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Indecomposable objects and Lusztig's canonical basis

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We compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$ — the categorification of $U_q^+(\mathfrak{sl}_3)$, the positive half of quantum $\mathfrak{sl}_3$ — and we decompose an arbitrary object into indecomposable ones. On the decategorified level, we obtain Lusztig's canonical basis of $U_q^+(\mathfrak{sl}_3)$. We also categorify the higher quantum Serre relations in $U_q^+(\mathfrak{sl}_3)$, by defining a certain complex in the homotopy category of $\dot{\mathcal{U}}_3^+$ that is homotopic to zero. All our work is done over the ring of integers. This paper is based on the extended diagrammatic calculus introduced to categorify quantum groups.

1. Introduction

In recent years, there has been a lot of work on diagrammatic categorification of quantum groups, initiated by Lauda's diagrammatic categorification [7] (see, also [3]) of Lusztig's idempotented version of $\dot{U}_q(\mathfrak{sl}_2)$. This was extended by Khovanov and Lauda in [5] to $\dot{U}_q(\mathfrak{sl}_n)$ and also in [4] to the positive half of an arbitrary quantum group $U_q^+(\mathfrak{g})$.

The general framework of these constructions is to define a certain 2-category $\mathcal{U}$ whose 1-morphisms categorify generators of a quantum group, and whose 2-morphisms are $\mathbb{K}$-linear combinations of certain planar diagrams modulo local relations, with $\mathbb{K}$ being a field. Then a 2-category $\dot{\mathcal{U}}$ is defined as the Karoubi envelope of the 2-category $\mathcal{U}$, i.e., the smallest category containing $\mathcal{U}$ in which all idempotent 2-morphisms split. Finally, it is shown that the split Grothendieck group of $\dot{\mathcal{U}}$ is isomorphic to the corresponding quantum group.

For the categorification of $U_q^+(\mathfrak{g})$, the 2-categories $\mathcal{U}$ and $\dot{\mathcal{U}}$ have a single object. Thus, one can see them as monoidal 1-categories. Since, in this paper, we are interested in categorifications of positive halves of quantum groups, we shall always assume that $\mathcal{U}$ and $\dot{\mathcal{U}}$ are monoidal (1-)categories.

Keywords and phrases. categorification, Lusztig's canonical basis, quantum groups, diagrammatic calculus.

The extension of the diagrammatic calculus — so-called thick calculus — was introduced in [6] in the case of quantum $\mathfrak{sl}_2$. With thick calculus one can work directly in $\dot{\mathcal{U}}$, and not just in $\mathcal{U}$. The thick calculus can be extended directly to include the categorification of $\mathfrak{sl}_n$ (see [11]).

The consequence of [6], and of the thick calculus, is that now one can take $\mathbb{Z}$-linear combinations of planar diagrams as morphisms of $\mathcal{U}$, because the idempotents being added have no denominators. In this paper, we use thick calculus to study the properties of the category $\dot{\mathcal{U}}_n^+$ that categorifies the positive half of the quantum $\mathfrak{sl}_n$. In particular, the category $\dot{\mathcal{U}}_n^+$ is defined over the ring of integers.

In Section 4 of this paper, we compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$. More precisely, in Theorem 2, we show that these are the following:

$$\mathcal{B} = \left\{ \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}, \mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)}\{t\} \mid b \geq a+c, \quad a,c \geq 0, \ t \in \mathbb{Z} \right\}$$

with $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}\{t\} \cong \mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}\{t\}$, for $a, c \geq 0$, $t \in \mathbb{Z}$. Moreover, we prove that an arbitrary object of $\dot{\mathcal{U}}_3^+$ can be decomposed as a direct sum of elements of $\mathcal{B}$, using idempotents defined over $\mathbb{Z}$.

