this Martin–Solovay semiscale is a scale in general (see Jackson [1, Remark 3.16]).

By a theorem of Jackson [2], the tree T has a *stability* property that gives a subtree $T' \subset T$ on which each measure μ_i in the weak homogeneity system still concentrates, and such that the Martin–Solovay semiscale from T' and $\vec{\mu}$ is a scale.

Passing from T to the subtree T' introduces a real parameter to the calculation of the norms, so Jackson's stability result does not directly imply the main theorem (or Corollary 1.4). To establish the main theorem, we will need to combine Woodin's semiscale construction [14] with ideas from Jackson's stability proof. Combining these methods also allows us to use a more direct argument that bypasses the notion of weak homogeneity.

We remark that Woodin has already proved, by a entirely different method, that the conclusion of the main theorem follows from the hypothesis

$\mathsf{AD}^+ + \theta_0 < \Theta$ augmented by an additional "mouse capturing" hypothesis. That work is unpublished.

2. Towers of measures

In this section we recall some basic concepts regarding towers of measures. A good reference for this material is Larson [5]. The background theory for these concepts is $\mathsf{ZF} + \mathsf{DC}$, where DC denotes the Axiom of Dependent Choices. This suits our needs because of the following observation.[3]

Remark 2.1. In proving the main theorem, we may assume DC without loss of generality. This is because both the hypothesis and the conclusion of the main theorem are absolute to an inner model where DC holds. More precisely, assume $\mathsf{AD}^+ + \theta_0 < \Theta$, let B be a set of reals of Wadge rank $\beta \geq \theta_0$, and define the model $\mathcal{M} = L(B, \mathbb{R})$.

The model $\mathcal{M}$ satisfies AD^+ because AD^+ is downward absolute to inner models with the same reals. Moreover, $\mathcal{M}$ satisfies DC because it satisfies $\mathsf{DC}_{\mathbb{R}}$, which is one of the clauses of AD^+, as well as the statement "every set is ordinal-definable from the parameter B and a real." Also, $\mathcal{M}$ satisfies $\theta_0 < \Theta$: it is definable from β, so every set that is ordinal-definable in $\mathcal{M}$ is ordinal-definable in V and we have $\theta_0^{\mathcal{M}} \leq \theta_0 \leq \beta < \Theta^{\mathcal{M}}$. (In fact $\theta_0^{\mathcal{M}} = \theta_0$ but we will not need this.)

It remains to observe that every Π^2_1 set of reals is Π^2_1 in $\mathcal{M}$. This is because Σ^2_1 statements about reals are absolute to $\mathcal{M}$ by Woodin's Σ^2_1 basis theorem (see Steel [12, Lemma 8.2]): every true $\Sigma^2_1(x)$ statement has a $\Delta^2_1(x)$ witness, and every such witness is in $\mathcal{M}$.

We let $\mathrm{meas}(\kappa^{<\omega})$ denote the set of countably complete measures on the set $\kappa^{<\omega}$. Each such measure concentrates on κ^n for some $n < \omega$. Given a measure $\mu \in \mathrm{meas}(\kappa^{<\omega})$, we define the corresponding ultrapower $\mathrm{Ult}(V, \mu)$ of V using all functions in V. The well-foundedness of this ultrapower follows from DC.

We let $j_\mu : V \to \mathrm{Ult}(V, \mu)$ denote the corresponding ultrapower embedding, and for a function $F \in V$ with domain $\kappa^{<\omega}$ we let $[F]_\mu$ denote the set represented by F in $\mathrm{Ult}(V, \mu)$. In particular, for every well-founded tree W

[3]Another approach to the proof of the main theorem, which was used in an earlier draft of this paper, is to note that the fragment $\mathsf{DC}_{\wp(\kappa^{<\omega})}$ of DC suffices for the argument provided that we use only functions in the collection $\bigcup\{L[S] : S \subset \kappa^{<\omega}\}$ when taking ultrapowers. Although that approach is more elementary, it complicates the exposition.

on κ we have a corresponding rank function $\mathrm{rank}_W : \kappa^{<\omega} \to \mathrm{Ord}$ and we can define the ordinal $[\mathrm{rank}_W]_\mu$.

