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On a remarkable formula of Jerison and Lee in CR geometry

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We discuss a remarkable formula discovered by Jerison and Lee to classify constant scalar curvature pseudohermitian structures on the sphere. We show that the formula is valid in the wider context of Einstein pseudohermitian manifolds. As an application we prove a uniqueness result that generalizes the theorem of Jerison and Lee.

1. Introduction

Recall that a Riemannian manifold (Σ^n, g) is called Einstein if its Ricci curvature is constant, i.e., $\mathrm{Ric} = cg$ for some constant c. In this case, the scalar curvature R is then obviously a constant. In general consider the trace-less Ricci tensor $T = \mathrm{Ric} - \frac{R}{n} g$. By the second Bianchi identity we have

$$\operatorname{div} T = \left(\frac{1}{2} - \frac{1}{n}\right) dR.$$

As a corollary, we have the well-known fact that (Σ^n, g) with $n \geq 3$ is Einstein iff $T = 0$.

Given a closed Riemannian manifold (Σ^n, g) the famous Yamabe problem seeks to conformally deform g to get a new metric $\widetilde{g}$ of constant scalar curvature. If we write $\widetilde{g} = u^{4/(n-2)} g$, where u is a positive smooth function, then the scalar curvatures are related by the following equation:

$$-\frac{4\,(n-1)}{n-2} \Delta_g u + Ru = \widetilde{R} u^{(n+2)/(n-2)}. \tag{1.1}$$

The Yamabe problem was solved by Yamabe [Y], Trudinger [T], Aubin [A] and Schoen [S] by showing that there is always a minimizer $\overline{u}$ for the following variational problem:

$$Y(\Sigma,[g]) := \inf_{\substack{u\in C^\infty(\Sigma)\\ u>0}} \frac{\int_\Sigma \left(\frac{4(n-1)}{n-2}|\nabla u|^2 + Ru^2\right) dv_g}{\left(\int_\Sigma u^{2n/(n-2)} dv_g\right)^{(n-2)/n}}$$

as the metric $\overline{g} = \overline{u}^{4/(n-2)} g$ then has constant scalar curvature. When $Y(\Sigma,[g]) \leq 0$, we also have uniqueness: there is only one constant scalar curvature metric up to scaling in the conformal class $[g]$. When $Y(\Sigma,[g]) > 0$, uniqueness in general fails. But when there is an Einstein metric in the conformal class, we have the following beautiful and important theorem due to Obata [O2].

Theorem 1. *Suppose $(\Sigma^n, \widetilde{g})$ is a closed Einstein manifold and $g = \phi\widetilde{g}$ is a conformal metric with constant scalar curvature, where ϕ is a positive smooth function. Then*

- *g is Einstein as well,*
- *furthermore ϕ must be constant unless $(\Sigma^n, \widetilde{g})$ is isometric to the standard sphere $(\mathbb{S}^n, g_c)$ up to a scaling and ϕ corresponds to the following function on $\mathbb{S}^n$*

$$\phi(x) = c(\cosh t + \sinh t x\cdot a)^{-2}$$

for some $c > 0, t \geq 0$ and $a \in \mathbb{S}^n$.

When $n = 2$ this is classic. Obata's proof for $n \geq 3$ is very elegant and is based on the following formula:

$$\widetilde{T} = T + (n-2)\phi^{-1}\left(D^2\phi - \frac{\Delta\phi}{n} g\right).$$

Since $\widetilde{g}$ is Einstein, we have $\widetilde{T} = 0$. Thus $-(n-2)\phi^{-1}\left(D^2\phi - \frac{\Delta\phi}{n}g\right) = T$. Pairing with T yields and using the fact the g has constant scalar curvature we obtain

$$-(n-2)\operatorname{div}(T(\nabla\phi,\cdot)) = \phi|T|^2.$$

As a corollary we have the complete classification of positive solutions of a non-linear PDE (stated only for $n \geq 3$ for brevity).

Corollary 1. *On $(\mathbb{S}^n, g_c)$ all positive solutions of the equation*

$$-\frac{4}{n(n-2)}\Delta u + u = u^{(n+2)/(n-2)}$$

are of the form

$$u(x) = (\cosh t + (\sinh t)\, x \cdot a)^{-(n-2)/2}$$

for some $t \geq 0$ and $a \in \mathbb{S}^n$.

Equivalently one can through the stereographic projection consider the following equation on $\mathbb{R}^n$

$$-\Delta v = v^{(n+2)/(n-2)}, v > 0.$$

This equation has been studied intensively from the PDE perspective via the method of moving planes or moving spheres, cf. [GNN, CGS] and the more recent [LZ].

In this paper, we consider the analogues of these uniqueness results in CR geometry. Let (M, θ) be a pseudohermitian manifold of dimension $2m+1$ and T the Reeb vector field. We always work with a local unitary frame $\{T_\alpha : \alpha = 1, \ldots, m\}$ for $T^{1,0}(M)$ and its dual frame $\{\theta^\alpha\}$. Thus

$$d\theta = \sqrt{-1}\sum_\alpha \theta^\alpha \wedge \overline{\theta^\alpha}.$$

We will often denote T by T_0. Let $B_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} - \frac{R}{m}\delta_{\alpha\beta}$ be the trace-less pseudohermitian Ricci tensor, where R is the pseudohermitian scalar curvature. We say that θ is pseudo-Einstein if $R_{\alpha\overline{\beta}} = \frac{R}{m}\delta_{\alpha\beta}$ or $B_{\alpha\overline{\beta}} = 0$. This is always true when $m = 1$. Pseudo-Einstein manifolds were first introduced by Lee [Lee]. By the Bianchi identity in CR geometry we have

$$B_{\alpha\overline{\beta},\beta} = \left(1 - \frac{1}{m}\right) R_\alpha - \sqrt{-1}\,(m-1)\, A_{\alpha\beta,\overline{\beta}}.$$

If θ is pseudo-Einstein and $m \geq 2$ then

$$R_\alpha = \sqrt{-1}\,(m-1)\, A_{\alpha\beta,\overline{\beta}}.$$

Therefore, a pseudo-Einstein θ does not necessarily have constant scalar curvature due to the presence of the torsion A. If the torsion A vanishes, then a pseudo-Einstein (M, θ) of dimension $2m+1 \geq 5$ is of constant scalar curvature. Slightly more generally, we have

Proposition 1. *Suppose $\left(M^{2m+1},\theta\right)$ with $m\geq 2$ is pseudo-Einstein and its torsion has zero divergence. Then its scalar curvature is constant.*

Proof. By the formula above, $R_\alpha=0$. By taking conjugate, we also have $R_{\overline{\beta}}=0$. Thus

$$\sqrt{-1}\delta_{\alpha\overline{\beta}}R_0=R_{\alpha,\overline{\beta}}-R_{\overline{\beta},\alpha}=0.$$

It follow $R_0=0$ as well and hence R is constant. □

Definition 1. *A pseudohermitian manifold $\left(M^{2m+1},\theta\right)$ is called Einstein if it is torsion-free and the pseudohermitian Ricci tensor is constant, i.e., $R_{\alpha\overline{\beta}}=\lambda\delta_{\alpha\beta}$ for some constant λ.*

When $m\geq 2$, $\left(M^{2m+1},\theta\right)$ is Einstein iff it is pseudo-Einstein and torsion-free by Proposition 1.