The main result (Theorem 3) is a categorification of the $U_q^+(\mathfrak{sl}_3)$ relation

$$E_1^{(a)}E_2^{(b)}E_1^{(c)} = \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)}E_1^{(a+c)}E_2^{(r)}, \quad \text{for } b \leq a+c. \tag{1}$$

By decategorifying the set of indecomposables from $\mathcal{B}$ with no shifts, we obtain the set

$$B = \left\{ E_1^{(a)}E_2^{(b)}E_1^{(c)}, \ E_2^{(a)}E_1^{(b)}E_2^{(c)} \mid b \geq a+c, \quad a,c \geq 0 \right\}$$

with $E_1^{(a)}E_2^{(a+c)}E_1^{(c)} = E_2^{(c)}E_1^{(a+c)}E_2^{(a)}$, for $a, c \geq 0$. The set B is Lusztig's canonical basis of $U_q^+(\mathfrak{sl}_3)$ (see, [9]), and one of its remarkable properties is that its structure constants live in $\mathbb{N}[q, q^{-1}]$. In this way, we have proven that the indecomposable objects of $\dot{\mathcal{U}}_3^+$ lift the canonical basis of $U_q^+(\mathfrak{sl}_3)$. We note once again that we are working in the category that is defined over the ring of integers $\mathbb{Z}$.

Previous results on this topic were obtained when the category is defined over a field, i.e., when 1-morphisms are $\mathbb{K}$-linear combinations of planar diagrams, for some characteristic zero field $\mathbb{K}$. The result that the indecomposable objects lift the Lusztig canonical basis for $\mathfrak{sl}_3$ was obtained by Khovanov and Lauda [3]. Furthermore, Brundan and Kleshchev [1] have extended that result to the case of affine $\mathfrak{sl}_n$. Finally, for $\mathbb{K} = \mathbb{C}$, Varagnolo and Vasserot [12] proved this fact for any simply-laced $\mathfrak{g}$.

All of these results require $\mathbb{K}$ to be a field of characteristic zero. It has previously been proven that the indecomposables of $\mathcal{U}_n^+$ descend to the canonical basis, but only after base change from $\mathbb{Z}$ to a field $\mathbb{K}$. For example, Varagnolo and Vasserot [12] have shown that $\mathcal{U}_n^+$ is equivalent to a category of perverse sheaves on quiver varieties, where the analogous result is proven over a field $\mathbb{K}$ of characteristic zero using the decomposition theorem. However, by constructing all the relevant idempotents over $\mathbb{Z}$, we have shown a stronger statement: that taking the Karoubi envelope commutes with base change, and that the behavior of $\dot{\mathcal{U}}_3^+$ does not depend on the underlying characteristic. Said another way, the decomposition theorem holds in finite characteristic, for particular maps of quiver varieties. This is in marked difference to other contexts appearing in geometry and categorification (such as flag varieties and categorifications of the Hecke algebra) where the decomposition theorem fails in finite characteristic. This presents one strong application of the diagrammatic calculus.

The other goal of this paper is to categorify the higher quantum Serre relations for E_1 and E_2. The higher quantum Serre relations for the generators E_1 and E_2 are

$$\sum_{i=0}^{m}(-1)^i q^{+(m-n-1)i} E_1^{(m-i)} E_2^{(n)} E_1^{(i)} = 0, \qquad \text{for } m > n > 0, \tag{2}$$

$$\sum_{i=0}^{m}(-1)^i q^{-(m-n-1)i} E_1^{(m-i)} E_2^{(n)} E_1^{(i)} = 0, \qquad \text{for } m > n > 0. \tag{3}$$

The relations (2) and (3) can be obtained by summing appropriately some relations of the form (1). Thus from the categorification of (1) (Theorem 3), one can obtain a decomposition that lifts (2) and (3).

However, in Section 5, we give a direct and rather simple categorification of the higher quantum Serre relation in the homotopy category of $\dot{\mathcal{U}}_3^+$ — the category of complexes in $\dot{\mathcal{U}}_3^+$, modulo homotopies. Since higher quantum Serre relations have the form of alternating sums, it is natural to look for their categorification in the form of a complex of objects of $\dot{\mathcal{U}}_3^+$ that lift the summands of (2) and (3). In Theorem 6, we define such complexes and show that they are homotopic to zero. What is particularly interesting is not just that such complexes exist but that the differentials and the homotopies have a very simple form.