In the absence of the Axiom of Choice we cannot prove elementarity for the embeddings j_μ, but for any set $S \subset \kappa^{<\omega}$ the model $L[S]$ is well-ordered, so the restriction $j_\mu \restriction L[S]$ is an elementary embedding from $L[S]$ to the ultrapower $\mathrm{Ult}(L[S], \mu)$ where again we take the ultrapower using all functions in V (not just in $L[S]$.) It is not the map j_μ itself that will be useful, but only such restrictions of it.

We will use the standard definitions for projections of measures, towers of measures, and countable completeness of towers as given by Larson [5, Section 1.2]. If the measure μ' projects to μ we let $j_{\mu,\mu'}$ denote the natural factor map, so we have $j_{\mu'} = j_{\mu,\mu'} \circ j_\mu$.

Recall that a *tower* of measures on $\kappa^{<\omega}$ is a sequence $(\mu_i : i < \omega)$ of measures such that for each $i < \omega$ the measure μ_i concentrates on κ^i, and whenever $i < j < \omega$ the measure μ_j projects to μ_i. Recall also that such a tower $(\mu_i : i < \omega)$ is called *countably complete* if, for every sequence $(X_i : i < \omega)$ such that $X_i \in \mu_i$ for all $i < \omega$, there is a sequence $f \in \kappa^\omega$ with the property that $f \restriction i \in X_i$ for all $i < \omega$. We will use the following standard fact, which can be found in Larson [5, Proposition 1.2.2 and Theorem 1.2.3]:

Lemma 2.2. *For a tower of measures $\vec{\mu} = (\mu_i : i < \omega)$ from* $\mathrm{meas}(\kappa^{<\omega})$ *the following statements are equivalent:*

(1) *The direct limit of the ordinals under the system of ultrapower maps $(j_{\mu_i,\mu_j} : i \leq j < \omega)$ is ill-founded.*

(2) *The tower $\vec{\mu}$ is not countably complete.*

(3) *There is a sequence of ordinals $(h_i : i < \omega)$ such that $j_{\mu_i,\mu_{i+1}}(h_i) > h_{i+1}$ for all $i < \omega$.*

In light of the equivalence of these three statements, if the tower $\vec{\mu}$ is not countably complete, we call $\vec{\mu}$ itself *ill-founded* and we call a sequence of ordinals $(h_i : i < \omega)$ as in statement (2.2) a *witness to the ill-foundedness of $\vec{\mu}$*. We also consider functions that witness the ill-foundedness of many towers simultaneously:

Definition 2.3. Given a set σ of countably complete measures on $\kappa^{<\omega}$, a *continuous witness to the ill-foundedness of all towers of measures from σ* is a function $H : \sigma \to \mathrm{Ord}$ such that for any two distinct measures μ and μ' in σ with μ' projecting to μ, we have $j_{\mu,\mu'}(H(\mu)) > H(\mu')$.

This terminology is justified by the observation that if σ and H are as above then there can be no well-founded tower of measures from σ. In the context of $\mathsf{ZF} + \mathsf{DC}$ it is not known whether the non-existence of well-founded towers of measures from σ implies the existence of such a continuous witness H, although in two special cases it is known to be true: if AC holds and $|\sigma|$ is less than the completeness of the measures in σ (see Larson [5, Lemma 1.3.8]) and if AD holds and σ is countable. The latter case is an observation of Martin and possibly others; a similar but more complicated argument appears in a proof of Martin and Woodin [9, Lemma 2.2].

Note that Definition 2.3 includes the possibility that σ is not large enough to form any towers of measures at all (for example, if it is finite or even empty), but we will not typically consider such trivial σ.

For any set of measures $\sigma \subset \operatorname{meas}(\kappa^{<\omega})$, if W is a well-founded tree on κ then the function H defined on the subset of measures $\{\mu \in \sigma : W \in \mu\}$ by $H(\mu) = [\operatorname{rank}_W]_\mu$ is a continuous witness to the ill-foundedness of all towers of measures from this subset. (We abbreviate this special case of Definition 2.3 by saying "the function H is a continuous witness to the ill-foundedness of all towers of measures from σ concentrating on the tree W.")

To verify this standard fact, note that whenever we have measures μ and μ' concentrating on levels n and $n+1$ of W, respectively, and such that μ' projects to μ, we can represent the ordinal $j_{\mu,\mu'}(H(\mu))$ in the μ'-ultrapower by the function that maps a node in level $n+1$ of W to the rank of its predecessor. Because the rank of a node in a well-founded tree is less than the rank of its predecessor, we have $j_{\mu,\mu'}(H(\mu)) > H(\mu')$.