If $\widetilde{\theta}=f^{2/m}\theta$ is another pseudohermitian structure, where f is a smooth and positive function, then the pseudohermitian scalar curvatures of θ and $\widetilde{\theta}$ are related by the following formula:

$$-\frac{2\left(m+1\right)}{m}\Delta_b f+Rf=\widetilde{R}f^{(m+2)/m}.$$

The CR Yamabe problem, initiated by Jerison and Lee [JL1], seeks to conformally deform θ to get a new pseudohermitian structure $\widetilde{\theta}$ of constant scalar curvature. Like the Riemannian case, for a closed strictly pseudoconvex CR manifold M^{2m+1} one considers the Yamabe functional

$$Y\left(M,\theta\right)=\frac{\int_M R\,dv_\theta}{\left(\int_M dv_\theta\right)^{m/(m+1)}},$$

where θ is any contact form associated to the CR structure and $dv_\theta=\theta\wedge\left(d\theta\right)^m$ is the volume form. Set

$$Y\left(M\right)=\inf_\theta Y\left(M,\theta\right).$$

This defines a CR invariant. The CR Yamabe problem, interpreted narrowly, is whether the infimum is achieved. As in the Riemannian case, the unit sphere $\mathbb{S}^{2m+1}=\left\{z\in\mathbb{C}^{m+1}:|z|=1\right\}$ with its canonical pseudohermitian structure $\theta_c=\left(2\sqrt{-1}\partial\left|z\right|^2\right)|_{\mathbb{S}^{2m+1}}$ plays a fundamental role. $(\mathbb{S}^{2m+1},\theta_c)$ is of constant pseudohermitian curvature with $R_{\alpha\overline{\beta}}=\left(m+1\right)/2\delta_{\alpha\beta}$.

In their fundamental work [JL1], Jerison and Lee proved the following:

(1) $Y\left(\mathbb{S}^{2m+1}\right) = Y\left(\mathbb{S}^{2m+1}, \theta_{\mathrm{c}}\right) = 2\pi m\left(m+1\right)$, or equivalently the following sharp Sobolev inequality holds on $\mathbb{S}^{2m+1}$

$$\begin{aligned}\int_{\mathbb{S}^{2m+1}} \left(\frac{2\left(m+1\right)}{m}\left|\nabla_b f\right|^2 + \frac{m\left(m+1\right)}{2} f^2\right) dv_{\mathrm{c}} \\ \geq 2\pi m\left(m+1\right)\left(\int_{\mathbb{S}^{2m+1}} \left|f\right|^{2(m+1)/m} dv_{\mathrm{c}}\right)^{m/(m+1)},\end{aligned} \tag{1.2}$$

where $\nabla_b f$ is the horizontal gradient.

(2) For any closed strictly pseudoconvex CR manifold M^{2m+1}

$$Y\left(M\right) \leq Y\left(\mathbb{S}^{2m+1}\right).$$

(3) The CR Yamabe problem has a solution if $Y\left(M\right) < Y\left(\mathbb{S}^{2m+1}\right)$.

Moreover, they proved $Y\left(M\right) < Y\left(\mathbb{S}^{2m+1}\right)$ when $m \geq 2$ and M is not locally CR equivalent to $\mathbb{S}^{2m+1}$. To our best knowledge, the conjecture that $Y\left(M\right) < Y\left(\mathbb{S}^{2m+1}\right)$ unless M is CR equivalent to $\mathbb{S}^{2m+1}$ is still open in the remaining cases. However, it is now known that for all compact strictly pseudoconvex CR manifold M there is always a pseudohermitian structure θ on M whose scalar curvature is constant by the more recent work Gamara [G] and Gamara and Yacoub [GY].

Similar to the Riemannian case, there is a unique constant scalar curvature pseudohermitian structure on M up to scaling when $Y\left(M\right) \leq 0$. But uniqueness in general fails if $Y\left(M\right) > 0$. For the CR sphere $\mathbb{S}^{2m+1}$ Jerison and Lee [JL2] classified all pseudohermitian structures with constant scalar curvature. This is of fundamental importance for the whole program of CR Yamabe problem.

Theorem 2. *Suppose $\theta = \phi\theta_c$ is a pseudohermitian structure on $\mathbb{S}^{2m+1}$. Then θ has constant scalar curvature iff ϕ is of the following form:*

$$\phi\left(z\right) = c\left|\cosh t + \left(\sinh t\right) z \cdot \overline{\xi}\right|^{-2}$$

for some $c > 0, t \geq 0$ and $\xi \in \mathbb{S}^{2m+1}$.

The key ingredient in the proof is the following remarkable, highly nontrivial identity on $\left(\mathbb{S}^{2m+1},\theta\right)$:

$$\begin{aligned}
&\operatorname{Re}\left(gD_\alpha+\overline{g}E_\alpha-3\phi_0\sqrt{-1}U_\alpha\right)_{,\overline{\alpha}}\\
&\quad=\left(\frac{1}{2}+\frac{1}{2}\phi\right)\left(\left|D_{\alpha\beta}\right|^2+\left|E_{\alpha\overline{\beta}}\right|^2\right)\\
&\qquad+\phi\left[\left|D_\alpha-U_\alpha\right|^2+\left|U_\alpha+E_\alpha-D_\alpha\right|^2+\left|U_\alpha+E_\alpha\right|^2+\left|\phi^{-1}\phi_{\overline{\gamma}}D_{\alpha\beta}\right.\right.\\
&\qquad\left.\left.+\ \phi^{-1}\phi_\beta E_{\alpha\overline{\gamma}}\right|^2\right],
\end{aligned}$$

where

$$\begin{aligned}
&D_{\alpha\beta}=\phi^{-1}\phi_{\alpha,\beta},D_\alpha=\phi^{-1}\phi_{\overline{\beta}}D_{\alpha\beta},E_\alpha=\phi^{-1}\phi_\gamma E_{\alpha\overline{\gamma}},\\
&E_{\alpha\overline{\beta}}=\phi^{-1}\phi_{\alpha,\overline{\beta}}-\phi^{-2}\phi_\alpha\phi_{\overline{\beta}}-\frac{1}{2}\left(\frac{1}{2}\phi^{-1}-\frac{1}{2}+\phi^{-2}\left|\partial\phi\right|^2+\sqrt{-1}\phi^{-1}\phi_0\right)\delta_{\alpha\overline{\beta}},\\
&U_\alpha=-\frac{2}{m+2}\sqrt{-1}A_{\alpha\beta,\overline{\beta}},g=\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2+\sqrt{-1}\phi_0.
\end{aligned}$$

We should also mention the recent deep work [FL] by Frank and Lieb in which the sharp Hardy–Littlewood–Sobolev inequality, of which (1.2) is a special case, on the Heisenberg group is established. Their paper also contains a new and shorter proof of the Sobolev inequality (1.2) as well as a nice argument which yields the classification of all the minimizers quickly.

The purpose of this work is to point out that the Jerison–Lee identity is valid on any closed Einstein pseudohermitian manifold $\left(M^{2m+1},\theta\right)$ and as an application prove the following uniqueness theorem which generalizes the above result of Jerison and Lee.