Remark 1. We note that all results from this paper about generators E_1 and E_2, and objects $\mathcal{E}_1$ and $\mathcal{E}_2$, are valid in an arbitrary quantum group $U_{\mathrm{q}}(\mathfrak{g})$ and in the categorification of its positive half, for generators E_r and E_s, and for objects $\mathcal{E}_r$ and $\mathcal{E}_s$, respectively, with r and s satisfying $r \cdot s = -1$.

2. $U_q^+(\mathfrak{sl}_n)$

In this section, we define the positive half of quantum $\mathfrak{sl}_n$ — denoted $U_q^+(\mathfrak{sl}_n)$. We also give some of its combinatorial properties in the case $n = 3$.

Let $n \geq 2$ be fixed. The index set of quantum $\mathfrak{sl}_n$ is $I = \{1, 2, \ldots, n-1\}$. An inner product is defined on $\mathbb{Z}[I]$ by setting

$$i \cdot j = \begin{cases} 2, & i = j, \\ -1, & |i-j| = 1, \\ 0, & |i-j| \geq 2, \end{cases}$$

for $i, j \in I$.

$U_q^+(\mathfrak{sl}_n)$ is a $\mathbb{Q}(q)$-algebra generated by $E_1, E_2, \ldots, E_{n-1}$ modulo relations

$$E_i^2 E_j + E_j E_i^2 = [2] E_i E_j E_i, \quad i \cdot j = -1, \tag{4}$$

$$E_i E_j = E_j E_i, \qquad i \cdot j = 0. \tag{5}$$

The quantum integers and binomial coefficients are given by

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]! = [n][n-1]\cdots[2][1],$$
$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

The divided powers of the generators are defined by

$$E_i^{(a)} := \frac{E_i^a}{[a]!}, \quad a \geq 0, \quad i = 1, \ldots, n-1.$$

The divided powers satisfy

$$E_i^{(a)} E_j^{(b)} = E_j^{(b)} E_i^{(a)}, \quad i \cdot j = 0, \tag{6}$$

$$E_i^{(a)} E_i^{(b)} = \begin{bmatrix} a+b \\ a \end{bmatrix} E_i^{(a+b)} \tag{7}$$

and the quantum Serre relations

$$E_i^{(2)} E_j + E_j E_i^{(2)} = E_i E_j E_i, \quad i \cdot j = -1. \tag{8}$$

The integral form ${}_{\mathbb{Z}}U_q^+(\mathfrak{sl}_n)$ is the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q^+(\mathfrak{sl}_n)$ generated by $E_i^{(a)}$, for all $i = 1, \ldots, n-1$ and $a \geq 0$.

2.1. Combinatorics of $U_q^+(\mathfrak{sl}_3)$

The higher quantum Serre relations (Chapter 7 of [8]) for E_1 and E_2 are

$$\sum_{r=0}^{m} (-1)^r q^{+(m-n-1)r} \; E_1^{(m-r)} E_2^{(n)} E_1^{(r)} = 0, \quad m > n > 0, \tag{9}$$

$$\sum_{r=0}^{m} (-1)^r q^{-(m-n-1)r} \; E_1^{(m-r)} E_2^{(n)} E_1^{(r)} = 0, \quad m > n > 0. \tag{10}$$

In particular, the quantum Serre relations are obtained for $m = 2$ and $n = 1$.

Proposition 1 ([8, Lemma 42.1.2.(d)]). *For any three non-negative integers a, b, c, with $b \leq a + c$, we have*

$$E_1^{(a)} E_2^{(b)} E_1^{(c)} = \sum_{\substack{p+r=b \\ p \leq c \\ r \leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)} E_1^{(a+c)} E_2^{(r)}. \tag{11}$$