3. Scales from sets of measures

The new result that allows us to prove the main theorem is the following lemma, whose proof we defer until the next section. The statement of the lemma uses the following standard notation. Given a positive integer m, an ordinal κ, a tree T on $\omega^m \times \kappa$, and a real $x \in \mathcal{N}^m$, we define the tree T_x on κ by

$$T_x = \{s \in \kappa^{<\omega} : (x \upharpoonright |s|, s) \in T\},$$

so we have $x \in p[T]$ if and only if T_x is ill-founded. (In taking the restriction $x \upharpoonright |s|$ we are considering x as an infinite sequence of m-tuples rather than as an m-tuple of infinite sequences.) If $x \notin p[T]$ then the function given by $\mu \mapsto [\operatorname{rank}_{T_x}]_\mu$ is a continuous witness to the ill-foundedness of all towers of measures from $\operatorname{meas}(\kappa^{<\omega})$ concentrating on T_x.

Lemma 3.1 ($\mathsf{ZF}+\mathsf{DC}$). *Let T be a tree on $\omega^m \times \kappa$ for some positive integer m and ordinal κ. Let μ be a countably complete measure on $\kappa^{<\omega}$. Assume that there is a countably complete fine measure on $\wp_{\omega_1}(\mathrm{meas}(\kappa^{<\omega}))$. Let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order, so that μ_0 is the trivial measure on κ^0 and μ_n is μ itself.*

Then there is a countable set of measures $\sigma \subset \mathrm{meas}(\kappa^{<\omega})$ containing $\mu_0, \ldots, \mu_n$ and such that for any real $x \in \mathcal{N}^m$ with $T_x \in \mu$, and any continuous witness H to the ill-foundedness of all towers of measures from σ concentrating on T_x, we have $[\mathrm{rank}_{T_x}]_\mu \le H(\mu)$.

In the lemma we pretend that an ill-founded tree has a rank, denoted by ∞, that is greater than every ordinal. So letting μ be the trivial measure μ_0 on κ^0, the lemma says that if $x \in p[T]$ then no such continuous witness H can exist. This implies that for any enumeration $\vec{\mu}$ of σ the Martin–Solovay construction from the pair $(T, \vec{\mu})$ yields a semiscale on the set of reals $\mathcal{N}^m \setminus p[T]$, although in general it is not known to imply that $\vec{\mu}$ is actually a weak homogeneity system for T. With a bit more work we can get a scale from the lemma, proving the main theorem.

Proof of the main theorem. Assume $\mathsf{ZF}+\mathsf{AD}^+ + \theta_0 < \Theta$. By Remark 2.1 we may also assume DC, so that the lemma applies. Let A be a Π^2_1 set of reals. We want to construct a scale on A whose norms are ordinal-definable. By Woodin's theorem we have $A = \mathcal{N}^m \setminus p[T]$ for some definable tree T on $\omega^m \times \kappa$ where κ is the ordinal $\boldsymbol{\delta}^2_1$.

The coding lemma of Moschovakis [10] gives us a definable surjection from $\mathbb{R}$ onto $\wp(\kappa^{<\omega})$. We can code any measure on $\kappa^{<\omega}$ by an ordinal-definable set of reals, namely its pre-image under this surjection. We are assuming that $\theta_0 < \Theta$, so there are fewer than Θ many ordinal-definable sets of reals, and this fact together with the coding of measures yields a surjection from $\mathbb{R}$ onto $\mathrm{meas}(\kappa^{<\omega})$.[4]

Now we consider Martin's measure on the set $\wp_{\omega_1}(\mathbb{R})$, which is defined to consist of all subsets $\mathcal{X} \subset \wp_{\omega_1}(\mathbb{R})$ such that $\{x \in \mathbb{R} : x \le_{\mathrm{T}} d\} \in \mathcal{X}$ for a cone of Turing degrees d. It is a countably complete fine measure on $\wp_{\omega_1}(\mathbb{R})$, and pushing it forward by a surjection $\mathbb{R} \to \mathrm{meas}(\kappa^{<\omega})$ yields a countably complete fine measure on $\wp_{\omega_1}(\mathrm{meas}(\kappa^{<\omega}))$.