Theorem 3. *Let $(M^{2m+1},\widetilde{\theta})$ be a closed Einstein pseudohermitian manifold. Suppose $\theta=\phi\widetilde{\theta}$ is another pseudohermitian structure with constant pseudohermitian scalar curvature. Then*

- *θ is Einstein as well,*
- *furthermore ϕ must be constant unless $(M^{2m+1},\widetilde{\theta})$ is CR isometric to $(\mathbb{S}^{2m+1},\theta_c)$ up to a scaling and ϕ corresponds to the following function on $\mathbb{S}^{2m+1}$:*

$$\phi(z)=c\left|\cosh t+(\sinh t)\,z\cdot\overline{\xi}\right|^{-2}$$

for some $c>0,t\geq 0$ and $\xi\in\mathbb{S}^{2m+1}$.

More precisely, the second part means that if ϕ is not constant then there exists a CR diffeomorphism $F : M \to \mathbb{S}^{2m+1}$ s.t. $F^*\theta_c = \lambda\widetilde{\theta}$ for some $\lambda > 0$ and $\phi \circ F^{-1}$ is of the form above on $\mathbb{S}^{2m+1}$.

The paper is organized as follows. In Section 2, we present the Jerison–Lee identity on any closed Einstein pseudohermitian manifold $\left(M^{2m+1}, \theta\right)$. Using this identity we will prove the above theorem in Section 3. There is an appendix in which we collect several formulas in CR geometry that are needed in Section 3.

2. The Jerison–Lee identity

In this section, we discuss the Jerison–Lee identity from [JL2]. Though it is only stated for the CR sphere $\mathbb{S}^{2m+1}$ there, the identity and its proof are valid on any closed Einstein pseudohermitian manifold. To publicize this remarkable identity in the wider context and also for completeness, we present a detailed proof following [JL2] faithfully. Therefore, this section is expository. We use slightly different notation and provide more details at several places to make the proof easier to follow.

Let $\left(M^{2m+1}, \theta\right)$ be a pseudohermitian manifold and ϕ a smooth and positive function. Consider $\widetilde{\theta} = \phi^{-1}\theta$. The pseudohermitian invariants transform as follows [Lee]:

$$\begin{aligned}
\widetilde{A}_{\alpha\beta} &= A_{\alpha\beta} - \sqrt{-1}\phi^{-1}\phi_{\alpha,\beta}, \\
\widetilde{B}_{\alpha\overline{\beta}} &= B_{\alpha\overline{\beta}} + (m+2)\left(\phi^{-1}\phi_{\alpha,\overline{\beta}} - \phi^{-2}\phi_\alpha\phi_{\overline{\beta}}\right) \\
&\quad - \frac{m+2}{m}\left(\phi^{-1}\phi_{\gamma,\overline{\gamma}} - \phi^{-2}\left|\partial\phi\right|^2\right)h_{\alpha\overline{\beta}}, \\
\widetilde{R} &= \phi R + (m+1)\,\Delta_b\phi - (m+1)\,(m+2)\,\phi^{-1}\left|\partial\phi\right|^2.
\end{aligned}$$

Here we are working with a local unitary frame $\{T_\alpha : \alpha = 1, \ldots, m\}$ w.r.t. θ.

Proposition 2. *Let θ and $\widetilde{\theta} = \phi^{-1}\theta$ be two pseudohermitian structures on a closed manifold M^{2m+1}. Suppose that both θ and $\widetilde{\theta}$ have constant scalar curvature $m\,(m+1)\,/2$ and $\widetilde{\theta}$ is Einstein. Set*

$$\begin{aligned}
D_{\alpha\beta} &= -\sqrt{-1}A_{\alpha\beta}, D_\alpha = \phi^{-1}\phi_{\overline{\beta}}D_{\alpha\beta}, \\
E_{\alpha\overline{\beta}} &= -\frac{1}{m+2}B_{\alpha\overline{\beta}}, E_\alpha = \phi^{-1}\phi_\beta E_{\alpha\overline{\beta}},
\end{aligned}$$

$$U_\alpha = -\frac{2}{m+2}\sqrt{-1}A_{\alpha\beta,\overline{\beta}},$$
$$g = \frac{1}{2} + \frac{1}{2}\phi + \phi^{-1}\left|\partial\phi\right|^2 + \sqrt{-1}\phi_0.$$

Then

$$\begin{aligned} &\operatorname{Re}\left(gD_\alpha + \overline{g}E_\alpha - 3\phi_0\sqrt{-1}U_\alpha\right)_{,\overline{\alpha}} \\ = &\left(\frac{1}{2}+\frac{1}{2}\phi\right)\left(\left|D_{\alpha\beta}\right|^2 + \left|E_{\alpha\overline{\beta}}\right|^2\right) \\ &+ \phi\Big[\left|D_\alpha - U_\alpha\right|^2 + \left|U_\alpha + E_\alpha - D_\alpha\right|^2 + \left|U_\alpha + E_\alpha\right|^2 \\ &+ \phi^{-2}\left|\phi_{\overline{\gamma}}D_{\alpha\beta} + \phi_\beta E_{\alpha\overline{\gamma}}\right|^2\Big]. \end{aligned} \tag{2.1}$$

Remark 1. *Jerison–Lee [JL2] use the normalization $R = \widetilde{R} = m\,(m+1)$. We instead use the normalization $R = \widetilde{R} = m\,(m+1)\,/2$. This is why some of our coefficients are different. Our normalization has the advantage that the adapted metric for θ_c on $\mathbb{S}^{2m+1}$ is round.*

Since $\widetilde{\theta}$ is Einstein and $R = \widetilde{R} = m\,(m+1)\,/2$, we have

$$A_{\alpha\beta} = \sqrt{-1}\phi^{-1}\phi_{\alpha,\beta}, \tag{2.2}$$

$$\begin{aligned} B_{\alpha\overline{\beta}} = &-(m+2)\left(\phi^{-1}\phi_{\alpha,\overline{\beta}} - \phi^{-2}\phi_\alpha\phi_{\overline{\beta}}\right) \\ &+ \frac{m+2}{m}\left(\phi^{-1}\phi_{\gamma,\overline{\gamma}} - \phi^{-2}\left|\partial\phi\right|^2\right)\delta_{\alpha\overline{\beta}}, \end{aligned} \tag{2.3}$$

$$\phi_{\gamma,\overline{\gamma}} = \frac{m}{4} - \frac{m}{4}\phi + \frac{m+2}{2}\phi^{-1}\left|\partial\phi\right|^2 + \frac{m}{2}\sqrt{-1}\phi_0. \tag{2.4}$$

Thus

$$D_{\alpha\beta} = \phi^{-1}\phi_{\alpha,\beta}, \tag{2.5}$$

$$\begin{aligned} E_{\alpha\overline{\beta}} = &\ \phi^{-1}\phi_{\alpha,\overline{\beta}} - \phi^{-2}\phi_\alpha\phi_{\overline{\beta}} \\ &-\frac{1}{2}\left(\frac{1}{2}\phi^{-1} - \frac{1}{2} + \phi^{-2}\left|\partial\phi\right|^2 + \sqrt{-1}\phi^{-1}\phi_0\right)\delta_{\alpha\overline{\beta}}. \end{aligned} \tag{2.6}$$