In particular, for $b = a + c$, we have

$$E_1^{(a)} E_2^{(a+c)} E_1^{(c)} = E_2^{(c)} E_1^{(a+c)} E_2^{(a)}. \tag{12}$$

We note that the higher quantum Serre relations (9) follow from (11), together with the following well-known relation among quantum binomial coefficients, valid for any non-negative integer N:

$$\sum_{k=0}^{N} (-1)^k q^{(N-1)k} \begin{bmatrix} N \\ k \end{bmatrix} = 0. \tag{13}$$

2.2. Monomials in $U_q^+(\mathfrak{sl}_3)$

By a monomial, we mean a product of the form $E_1^{(a_1)}E_2^{(b_2)}E_1^{(a_2)}\dots E_1^{(a_n)}E_2^{(b_n)}$. The number of non-zero exponents a_i and b_i we call the length of the monomial. Let B be the following set of monomials:

$$B = \{E_1^{(a)}E_2^{(b)}E_1^{(c)},\ \ E_2^{(a)}E_1^{(b)}E_2^{(c)}|\quad b \geq a+c,\ \ a,b,c \geq 0\}, \tag{14}$$

where we have $E_1^{(a)}E_2^{(a+c)}E_1^{(c)} = E_2^{(c)}E_1^{(a+c)}E_2^{(a)}$, for all $a, c \geq 0$.

The set B is Lusztig's canonical basis of $U_q^+(\mathfrak{sl}_3)$, and its structure constants are from $\mathbb{N}[q,q^{-1}]$.

We also have the following.

Theorem 1. *Every monomial from $U_q^+(\mathfrak{sl}_3)$ can be expressed as a linear combination of monomials from B, with coefficients from $\mathbb{N}[q,q^{-1}]$.*

Proof. By induction on length. For monomials of length at most 2, the statement of the theorem is obvious. By Proposition 1, any monomial of length at most 3 can be expressed as a linear combination of the monomials from B, which proves the base of induction.

Suppose that a monomial v has length l, with $l \geq 4$. Then it contains a piece of the form $E_1^{(a)}E_2^{(b)}E_1^{(c)}E_2^{(d)}$, with $a,b,c,d > 0$ (or a piece of the form $E_2^{(a)}E_1^{(b)}E_2^{(c)}E_1^{(d)}$, with $a,b,c,d > 0$, which is done completely analogously). Then at least one of the inequalities $b < a+c$ or $c < b+d$ is satisfied. Suppose that the first one is satisfied (the second one is done in the same way). Then, by Proposition 1, we have

$$\begin{aligned} E_1^{(a)}E_2^{(b)}E_1^{(c)}E_2^{(d)} &= \sum_{\substack{p+r=b\\ p\leq c\\ r\leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} E_2^{(p)}E_1^{(a+c)}E_2^{(r)}E_2^{(d)} \\ &= \sum_{\substack{p+r=b\\ p\leq c\\ r\leq a}} \begin{bmatrix} a+c-b \\ c-p \end{bmatrix} \begin{bmatrix} r+d \\ r \end{bmatrix} E_2^{(p)}E_1^{(a+c)}E_2^{(r+d)}, \end{aligned}$$

i.e., v can be written as a linear combination of the monomials of length at most $l-1$, which proves the first part of the theorem.

As for the coefficients, they are all sums and products of quantum binomial coefficients, and so they live in $\mathbb{N}[q,q^{-1}]$, as desired. □

3. The category $\dot{\mathcal{U}}_n^+$

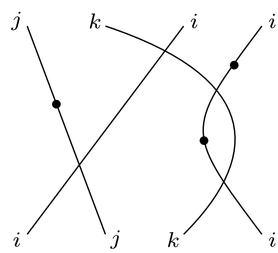
A categorification of the positive half of quantum $\mathfrak{sl}_n$ (and also of an arbitrary quantum group $U_q^+(\mathfrak{g})$) was defined in [4], though in this paper we prefer the description found in [5] in terms of a diagrammatic category $\mathcal{U}_n^+$. Before going to the definition of $\mathcal{U}_n^+$, first we recall some notation and explain the diagrams that appear in its definition (see also [5]).