Therefore all the conditions of the lemma are met. Using DC, we build an increasing sequence of countable sets of measures $(\sigma_j : j < \omega)$ as follows.

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[4]If instead of $\theta_0 < \Theta$ we assume the hypothesis $\mathsf{AD}^{1/2}_{\mathbb{R}}$, which postulates determinacy for games where one player plays reals and the other player plays integers, then we can use the coding of measures from Kechris [3] to achieve the same result.

Let $\sigma_0 = \{\mu_0\}$ where μ_0 is the trivial measure on κ^0. Given σ_j, let σ_{j+1} be a countable set of measures having the property of the lemma with respect to every measure $\mu \in \sigma_j$, which is possible because adding more measures preserves this property.

Now we let $\sigma = \bigcup\{\sigma_j : j < \omega\}$. This set has the property of the lemma with respect to each of its elements. That is, for any real $x \in \mathcal{N}^m$ and any continuous witness H to the ill-foundedness of all towers of measures from σ concentrating on T_x, we have $[\text{rank}_{T_x}]_\mu \leq H(\mu)$ for all measures $\mu \in \sigma$ concentrating on T_x. Enumerating σ as a sequence of measures $(\mu_i : i < \omega)$, we claim that the corresponding sequence of norms $\vec{\varphi} = (\varphi_i : i < \omega)$ given by

$$(*) \qquad \varphi_i(x) = \begin{cases} [\text{rank}_{T_x}]_{\mu_i} & \text{if } T_x \in \mu_i, \\ 0 & \text{if } T_x \notin \mu_i \end{cases}$$

is a scale on the set $\mathcal{N}^m \setminus p[T]$ (which is A). The choice of enumeration is not important because permuting the norms of a scale yields another scale. We call this sequence of norms the *Martin–Solovay semiscale* from $(T, \vec{\mu})$. Recall that each of the norms φ_i is ordinal-definable by Kunen's theorem.

We caution the reader that one normally speaks of the Martin–Solovay semiscale from $(T, \vec{\mu})$ only in the case that $\vec{\mu}$ is a weak homogeneity system for T, meaning that whenever $x \in p[T]$ there is a well-founded tower of measures from $\vec{\mu}$ concentrating on T_x. We have not verified this — and we do not need to — but we remark that under AD it follows from the non-existence of a continuous witness to the ill-foundedness of all such towers. (See the remarks following Definition 2.3.)

To see that the sequence of norms $\vec{\varphi}$ is a scale, let $(x_k : k < \omega)$ be a sequence of reals from A converging to a real x, and assume that for all $i < \omega$ the sequence of ordinals $(\varphi_i(x_k) : k < \omega)$ is eventually constant. Define a function H on the set $\{\mu \in \sigma : T_x \in \mu\}$ by $H(\mu) = \lim_{k\to\omega}[\text{rank}_{T_{x_k}}]_\mu$.

This function H is a continuous witness to the ill-foundedness of all towers of measures from σ concentrating on the tree T_x: if μ concentrates on T_x then it also concentrates on the well-founded tree T_{x_k} for all sufficiently large k, and the rank function $\text{rank}_{T_{x_k}}$ is a strictly decreasing function on T_{x_k}.

Therefore for every $i < \omega$, if the measure μ_i concentrates on T_x then by our choice of σ we have $[\text{rank}_{T_x}]_{\mu_i} \leq H(\mu_i)$, or in other words, $\varphi_i(x) \leq H(\mu_i)$. If μ_i does not concentrate on T_x then we simply have $\varphi_i(x) = 0$, so in any case we have $\varphi_i(x) \leq H(\mu_i) = \lim_{k\to\omega} \varphi_i(x_k)$, establishing the lower semicontinuity property in the definition of scales.