We compute

$$\begin{aligned} \phi_{\alpha\beta,\overline{\beta}} = &\ \phi_{\beta,\overline{\beta}\alpha} + \sqrt{-1}\delta_{\alpha\overline{\beta}}\phi_{\beta,0} + R_{\alpha\overline{\beta}}\phi_\beta \\ = &-\frac{m}{4}\phi_\alpha - \frac{m+2}{2}\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{m+2}{2}\phi^{-1}\phi_{\overline{\beta}}\phi_{\alpha,\beta} + \frac{m+2}{2}\phi^{-1}\phi_{\overline{\beta},\alpha}\phi_\beta \\ &+ \frac{m}{2}\sqrt{-1}\phi_{0,\alpha} + \sqrt{-1}\phi_{\alpha,0} + R_{\alpha\overline{\beta}}\phi_\beta \end{aligned}$$

$$= -\frac{m}{4}\phi_\alpha - \frac{m+2}{2}\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{m+2}{2}\phi^{-1}\phi_{\overline{\beta}}\phi_{\alpha,\beta} + \frac{m+2}{2}\phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta$$
$$- \frac{m+2}{2}\sqrt{-1}\phi^{-1}\phi_\alpha\phi_0 + \frac{m+2}{2}\sqrt{-1}\phi_{0,\alpha} - \sqrt{-1}A_{\alpha\beta}\phi_{\overline{\beta}} + R_{\alpha\overline{\beta}}\phi_\beta.$$

Using the decomposition $R_{\alpha\overline{\beta}} = B_{\alpha\overline{\beta}} + \frac{R}{m}\delta_{\alpha\beta}$, (2.3), and the fact $R = m(m+1)/2$ and simplifying we obtain

$$R_{\alpha\overline{\beta}}\phi_\beta = \frac{m+2}{2}\Big(-2\phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta + 3\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{1}{2}\phi^{-1}\phi_\alpha + \sqrt{-1}\phi^{-1}\phi_0\phi_\alpha\Big) + \frac{m}{4}\phi_\alpha.$$

Plugging it into the previous formula yields

$$\begin{aligned}
\phi_{\alpha\beta,\overline{\beta}} &= -\frac{m}{4}\phi_\alpha - \frac{m+2}{2}\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{m+2}{2}\phi^{-1}\phi_{\overline{\beta}}\phi_{\alpha,\beta} \\
&\quad + \frac{m+2}{2}\phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta - \sqrt{-1}\frac{m+2}{2}\phi^{-1}\phi_\alpha\phi_0 \\
&\quad + \frac{m+2}{2}\sqrt{-1}\phi_{0,\alpha} - \sqrt{-1}A_{\alpha\beta}\phi_{\overline{\beta}} + \frac{m+2}{2}\Big(-2\phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta \\
&\quad + 3\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{1}{2}\phi^{-1}\phi_\alpha + \sqrt{-1}\phi^{-1}\phi_0\phi_\alpha\Big) + \frac{m}{4}\phi_\alpha \\
&= \frac{m+2}{2}\Big(\sqrt{-1}\phi_{0,\alpha} + \phi^{-1}\phi_{\alpha,\beta}\phi_{\overline{\beta}} - \phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta + 2\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha \\
&\quad + \frac{1}{2}\phi^{-1}\phi_\alpha\Big) + \phi^{-1}\phi_{\alpha,\beta}\phi_{\overline{\beta}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
U_\alpha &= -\frac{2}{m+2}\sqrt{-1}A_{\alpha\beta,\overline{\beta}} \\
&= \phi^{-1}\left[\sqrt{-1}\phi_{0,\alpha} + 2\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \phi^{-1}\left(\phi_{\alpha,\beta}\phi_{\overline{\beta}} - \phi_{\alpha,\overline{\beta}}\phi_\beta + \frac{1}{2}\phi_\alpha\right)\right] \qquad (2.7)\\
&= \phi^{-1}\left[\sqrt{-1}\phi_{0,\alpha} + 2\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \phi D_\alpha - \phi^{-1}\phi_{\alpha,\overline{\beta}}\phi_\beta + \frac{1}{2}\phi^{-1}\phi_\alpha\right].
\end{aligned}$$

Lemma 1. *We have*

$$U_\alpha = \phi^{-1}\left[\sqrt{-1}\phi_{0,\alpha} + \phi\left(D_\alpha - E_\alpha\right) + \frac{1}{2}\phi^{-1}\overline{g}\phi_\alpha\right]. \qquad (2.8)$$

Proof. Replacing $\phi_{\alpha,\overline{\beta}}$ in (2.7) using identity (2.6) we obtain

$$\begin{aligned}
U_\alpha = \phi^{-1}\Bigg[&\sqrt{-1}\phi_{0,\alpha} + 2\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \phi D_\alpha + \frac{1}{2}\phi^{-1}\phi_\alpha \\
&-E_{\alpha,\overline{\beta}}\phi_\beta - \phi^{-2}\left|\partial\phi\right|^2\phi_\alpha - \frac{1}{2}\left(\frac{1}{2}\phi^{-1} - \frac{1}{2} + \phi^{-2}\left|\partial\phi\right|^2\right. \\
&\left.+\ \sqrt{-1}\phi^{-1}\phi_0\right)\phi_\alpha\Bigg] = \phi^{-1}\left[\sqrt{-1}\phi_{0,\alpha} + \phi\left(D_\alpha - E_\alpha\right)\right. \\
&\left.+\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\phi^{-1} + \phi^{-2}\left|\partial\phi\right|^2 - \sqrt{-1}\phi^{-1}\phi_0\right)\phi_\alpha\right] \\
= \phi^{-1}&\left[\sqrt{-1}\phi_{0,\alpha} + \phi\left(D_\alpha - E_\alpha\right) + \frac{1}{2}\phi^{-1}\overline{g}\phi_\alpha\right].
\end{aligned}$$

□

As $D_\alpha = -\sqrt{-1}A_{\alpha\gamma}\phi^{-1}\phi_{\overline{\gamma}}$, we have

$$\begin{aligned}
D_{\alpha,\overline{\alpha}} &= -\sqrt{-1}A_{\alpha\gamma,\overline{\alpha}}\phi^{-1}\phi_{\overline{\gamma}} - \sqrt{-1}A_{\alpha\gamma}\phi^{-1}\phi_{\overline{\alpha\gamma}} + \sqrt{-1}A_{\alpha\gamma}\phi^{-2}\phi_{\overline{\alpha}}\phi_{\overline{\gamma}} \\
&= \frac{m+2}{2}\phi^{-1}U_\alpha\phi_{\overline{\alpha}} + \left|D_{\alpha\beta}\right|^2 - D_{\alpha\beta}\phi^{-2}\phi_{\overline{\alpha}}\phi_{\overline{\beta}} \qquad (2.9)\\
&= \frac{m+2}{2}\phi^{-1}U_\alpha\phi_{\overline{\alpha}} + \left|D_{\alpha\beta}\right|^2 - \phi^{-1}D_\alpha\phi_{\overline{\alpha}}.
\end{aligned}$$