Let $n \geq 2$ be fixed. We refer to the elements of the set $\{1, \dots, n-1\}$ as *colors*. Let Seq denote the set of all finite sequence of colors. For $\nu = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{N}^{n-1}$ let $\mathrm{Seq}(\nu)$ denote the set of all sequences $\underline{i} = (i_1, \dots, i_k) \in \mathrm{Seq}$ such that $\sharp\{j | i_j = l\} = \nu_l$, for all $l = 1, \dots, n-1$. Note that $k = \sum_l \nu_l$.

We will use the following notion of planar diagrams: we consider collections of arcs on the plane connecting the points $\{1, 2, \dots, k\} \subset \mathbb{R}$ in one horizontal line to the points $\{1, 2, \dots, k\} \subset \mathbb{R}$ in another horizontal line. Each arc is labeled by a number from the set $\{1, \dots, n-1\}$ (called the *color* of the arc). We require that arcs have no critical points when projected to y-axis. Arcs can intersect, but no triple intersections are allowed. Finally, an arc can carry dots.

The following is an example of a planar diagram:

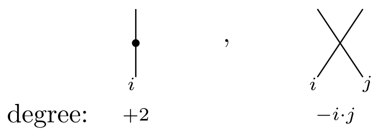
(15)



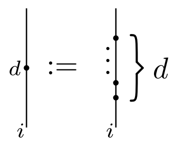
We identify two planar diagrams if there exists an isotopy between them that does not create critical points for the projection onto the y-axis.

Since we are not allowing the arcs to have critical points when projected to the y-axis, we can assume that they are always oriented upwards. We think of a planar diagram as going from its bottom boundary (a sequence of colors) to its top boundary. We read the colors on each boundary from left to right.

Each diagram has a degree defined as follows. The degree of a dot is equal to 2. The degree of a crossing between two arcs that are colored i and j is equal to $-i \cdot j$. In other words, for $i = j$ the degree of a crossing is equal to -2, for $|i - j| = 1$ (adjacent colors) the degree of a crossing is equal to 1, while for $|i-j| \geq 2$ (distant colors) the degree of a crossing is equal to 0. Finally, the degree of a diagram is obtained by summing the contributions coming from all dots and all crossings



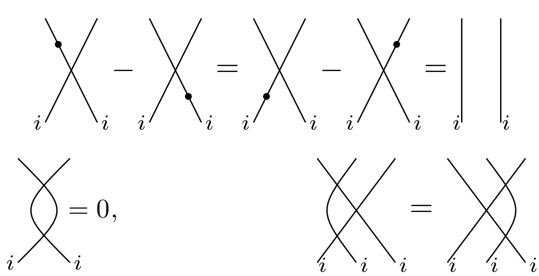
We also use the following shorthand for a collection of dots on a strand:



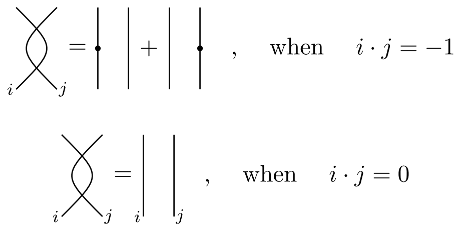
3.1. The category $\mathcal{U}_n^+$

$\mathcal{U}_n^+$ is the monoidal $\mathbb{Z}$-linear additive category whose objects and morphisms are the following:

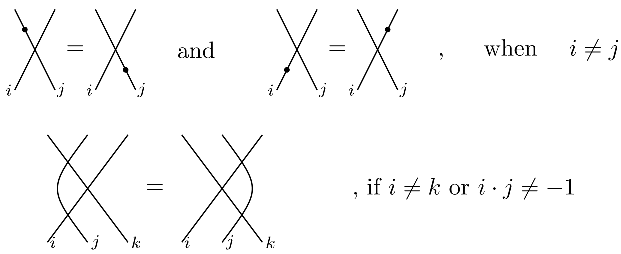
- Objects: for each $\underline{i} = (i_1, \ldots, i_k) \in \mathrm{Seq}$ and $t \in \mathbb{Z}$, we define $\mathcal{E}_{\underline{i}}\{t\} := \mathcal{E}_{i_1} \ldots \mathcal{E}_{i_k}\{t\}$. An object of $\mathcal{U}_n^+$ is a formal finite direct sum of $\mathcal{E}_{\underline{i}}\{t\}$, with $i \in \mathrm{Seq}$ and $t \in \mathbb{Z}$.
- Morphisms: for $\underline{i} = (i_1, \ldots, i_k) \in \mathrm{Seq}(\nu)$ and $\underline{j} = (j_1, \ldots, j_l) \in \mathrm{Seq}(\mu)$ the set $\mathrm{Hom}(\mathcal{E}_{\underline{i}}\{t\}, \mathcal{E}_{\underline{j}}\{t'\})$ is empty, unless $\nu = \mu$. If $\nu = \mu$ (and consequently $k = l$), the morphisms from $\mathcal{E}_{\underline{i}}\{t\}$ to $\mathcal{E}_{\underline{j}}\{t'\}$ consist of finite $\mathbb{Z}$-linear combinations of planar diagrams going from $\underline{i}$ to $\underline{j}$, of degree $t - t'$, modulo the following set of homogeneous local relations:



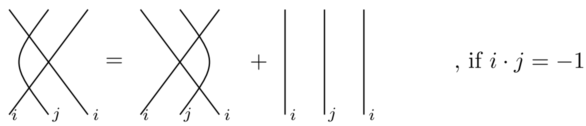
(16)



(17)



(18)

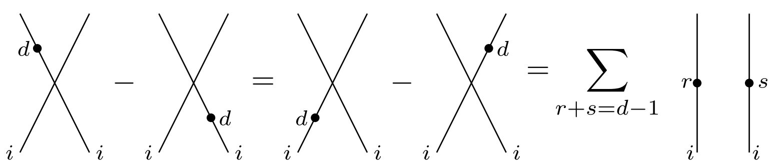


This ends the definition of $\mathcal{U}_n^+$.

As an example of a morphism, a diagram from (15) represents a morphism in $\mathrm{Hom}(\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k\mathcal{E}_i\{t\}, \mathcal{E}_j\mathcal{E}_k\mathcal{E}_i\mathcal{E}_i\{t+2\})$.

We have the following relation in $\mathcal{U}_n^+$:

Proposition 2 (Dot migration [7, Proposition 5.2]). *We have*



3.2. The category $\dot{\mathcal{U}}_n^+$ and thick calculus

In [6], the extension of the calculus to thick edges have been introduced. Thick lines categorify the divided powers $E_i^{(a)}$ (see below and Section 4 of [6]).

For a category $\mathcal{C}$, the Karoubi envelope $\mathrm{Kar}(\mathcal{C})$ is the smallest category containing $\mathcal{C}$, such that all idempotents split (for more details, see, e.g., Section 3.4 of [6]).

We define the category $\dot{\mathcal{U}}_n^+$ as the Karoubi envelope of the category $\mathcal{U}_n^+$.

As in [6], the category $\dot{\mathcal{U}}_n^+$ categorifies $U_{\mathrm{q}}^+(\mathfrak{sl}_n)$, in a sense that its split Grothendieck group is isomorphic to the integral form of $U_{\mathrm{q}}^+(\mathfrak{sl}_n)$. The isomorphism sends the class of $\mathcal{E}_i^{(a)}$ to the generator $E_i^{(a)}$ of $U_{\mathrm{q}}^+(\mathfrak{sl}_n)$.

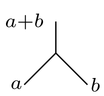
In the category $\dot{\mathcal{U}}_n^+$, the planar diagrams with thick edges from above can be interpreted as morphisms whose bottom and top end correspond to certain objects of $\dot{\mathcal{U}}_n^+$. In particular, the object corresponding to bottom (or the top end) of an arc of color i and thickness a is denoted $\mathcal{E}_i^{(a)}$.