In particular, we have $\mathrm{rank}(T_x) = [\mathrm{rank}_{T_x}]_{\mu_0} \leq H(\mu_0) < \infty$ for the trivial measure μ_0 on κ^0. This implies that the tree T_x is well-founded, so $x \in A$. $\Box$

4. Proof of Lemma 3.1

Let m, κ, T, and μ be as in the lemma. We will assume $m = 1$ for simplicity of notation; the argument for higher dimensions is similar. We let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order. So μ_0 denotes the trivial measure on κ^0 and μ_n denotes μ itself, and for each $i \leq n$ the measure μ_i concentrates on κ^i. For a countable set of measures $\sigma \subset \mathrm{meas}(\kappa^{<\omega})$ containing $\mu_0, \ldots, \mu_n$ we define the following game, where for convenience of notation the moves are numbered starting with n.

$$(\mathcal{G}_T^{\sigma,\mu}) \qquad \begin{array}{lccccc} \mathrm{I} & r_n, s_n, h_n & & r_{n+1}, s_{n+1}, h_{n+1} & & \ldots \\ \mathrm{II} & & \mu_{n+1} & & \mu_{n+2} & \ldots \end{array}$$

Rules for Player I:

- for every $i \geq n$ we have $r_i \in \omega^{i+1}$ and $s_i \in j_{\mu_i}(\kappa)^{i+1}$,
- $[\mathrm{id}]_{\mu_n} \subset s_n$, and
- for every $i \geq n$,

$$\begin{aligned} (r_i, s_i) &\in j_{\mu_i}(T), & j_{\mu_i,\mu_{i+1}}(r_i, s_i) &\subsetneq (r_{i+1}, s_{i+1}), \\ h_i &\in \mathrm{Ord}, & j_{\mu_i,\mu_{i+1}}(h_i) &> h_{i+1}. \end{aligned}$$

Rules for Player II: for every $i \geq n$,

- $\mu_{i+1} \in \sigma$,
- μ_{i+1} projects to μ_i, and
- μ_{i+1} concentrates on the set $T_{r_i} \subset \kappa^{i+1}$.

The first player to deviate from these rules loses, and if both players follow the rules for all ω moves then Player I wins. An infinite run of the game yields a real

$$x = \bigcup \{r_i : i \geq n\} \in \mathcal{N},$$

a tower of measures $\vec{\mu} = (\mu_i : i < \omega)$ from σ extending the given sequence $\mu_0, \ldots, \mu_n$, and a witness $(h_i : i \geq n)$ to the ill-foundedness of this tower. (Technically speaking, to get a witness to ill-foundedness we must fill in the first n ordinals $h_0, \ldots, h_{n-1}$, but it is always possible to do this and it doesn't matter how we do it.) Note that every measure μ_i in the tower concentrates on the tree T_x. For $i \leq n$ this is guaranteed by the rules for Player I because $[\mathrm{id}]_{\mu_n} \subsetneq s_n \in j_{\mu_n}(T_{r_n})$, and for $i > n$ this is guaranteed by the rules for Player II.

To illustrate the role of the moves s_i we consider an example. Suppose the Martin–Solovay construction from T and an enumeration of σ does not even yield a *semiscale* on the set of reals $\mathcal{N} \setminus p[T]$. Then there is some real $x \in p[T]$ and a continuous witness H to the ill-foundedness of all towers of measures from σ concentrating on T_x. Let $f \in \kappa^\omega$ witness $x \in p[T]$; that is, $(x, f) \in [T]$. Letting μ_0 denote the trivial measure on κ^0, Player I has a winning strategy in the game $\mathcal{G}_T^{\sigma,\mu_0}$ given by playing $r_i = x \upharpoonright (i+1)$, $s_i = j_{\mu_i}(f \upharpoonright (i+1))$, and $h_i = H(\mu_i)$.

The following claim shows more generally that we can get a winning strategy for Player I from a failure of lower semicontinuity.

Claim. Let $\sigma \subset \mathrm{meas}(\kappa^{<\omega})$ be a countable set of measures containing all the projections $\mu_0, \ldots, \mu_n$ of μ. Suppose that there is a real $x \in \mathcal{N}$ with $T_x \in \mu$ and a continuous witness H to the ill-foundedness of all towers of measures from σ concentrating on T_x, but $[\mathrm{rank}_{T_x}]_\mu > H(\mu)$. (The left hand side may be ∞.) Then Player I has a winning strategy for the game $\mathcal{G}_T^{\sigma,\mu}$.

Proof. As before we let $\mu_0, \ldots, \mu_n$ denote the projections of μ in order. On the first turn (which is numbered n), Player I plays $r_n = x \upharpoonright (n+1)$. We have

$$\mathrm{rank}_{j_{\mu_n}(T_x)}([\mathrm{id}]_{\mu_n}) = [\mathrm{rank}_{T_x}]_{\mu_n} > H(\mu_n),$$

so Player I can play a successor s_n of the node $[\mathrm{id}]_{\mu_n}$ in the tree $j_{\mu_n}(T_x)$ with $\mathrm{rank}_{j_{\mu_n}(T_x)}(s_n) \geq H(\mu_n)$. For the ordinal part of the move, Player I plays $h_n = H(\mu_n)$.