As $E_\alpha = \phi^{-1}\phi_\gamma E_{\alpha\overline{\gamma}}$, we have

$$\begin{aligned}
E_{\alpha,\overline{\alpha}} &= \phi^{-1}\phi_\gamma E_{\alpha\overline{\gamma},\overline{\alpha}} + \left(\phi^{-1}\phi_{\gamma,\overline{\alpha}} - \phi^{-2}\phi_{\overline{\alpha}}\phi_\gamma\right)E_{\alpha\overline{\gamma}} \\
&= \frac{1-m}{2}\phi^{-1}\phi_\gamma U_{\overline{\gamma}} + \left(\phi^{-1}\phi_{\gamma,\overline{\alpha}} - \phi^{-2}\phi_{\overline{\alpha}}\phi_\gamma\right)E_{\alpha\overline{\gamma}} \qquad (2.10)\\
&= \frac{1-m}{2}\phi^{-1}\phi_\gamma U_{\overline{\gamma}} + \left|E_{\alpha\overline{\beta}}\right|^2.
\end{aligned}$$

We now compute the left-hand side of (2.1)

$$\begin{aligned}
\text{LHS} = &\operatorname{Re}\left[gD_{\alpha,\overline{\alpha}} + \overline{g}E_{\alpha,\overline{\alpha}}\right] - 3\phi_0\operatorname{Re}\sqrt{-1}U_{\alpha,\overline{\alpha}} \\
&+\operatorname{Re}\left[g_{\overline{\alpha}}D_\alpha + \overline{g}_{\overline{\alpha}}E_\alpha - 3\sqrt{-1}\phi_{0,\overline{\alpha}}U_\alpha\right]. \qquad (2.11)
\end{aligned}$$

By a Bianchi identity [Lee, (2.13)], we have $2\operatorname{Re}A_{\alpha\beta,\overline{\beta}\overline{\alpha}} = R_0 = 0$. Thus

$$\operatorname{Re}\sqrt{-1}U_{\alpha,\overline{\alpha}} = \frac{2}{m+2}\operatorname{Re}A_{\alpha\beta,\overline{\beta}\overline{\alpha}} = 0.$$

Plugging this identity as well as (2.9) and (2.10) into (2.11) we obtain

$$\begin{aligned}\text{LHS} =& \frac{3}{2}\phi^{-1}\operatorname{Re} gU_\alpha\phi_{\overline{\alpha}} + \left(\frac{1}{2}+\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2\right)\left(\left|D_{\alpha\beta}\right|^2+\left|E_{\alpha\overline{\beta}}\right|^2\right) \\ & - \phi^{-1}\operatorname{Re} gD_\alpha\phi_{\overline{\alpha}} + \operatorname{Re}\left[g_{\overline{\alpha}}D_\alpha + \overline{g}_{\overline{\alpha}}E_\alpha - 3\sqrt{-1}\phi_{0,\overline{\alpha}}U_\alpha\right]. \end{aligned} \tag{2.12}$$

Lemma 2. *We have*

$$\begin{aligned}\sqrt{-1}\phi_{0,\overline{\alpha}} &= \phi\left(D_{\overline{\alpha}} - U_{\overline{\alpha}} - E_{\overline{\alpha}}\right) + \frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}}, \\ g_{\overline{\alpha}} &= \phi^{-1}g\phi_{\overline{\alpha}} + \phi\left(2D_{\overline{\alpha}} - U_{\overline{\alpha}}\right), \\ \overline{g}_{\overline{\alpha}} &= \phi\left(2E_{\overline{\alpha}} + U_{\overline{\alpha}}\right).\end{aligned}$$

Proof. The first formula follows from (2.8) directly. We have

$$\begin{aligned}g_{\overline{\alpha}} &= \frac{1}{2}\phi_{\overline{\alpha}} - \phi^{-2}\phi_{\overline{\alpha}}\left|\partial\phi\right|^2 + \phi^{-1}\left(\phi_{\beta\overline{\alpha}}\phi_{\overline{\beta}} + \phi_\beta\phi_{\overline{\beta}\overline{\alpha}}\right) + \sqrt{-1}\phi_{0,\overline{\alpha}} \\ &= \left(\frac{1}{2} - \phi^{-2}\left|\partial\phi\right|^2\right)\phi_{\overline{\alpha}} + \phi D_{\overline{\alpha}} + \phi^{-1}\phi_{\beta\overline{\alpha}}\phi_{\overline{\beta}} + \sqrt{-1}\phi_{0,\overline{\alpha}}.\end{aligned}$$

We compute

$$\begin{aligned}\phi_{\beta\overline{\alpha}}\phi_{\overline{\beta}} = \overline{\phi_{\overline{\beta}\alpha}\phi_\beta} &= \overline{\left(\phi_{\alpha\overline{\beta}} - \sqrt{-1}\phi_0\delta_{\alpha\beta}\right)\phi_\beta} \\ &= \overline{\phi_{\alpha\overline{\beta}}\phi_\beta} + \sqrt{-1}\phi_0\phi_{\overline{\alpha}}.\end{aligned}$$

Thus

$$g_{\overline{\alpha}} = \left(\frac{1}{2} - \phi^{-2}\left|\partial\phi\right|^2 + \sqrt{-1}\phi^{-1}\phi_0\right)\phi_{\overline{\alpha}} + \phi D_{\overline{\alpha}} + \phi^{-1}\overline{\phi_{\alpha\overline{\beta}}\phi_\beta} + \sqrt{-1}\phi_{0,\overline{\alpha}}.$$

Replacing $\phi^{-1}\phi_{\alpha,\overline{\beta}}$ by the formula (2.6) we end up with

$$\begin{aligned}g_{\overline{\alpha}} =& \left(\frac{1}{2} - \phi^{-2}\left|\partial\phi\right|^2 + \sqrt{-1}\phi^{-1}\phi_0\right)\phi_{\overline{\alpha}} + \phi\left(D_{\overline{\alpha}} + E_{\overline{\alpha}}\right) + \sqrt{-1}\phi_{0,\overline{\alpha}} \\ & \overline{+\phi^{-2}\left|\partial\phi\right|^2\phi_\alpha + \frac{1}{2}\left(\frac{1}{2}\phi^{-1} - \frac{1}{2} + \phi^{-2}\left|\partial\phi\right|^2 + \sqrt{-1}\phi^{-1}\phi_0\right)\phi_\alpha} \\ =& \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\phi^{-1} + \phi^{-2}\left|\partial\phi\right|^2 + \sqrt{-1}\phi^{-1}\phi_0\right)\phi_{\overline{\alpha}} + \phi\left(D_{\overline{\alpha}} + E_{\overline{\alpha}}\right) + \sqrt{-1}\phi_{0,\overline{\alpha}} \\ =& \frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}} + \phi\left(D_{\overline{\alpha}} + E_{\overline{\alpha}}\right) + \sqrt{-1}\phi_{0,\overline{\alpha}}.\end{aligned}$$

By the same calculation

$$\overline{g}_{\overline{\alpha}} = \frac{1}{2}\phi^{-1} g\phi_{\overline{\alpha}} + \phi\left(D_{\overline{\alpha}} + E_{\overline{\alpha}}\right) - \sqrt{-1}\phi_{0,\overline{\alpha}}.$$