A thick line of color i is defined as the identity morphism

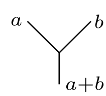
$$: \quad \mathcal{E}_i^{(a)} \longrightarrow \mathcal{E}_i^{(a)}.$$

It is given explicitly in terms of "ordinary" lines from above — see, [6, equation (2.18)], and drawn as a strand with an additional label (natural number) a, also called the *thickness* of a strand. In particular, the ordinary strands from above correspond to the case $a = 1$, and are also called *thin* edges or thin strands. We refer the reader to [6], in particular Sections 2 and 4, for more details. Here, we just recall the basic facts that will be used later on.

Trivalent vertices of a single color are now allowed in our planar diagrams, as long as the sum of thicknesses of the incoming edges is equal to the sum of thicknesses of the outgoing edges. These are called splitters in [6]. The trivalent vertices (for any color i — the labels on the pictures below represent thicknesses)



$$: \ \mathcal{E}_i^{(a)}\mathcal{E}_i^{(b)} \to \mathcal{E}_i^{(a+b)}$$



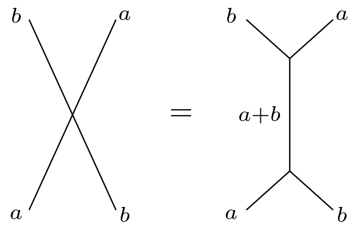
$$: \ \mathcal{E}_i^{(a+b)} \to \mathcal{E}_i^{(a)}\mathcal{E}_i^{(b)}$$

correspond to the projection and inclusion maps, respectively, obtained from the decomposition

$$\mathcal{E}_i^{(a)}\mathcal{E}_i^{(b)} \cong \bigoplus_{\left[\begin{smallmatrix} a+b \\ b \end{smallmatrix}\right]} \mathcal{E}_i^{(a+b)}.$$

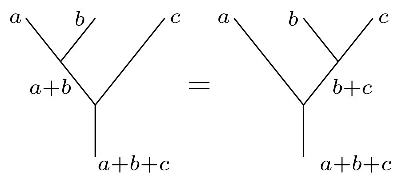
The degrees of both of these two vertices are equal to $-ab$, which explains which summands must be involved in these morphisms. The explicit definitions are given in [6, pp. 15].

These morphisms may be composed, and in particular they can be used to define the thick crossing

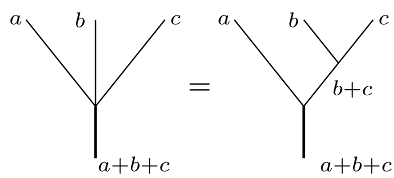


3.2.1. Some properties of the thick calculus. Below we give some of the basic properties of thick edges that we shall use in this paper (see [6] for more details). Note that the labels of the strands below denote thickness. All relations hold under horizontal and vertical flips, because of the symmetries on $\mathcal{U}$.

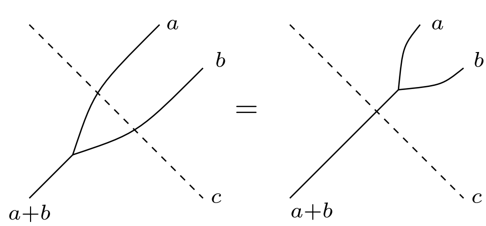
Proposition 3 (Associativity of splitters [6, Proposition 2.2.4]). *For arbitrary color i we have the following:*



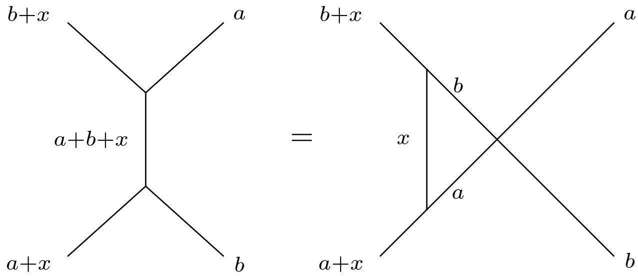
In particular, the above associativity of splitters imply that the multi-splitters, i.e., splitting of a thick line into three or more strands, is well defined



Proposition 4 (Pitchfork lemma [6]). *For any two colors i (drawn as a thick line) and j (drawn dashed line) we have*



Proposition 5 (Opening of a thick edge [6, Proposition 2.2.5]). *For any color i we have*



3.3. Schur polynomials and decorations of thick lines

Just as we can decorate thin strands with dots, we can decorate thick lines with symmetric polynomials. These correspond to symmetric polynomials in dots on thin edges involved in the definition of a thick line (for a precise definition, see, [6]). For notational convenience, we will only decorate thick strands with Schur polynomials, which form an additive basis of the ring of symmetric polynomials.