Now let $i \geq n$ and suppose that Player I has played the finite sequences $r_n, \ldots, r_i$ and $s_n, \ldots, s_i$ and ordinals $h_n, \ldots, h_i$ and that Player II has played the measures $\mu_{n+1}, \ldots, \mu_{i+1}$. Suppose that the rules of the game have been followed and moreover suppose that we have maintained the inequality

$$(**) \qquad \mathrm{rank}_{j_{\mu_i}(T_x)}(s_i) \geq H(\mu_i).$$

Applying the embedding $j_{\mu_i \mu_{i+1}}$ to both sides of this inequality, we get

$$\mathrm{rank}_{j_{\mu_{i+1}}(T_x)}(j_{\mu_i,\mu_{i+1}}(s_i)) \geq j_{\mu_i,\mu_{i+1}}(H(\mu_i)),$$

and then because the measure μ_{i+1} projects to the measure μ_i and both measures concentrate on the tree T_x, our hypothesis on H yields

$$\operatorname{rank}_{j_{\mu_{i+1}}(T_x)}(j_{\mu_i,\mu_{i+1}}(s_i)) > H(\mu_{i+1}),$$

so we can take a successor s_{i+1} of the node $j_{\mu_i,\mu_{i+1}}(s_i)$ in the tree $j_{\mu_{i+1}}(T_x)$ with

$$\operatorname{rank}_{j_{\mu_{i+1}}(T_x)}(s_{i+1}) \geq H(\mu_{i+1}),$$

maintaining the inequality $(**)$ for one more step. To complete the move, we play $r_{i+1} = x \restriction (i+2)$ and $h_{i+1} = H(\mu_{i+1})$. Player I can follow the rules in this manner for ω moves, thereby winning the game. $\square$

Now to prove the lemma we suppose toward a contradiction that we have a countably complete fine measure $\mathcal{U}$ on $\wp_{\omega_1}(\operatorname{meas}(\kappa^{<\omega}))$ and that Player I has a winning strategy in the game $\mathcal{G}^{\sigma,\mu}_T$ for every countable set of measures $\sigma \subset \operatorname{meas}(\kappa^{<\omega})$ containing $\mu_0, \ldots, \mu_n$. Because $\mathcal{U}$ is fine, this means that Player I has a winning strategy in the game $\mathcal{G}^{\sigma,\mu}_T$ for $\mathcal{U}$-almost every set of measures σ.

The game $\mathcal{G}^{\sigma,\mu}_T$ is a closed game and Player I's moves are finite sequences of ordinals, so for such σ we can define a *canonical* winning strategy $F^{\sigma,\mu}_T$. This strategy always makes the lexicographically least move that leads to a subgame for which Player I still has a winning strategy.

We will use these winning strategies $F^{\sigma,\mu}_T$ for $\mathcal{U}$-almost every σ, together with the measure $\mathcal{U}$, to build a single sequence of measures $(\mu_i : i \geq n)$ that is a winning play for Player II against the strategy $F^{\sigma,\mu}_T$ for $\mathcal{U}$-almost every set of measures σ. This will be a contradiction.

Let r^σ_n and s^σ_n denote the moves played as "r_n" and "s_n" respectively by the strategy $F^{\sigma,\mu}_T$ on the first turn. Define a measure $\mu_{n+1} \in \operatorname{meas}(\kappa^{<\omega})$ by

$$X \in \mu_{n+1} \iff \forall^*_{\mathcal{U}}\sigma\,(s^\sigma_n \in j_{\mu_n}(X)).$$

This measure is countably complete because the measure $\mathcal{U}$ is countably complete. More generally, for $i \geq n$ let r^σ_i and s^σ_i denote the moves played as "r_i" and "s_i" respectively by the strategy $F^{\sigma,\mu}_T$ on turn i against the play $(\mu_{n+1}, \ldots, \mu_i)$ by Player II, and define a measure $\mu_{i+1} \in \operatorname{meas}(\kappa^{<\omega})$ by

$$X \in \mu_{i+1} \iff \forall^*_{\mathcal{U}}\sigma\,(s^\sigma_i \in j_{\mu_i}(X)).$$

By the countable completeness of the measure $\mathcal{U}$ there is a single real x that is equal to $\bigcup\{r^\sigma_i : i \geq n\}$ for $\mathcal{U}$-almost every set of measures σ. (Apply countable completeness once for each i, and then once more at the end.)