Plugging the first formula into the above identities, we obtain the second and third formulas. □

Plugging these formulas into (2.12) we obtain

$$\begin{aligned}
\text{LHS} =& \frac{3}{2}\phi^{-1}\operatorname{Re} gU_\alpha\phi_{\overline{\alpha}} + \left(\frac{1}{2} + \frac{1}{2}\phi + \phi^{-1}\left|\partial\phi\right|^2\right)\left(\left|D_{\alpha\beta}\right|^2 + \left|E_{\alpha\overline{\beta}}\right|^2\right) \\
& - \phi^{-1}\operatorname{Re} gD_\alpha\phi_{\overline{\alpha}} + \operatorname{Re}\left[\phi^{-1}g\phi_{\overline{\alpha}} + \phi\left(2D_{\overline{\alpha}} - U_{\overline{\alpha}}\right)\right] D_\alpha \\
& + \operatorname{Re}\left[\phi\left(2E_{\overline{\alpha}} + U_{\overline{\alpha}}\right)\right] E_\alpha - 3\operatorname{Re}\left[\phi\left(D_{\overline{\alpha}} - U_{\overline{\alpha}} - E_{\overline{\alpha}}\right) + \frac{1}{2}\phi^{-1}g\phi_{\overline{\alpha}}\right] U_\alpha \\
=& \left(\frac{1}{2} + \frac{1}{2}\phi + \phi^{-1}\left|\partial\phi\right|^2\right)\left(\left|D_{\alpha\beta}\right|^2 + \left|E_{\alpha\overline{\beta}}\right|^2\right) \\
& + \phi\operatorname{Re}\left(2D_{\overline{\alpha}} - U_{\overline{\alpha}}\right) D_\alpha + \phi\operatorname{Re}\left(2E_{\overline{\alpha}} + U_{\overline{\alpha}}\right) E_\alpha \\
& + 3\phi\operatorname{Re}\left(U_{\overline{\alpha}} + E_{\overline{\alpha}} - D_{\overline{\alpha}}\right) U_\alpha.
\end{aligned}$$

It is then elementary to show that this equals the RHS.

3. Proof of the main theorem

We are now ready to prove our main theorem.

Theorem 4. *Let $\left(M^{2m+1}, \widetilde{\theta}\right)$ be a closed Einstein pseudohermitian manifold. Suppose $\theta = \phi\widetilde{\theta}$ is another pseudohermitian structure with constant pseudohermitian scalar curvature. Then*

- *θ is Einstein as well,*
- *furthermore ϕ must be constant unless $\left(M^{2m+1}, \widetilde{\theta}\right)$ is CR isometric to the standard sphere $\left(\mathbb{S}^{2m+1}, \theta_c\right)$ up to a scaling and ϕ corresponds to the following function on $\mathbb{S}^{2m+1}$*

$$\phi(z) = c\left|\cosh t + (\sinh t)\, z\cdot\overline{\xi}\right|^{-2}$$

for some $c > 0, t \geq 0$ and $\xi \in \mathbb{S}^{2m+1}$.

(Two pseudohermition manifolds $\left(\Sigma^{2m+1},\theta\right)$ and $\left(\widetilde{\Sigma}^{2m+1},\widetilde{\theta}\right)$ are said to be CR isometric if there exists a CR diffeomorphism $F:\Sigma\to\widetilde{\Sigma}$ such that $F^*\widetilde{\theta}=\theta$.)

The theorem is trivial if the pseudohermitian scalar curvature of $\widetilde{\theta}$ is zero or negative. Assume it is positive. By scaling both θ and $\widetilde{\theta}$, we may assume $R=\widetilde{R}=m\left(m+1\right)/2$. Integrating the Jerison–Lee identity (2.1) over M we have

$$\begin{aligned}0=&\int_M\left(\frac{1}{2}+\frac{1}{2}\phi\right)\left(\left|D_{\alpha\beta}\right|^2+\left|E_{\alpha\overline{\beta}}\right|^2\right)\\&+\int_M\phi\Big[\left|D_\alpha-U_\alpha\right|^2+\left|U_\alpha+E_\alpha-D_\alpha\right|^2+\left|U_\alpha+E_\alpha\right|^2\\&+\left|\phi^{-1}\phi_{\overline{\gamma}}D_{\alpha\beta}+\phi^{-1}\phi_\beta E_{\alpha\overline{\gamma}}\right|^2\Big].\end{aligned}$$

Therefore,

$$D_{\alpha\beta}=0,\ E_{\alpha\overline{\beta}}=0,U_\alpha=0,$$

i.e., more explicitly

$$\begin{aligned}&\phi_{\alpha,\beta}=0,\\&\phi_{\alpha,\overline{\beta}}=\phi^{-1}\phi_\alpha\phi_{\overline{\beta}}+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}\phi+\phi^{-1}\left|\partial\phi\right|^2+\sqrt{-1}\phi_0\right)\delta_{\alpha\overline{\beta}},\\&\phi_{0,\alpha}=\frac{\sqrt{-1}}{2}\left(\frac{1}{2}+\frac{1}{2}\phi^{-1}+\phi^{-2}\left|\partial\phi\right|^2-\sqrt{-1}\phi^{-1}\phi_0\right)\phi_\alpha.\end{aligned}$$

Recall that $D_{\alpha\beta}=-\sqrt{-1}A_{\alpha\beta}$ and $E_{\alpha\overline{\beta}}$ is a multiple of the trace-less Ricci by definition. Therefore, θ is pseudo-Einstein with constant scalar curvature and torsion free. This proves the first part.

To prove the second part, we now assume that ϕ is not constant. First observe that θ and $\widetilde{\theta}$ play symmetric roles in the statement. Therefore, it suffices to do it for θ. As $R_{\alpha\overline{\beta}}=\frac{m+1}{2}\delta_{\alpha\beta}$ and $A_{\alpha\beta}=0,$ it is easy to check by Proposition 4 in the appendix that the adapted Riemannian metric g_θ is Einstein: $\mathrm{Ric}\left(g_\theta\right)=\frac{m}{2}g_\theta$. Since the Ricci curvature is positive, M has a finite fundamental group. We can work on its universal covering $\widetilde{M}$, which is still a closed pseudohermitian manifold. For simplicity we will use the same letter for both the object on M and its pullback on $\widetilde{M}$. Let $u = \log\phi$. Then

$$\begin{aligned} u_{\alpha,\beta} &= -u_\alpha u_\beta, \\ u_{\alpha,\overline{\beta}} &= \frac{1}{2}\left(\frac{1}{2}e^{-u} - \frac{1}{2} + |\partial u|^2 + \sqrt{-1}u_0\right)\delta_{\alpha\overline{\beta}}, \\ u_{0,\alpha} &= -\frac{1}{2}u_0u_\alpha + \frac{\sqrt{-1}}{2}\left(\frac{1}{2} + \frac{1}{2}e^{-u} + |\partial u|^2\right)u_\alpha. \end{aligned}$$

We now claim that u is CR pluriharmonic. The argument is the same as in [JL2]. Indeed, when $m \geq 2$ this follows from the second equation. When $m = 1$, differentiating the second equation and simplifying using all three yields

$$u_{\overline{1},11} = \frac{1}{2}\left(-\frac{1}{2}e^{-u}u_1 + u_{1,1}u_{\overline{1}} + u_1u_{\overline{1},1} - \sqrt{-1}u_{0,1}\right) = 0.$$