Note that each of the measures μ_i concentrates on the tree T_x: for μ_i with $i \leq n$ this is guaranteed by the rules for Player I as remarked before, and for μ_{i+1} with $i \geq n$ this can easily be seen to follow from the definition of μ_{i+1}. Moreover, the sequence of measures $\vec{\mu} = (\mu_i : i < \omega)$ is a tower: for $i < n$ the measure μ_{i+1} projects to μ_i by definition, the measure μ_{n+1} projects to μ_n because $s_n \restriction n = [\mathrm{id}]_{\mu_n}$, and for $i \geq n$ the measure μ_{i+2} projects to μ_{i+1} because $s_{i+1} \restriction (i+1) = j_{\mu_i,\mu_{i+1}}(s_i)$.

Because the measure $\mathcal{U}$ is fine and countably complete, $\mathcal{U}$-almost every set of measures σ has, in addition to the above properties, the property that $\mu_i \in \sigma$ for every $i < \omega$. For such a set of measures σ, the sequence of measures $(\mu_{i+1} : i \geq n)$ is a legal play by Player II in the game $\mathcal{G}_T^{\sigma,\mu}$, so the moves played as $h_n, h_{n+1}, \ldots$ by the strategy $F_T^{\sigma,\mu}$ against this sequence $(\mu_{i+1} : i \geq n)$ form a sequence of ordinals witnessing that the tower $\vec{\mu}$ is ill-founded. The finitely many missing ordinals $h_0, \ldots, h_{n-1}$ are not important; we can always fill them in to get a witness $(h_i : i < \omega)$.

Let $(g_i : i < \omega)$ be a witness to the ill-foundedness of $\vec{\mu}$ such that the ordinal g_n takes the least possible value. By a standard argument similar to that used to prove Lemma 2.2 we can construct a well-founded tree W on κ such that each measure μ_i concentrates on W and $[\mathrm{rank}_W]_{\mu_n} = g_n$.[5]

By the countable completeness of the measure $\mathcal{U}$ and the meaning of the fact that $W \in \mu_{i+1}$ for every $i \geq n$, there is some set of measures σ with all the properties mentioned above as well as the property that $s_i^\sigma \in j_{\mu_i}(W)$ for every $i \geq n$. Define a sequence of ordinals $(g'_i : i \geq n)$ by

$$g'_i = \mathrm{rank}_{j_{\mu_i}(W)}(s_i^\sigma).$$

Then by the rules for Player I concerning the finite sequences s_i^σ we have

$$\begin{aligned} j_{\mu_i,\mu_{i+1}}(g'_i) &= \mathrm{rank}_{j_{\mu_{i+1}}(W)}(j_{\mu_i,\mu_{i+1}}(s_i^\sigma)) \\ &> \mathrm{rank}_{j_{\mu_{i+1}}(W)}(s_{i+1}^\sigma) = g'_{i+1} \end{aligned}$$

for every $i \geq n$, so the sequence of ordinals $(g'_i : i \geq n)$ witnesses the ill-foundedness of the tower $\vec{\mu}$; on the other hand, we have

$$g_n = \mathrm{rank}_{j_{\mu_n}(W)}([\mathrm{id}]_{\mu_n}) > \mathrm{rank}_{j_{\mu_n}(W)}(s_n) = g'_n,$$

contradicting the minimality of g_n. This contradiction proves the lemma. $\square$

[5]Here is the argument: choose a sequence of functions $(F_i : i < \omega)$ such that $F_i : \kappa^{<\omega} \to \mathrm{Ord}$ and $[F_i]_{\mu_i} = g_i$ for all $i < \omega$. We let the tree W consist of all nodes $s \in \kappa^{<\omega}$ such that $F_i(s \restriction i) > F_{i+1}(s \restriction (i+1))$ for all $i < |s|$. Then $[\mathrm{rank}_W]_{\mu_n} \leq g_n$ and we have equality by the minimality of g_n.

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