As $A_{11} = 0$, it follows that u is CR pluriharmonic by [Lee, Proposition 3.4]. As $\widetilde{M}$ is simply connected, u is the real part of a CR holomorphic function $u + \sqrt{-1}v$:

$$v_\alpha = -\sqrt{-1}u_\alpha, v_{\overline{\beta}} = \sqrt{-1}u_{\overline{\beta}}.$$

We also have

$$\begin{aligned} \sqrt{-1}v_0\delta_{\alpha\beta} = v_{\alpha,\overline{\beta}} - v_{\overline{\beta},\alpha} &= -\sqrt{-1}u_{\alpha,\overline{\beta}} - \sqrt{-1}u_{\overline{\beta},\alpha} \\ &= -2\sqrt{-1}u_{\alpha,\overline{\beta}} - u_0\delta_{\alpha\beta} \\ &= -\sqrt{-1}\left(\frac{1}{2}e^{-u} - \frac{1}{2} + |\partial u|^2\right)\delta_{\alpha\overline{\beta}}. \end{aligned}$$

Thus

$$v_0 = -\left(\frac{1}{2}e^{-u} - \frac{1}{2} + |\partial u|^2\right).$$

With this we can rewrite the equations satisfied by u as

$$\begin{aligned} u_{\alpha,\beta} &= -u_\alpha u_\beta, \\ u_{\alpha,\overline{\beta}} &= \frac{1}{2}\left(-v_0 + \sqrt{-1}u_0\right)\delta_{\alpha\overline{\beta}}, \\ u_{0,\alpha} &= -\frac{1}{2}u_0u_\alpha + \frac{\sqrt{-1}}{2}\left(1 - v_0\right)u_\alpha \end{aligned}$$

Let $f = \exp u/2 \cos v/2 - c$, where c is a constant such that $\int_{\widetilde{M}} f = 0$.

Proposition 3. *We have*

$$
\begin{aligned}
f_{\alpha,\beta} &= 0,\\
f_{\alpha,\overline{\beta}} &= \frac{1}{2}\left[-\left(e^{u/2}\sin v/2\right)_0 + \sqrt{-1}f_0\right]\delta_{\alpha\overline{\beta}},\\
f_{0,\alpha} &= \frac{\sqrt{-1}}{2}f_\alpha,\\
f_{0,0} &= -\frac{1}{2}\left(e^{u/2}\sin v/2\right)_0.
\end{aligned}
$$

Proof. These formulas are proved by direct calculations. For example, to prove the third one we first observe as $v_\alpha = -\sqrt{-1}u_\alpha$ and θ is torsion-free

$$v_{0,\alpha} = v_{\alpha,0} = -\sqrt{-1}u_{\alpha,0}.$$

Then we compute using the third equation for u

$$
\begin{aligned}
f_{0,\alpha} &= \frac{1}{2}e^{u/2}\left[\left(u_{0,\alpha} - \frac{1}{2}v_0v_\alpha + \frac{1}{2}u_0u_\alpha\right)\cos\frac{v}{2}\right.\\
&\quad \left. - \left(v_{0,\alpha} + \frac{1}{2}u_0v_\alpha + \frac{1}{2}v_0u_\alpha\right)\sin\frac{v}{2}\right] = \frac{1}{2}e^{u/2}\left[\left(u_{0,\alpha} + \frac{\sqrt{-1}}{2}v_0u_\alpha\right.\right.\\
&\quad \left.\left. + \frac{1}{2}u_0u_\alpha\right)\cos\frac{v}{2} - \left(-\sqrt{-1}u_{0,\alpha} - \frac{\sqrt{-1}}{2}u_0u_\alpha + \frac{1}{2}v_0u_\alpha\right)\sin\frac{v}{2}\right]\\
&= \frac{1}{2}e^{u/2}\left(\frac{\sqrt{-1}}{2}u_\alpha\cos\frac{v}{2} + \frac{1}{2}u_\alpha\sin\frac{v}{2}\right)\\
&= \frac{\sqrt{-1}}{2}e^{u/2}\left(\frac{1}{2}u_\alpha\cos\frac{v}{2} - \frac{1}{2}v_\alpha\sin\frac{v}{2}\right) = \frac{\sqrt{-1}}{2}f_\alpha.
\end{aligned}
$$

The first and second formulas can be proved similarly.

To prove the last identity, we differentiate the third one

$$
\begin{aligned}
\frac{\sqrt{-1}}{2}f_{\alpha,\overline{\beta}} = f_{0,\alpha\overline{\beta}} &= f_{0,\overline{\beta}\alpha} + \sqrt{-1}f_{0,0}\delta_{\alpha\overline{\beta}}\\
&= \overline{f_{0,\beta\overline{\alpha}}} + \sqrt{-1}f_{0,0}\delta_{\alpha\overline{\beta}}\\
&= -\frac{\sqrt{-1}}{2}\overline{f_{\beta,\overline{\alpha}}} + \sqrt{-1}f_{0,0}\delta_{\alpha\overline{\beta}}.
\end{aligned}
$$

Using the second identity we obtain

$$f_{0,0} = -\frac{1}{2}\left(e^{u/2}\sin\frac{v}{2}\right)_0.$$

□

Let $D^2 f$ denote the Hessian of f w.r.t. the adapted Riemannian metric g_θ. By Proposition A.3 in the appendix, we obtain from Proposition 3

$$D^2 f = -\frac{1}{2}\left(e^{u/2}\sin\frac{v}{2}\right)_0 g_\theta. \tag{3.1}$$

We pause to prove a simple lemma in Riemannian geometry.

Proposition 4. *Let (Σ^n, g) be a closed Riemannian manifold s.t. $Ric\,(g) = (n-1)\,c^2 g$ with $c > 0$ a constant. Suppose $u \in C^\infty(\Sigma)$ is a non-zero function s.t. $\int_\Sigma u = 0$ and*

$$D^2 u = -\chi g \tag{3.2}$$

for some $\chi \in C^\infty(\Sigma)$. Then (Σ^n, g) is isometric to the unit sphere $\mathbb{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$ with the metric $\frac{1}{c} g_0$ and u corresponds to a linear function on $\mathbb{S}^n$, where g_0 is the canonical metric on $\mathbb{S}^n$.

Proof. Taking trace of (3.2) yields $\Delta u = -n\chi$. Working with a local orthonormal frame we differentiate (3.2)

$$\begin{aligned} -\chi_i &= u_{ji,j} = u_{jj,i} + R_{ijlj} u_l \\ &= (\Delta u)_i + R_{il} u_l \\ &= -n\chi_i + (n-1)\,c^2 u_i. \end{aligned}$$

Thus $\chi_i - c^2 u_i = 0$ or $\chi - c^2 u$ is constant. Since $\int_\Sigma u = 0$, we have $\chi = c^2 u$. Therefore, $D^2 u = -c^2 u g$. The proposition then follows from the classic Obata theorem [O1]. □

Since $(\widetilde{M}, g_\theta)$ is Einstein with $\mathrm{Ric}(g_\theta) = \frac{m}{2} g$ and f satisfies (3.1), applying the above proposition we conclude that $(\widetilde{M}, g_\theta)$ is isometric to $\left(\mathbb{S}^{2m+1}, 4g_0\right)$ and f corresponds to a linear function on $\mathbb{S}^{2m+1}$. By an argument in [LW] we know that $(\widetilde{M}, \theta)$ is in fact CR isometric to $\left(\mathbb{S}^{2m+1}, \theta_c\right)$. For completeness, we repeat the proof here. Without loss of generality, we can take $(\widetilde{M}, g_\theta)$ to be $\left(\mathbb{S}^{2m+1}, 4g_0\right)$. Then θ is a pseudohermitian structure on $\mathbb{S}^{2m+1}$ whose adapted metric is $4g_0$ and the associated Tanaka–Webster connection is torsion-free. It is a well-known fact that the Reeb vector field T is then a Killing vector field for g_0. Therefore, there exists a skew-symmetric matrix A such that for all $X \in \mathbb{S}^{2m+1}, T(X) = AX$, here we use the obvious identification between $z = (z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1}$ and $X =$

$(x_1, y_1, \dots, x_{m+1}, y_{m+1}) \in \mathbb{R}^{2m+2}$. Changing coordinates by an orthogonal transformation we can assume that A is of the following form:

$$A = \begin{bmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{m+1} \\ & & & -a_{m+1} & 0 \end{bmatrix},$$

where $a_i \geq 0$. Therefore,

$$T = \sum_i a_i \left(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i} \right).$$

Since T is of unit length we must have

$$4 \sum_i a_i^2 (x_i^2 + y_i^2) = 1$$

on $\mathbb{S}^{2m+1}$. Therefore, all the a_i's are equal to $1/2$. It follows that

$$\theta = g_0(T, \cdot) = 2\sqrt{-1}\partial |z|^2.$$

Therefore, $\left(\widetilde{M}, \theta\right)$ is CR isometric to $\left(\mathbb{S}^{2m+1}, \theta_c\right)$.

Take $\left(\widetilde{M}, \theta\right)$ to be $\left(\mathbb{S}^{2m+1}, \theta_c\right)$. Then there exists a unit $\xi \in \mathbb{C}^{m+1}$ and $a > 0$ s.t.

$$f(z) = a \operatorname{Re} z \cdot \overline{\xi}.$$

Now $M = \mathbb{S}^{2m+1}/\Gamma$, where $\Gamma \subset U(m+1)$ is a finite group acting on $\mathbb{S}^{2m+1}$ freely. Since f must be invariant under Γ, it is easy to see that Γ must be trivial. Finally, we have

$$\begin{aligned} \operatorname{Re} e^{u/2 + \sqrt{-1}v/2} &= \exp u/2 \cos v/2 \\ &= c + a \operatorname{Re} z \cdot \overline{\xi} \\ &= \operatorname{Re}\left(c + a z \cdot \overline{\xi}\right). \end{aligned}$$

Thus $e^{u/2+\sqrt{-1}v/2} = \lambda + az \cdot \overline{\xi}$ with $\lambda = c + \sqrt{-1}c'$ for some $c' \in \mathbb{R}$. Then

$$\phi = e^u = \left|\lambda + az \cdot \overline{\xi}\right|^2.$$

This finishes the proof.

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Appendix

In this appendix, we collect some of the formulas in CR geometry used in the proof of the main theorem.

Let $\left(M^{2m+1},\theta\right)$ be pseudohermitian manifold and ∇ the Tanaka–Webster connection. Let $\widehat{\nabla}$ be the Levi–Civita connection of the adapted Riemannian metric g_θ. The following two propositions can be found, for example, in [DT] in equivalent forms.

Proposition A.1. *We have*

$$\begin{aligned}\widehat{\nabla}_X Y = \nabla_X Y + \theta\left(Y\right)AX + \frac{1}{2}\left(\theta\left(Y\right)\phi X + \theta\left(X\right)\phi Y\right)\\ - \left[\left\langle AX, Y\right\rangle + \frac{1}{2}\omega\left(X,Y\right)\right]T.\end{aligned}$$

With this formula, one can compare the curvature tensor R of ∇ and the curvature tensor $\widehat{R}$ of $\widehat{\nabla}$.

Proposition A.2. *Suppose X, Y are horizontal vector fields, then*

$$\begin{aligned}\widehat{R}\left(X,Y,X,Y\right) &= R\left(X,Y,X,Y\right) - \frac{3}{4}\left\langle JX,Y\right\rangle^2 + \left\langle AX,Y\right\rangle^2\\ &\quad - \left\langle AX,X\right\rangle\left\langle AY,Y\right\rangle,\\ \widehat{R}\left(X,T,Y,T\right) &= -\left\langle \nabla_T AX,Y\right\rangle - \left\langle AX,AY\right\rangle + \left\langle AX,JY\right\rangle + \frac{1}{4}\left\langle X,Y\right\rangle,\\ \widehat{R}\left(X,Y,Z,T\right) &= \left\langle \nabla_X AY,Z\right\rangle - \left\langle \nabla_Y AX,Z\right\rangle.\end{aligned}$$

Using Proposition A.1, it is easy to check by direct calculation the following:

Proposition A.3. *Let $u \in C^\infty(M)$ and D^2u be its Riemannian Hessian w.r.t. g_θ. We have the following formulas:*

$$\begin{aligned} D^2u\,(T,T) &= u_{0,0}, \\ D^2u\,(T,T_\alpha) &= u_{\alpha,0} - \frac{\sqrt{-1}}{2}u_\alpha, \\ D^2u\,(T_\alpha,T_\beta) &= u_{a,\beta} + A_{\alpha\beta}u_0, \\ D^2u\left(T_\alpha,T_{\overline{\beta}}\right) &= u_{\alpha,\overline{\beta}} - \frac{\sqrt{-1}}{2}\delta_{\alpha\beta}u_0. \end{aligned}$$

Taking trace using Proposition A.2 yields the Ricci curvature Ric of g_θ in terms of the pseudohermitian Ricci tensor $R_{\alpha\overline{\beta}}$ and the torsion A.

Proposition A.4. *Suppose $X = 2\operatorname{Re}\sum_{\alpha=1}^m c_\alpha T_\alpha$. We have*

$$\begin{aligned} \widehat{\operatorname{Ric}}\,(X,X) &= 2R_{\alpha\overline{\beta}}c_\alpha\overline{c_\beta} + \sqrt{-1}\,(m-1)\left(A_{\alpha\beta}c_\alpha c_\beta - A_{\overline{\alpha}\overline{\beta}}\overline{c_\alpha c_\beta}\right) \\ &\quad - \frac{1}{2}\left|X\right|^2 - \left\langle \nabla_T AX, X\right\rangle + \left\langle AX, JX\right\rangle, \\ \widehat{\operatorname{Ric}}\,(X,T) &= 2\left\langle X, \operatorname{Re}A_{\alpha\beta,\overline{\alpha}}T_{\overline{\beta}}\right\rangle, \\ \widehat{\operatorname{Ric}}\,(T,T) &= \frac{m}{2} - \left|A\right|^2. \end{aligned}$$

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