3.3.1. Schur polynomials. Here, we recall briefly the definition and some basic notation and properties of Schur polynomials. For more details, see, e.g., [2, 6, 10].

By a partition $\alpha = (\alpha_1, \dots, \alpha_k)$, we mean a non-increasing sequence of non-negative integers. We identify two partitions if they differ by a sequence of zeros at the end. We set $|\alpha| = \sum_i \alpha_i$. If for some a we have $\alpha_{a+1} = 0$, we say that α has at most a parts. We denote the set of all partitions with at most a parts by $P(a)$. Furthermore, by $P(a, b)$ we denote the subset of all partitions α from $P(a)$ such that $\alpha_1 \leq b$. In other words, $P(a, b)$ consists of partitions fitting inside a rectangle with a rows and b columns. The partition corresponding to this rectangle we denote by $K_{a,b}$, i.e., $K_{a,b} = (\underbrace{b, b, \dots, b}_{a})$.

We shall need to express quantum binomial coefficients as a sum over partitions fitting inside a rectangle. For any two non-negative integers a and b we have

$$\begin{bmatrix} a+b \\ a \end{bmatrix} = \sum_{\alpha \in P(a,b)} q^{2|\alpha| - ab}. \tag{19}$$

By $\bar{\alpha}$ we denotes the dual (conjugate) partition of α, i.e., $\bar{\alpha}_j = \sharp\{i|\alpha_i \geq j\}$. If $\alpha \in P(a,b)$, we define partition $\hat{\alpha}$ by $\hat{\alpha} = \overline{(b-\alpha_a,\dots,b-\alpha_1)}$. Note that if $\alpha \in P(a,b)$, then $\bar{\alpha} \in P(b,a)$ and $\hat{\alpha} \in P(b,a)$.

For any partition $\alpha \in P(a)$, the Schur polynomial π_α is given by the formula

$$\pi_\alpha(x_1,x_2,\dots,x_a) = \frac{|x_i^{\alpha_j+a-j}|}{\Delta},$$

where $\Delta = \prod_{1\leq r\leq s\leq a}(x_r - x_s)$, and $|x_i^{\alpha_j+a-j}|$ is the determinant of the $a \times a$ matrix whose (i,j) entry is $x_i^{\alpha_j+a-j}$. We extend our notation, so that π_α $(x_1,x_2,\dots,x_a) = 0$ is some entry of α is negative (α is not a partition then), or if $\alpha_{a+1} > 0$.

For two partitions α and γ, we say that $\alpha \subset \gamma$ if $\alpha_i \leq \gamma_i$ for all $i \geq 1$.

For three partitions α, β, and γ, the Littlewood–Richardson coefficients $c^\gamma_{\alpha,\beta}$ are given by

$$\pi_\alpha\pi_\beta = \sum_\gamma c^\gamma_{\alpha,\beta}\pi_\gamma.$$

The coefficients $c^\gamma_{\alpha,\beta}$ are non-negative integers that can be non-zero only when $|\gamma| = |\alpha| + |\beta|$. Also, $c^\gamma_{\alpha,\beta} \neq 0$ only when $\alpha \subset \gamma$ and $\beta \subset \gamma$. In particular

$$\alpha \in P(a,x), \beta \in P(b,y) \quad \text{and} \quad c^\gamma_{\alpha,\beta} \neq 0, \text{ imply } \gamma \in P(a+b,x+y). \tag{20}$$

The Littlewood–Richardson coefficients can be naturally extended for more than three partitions: for partitions $\alpha_1,\dots,\alpha_k$ and β, with $k \geq 2$, we define $c^\beta_{\alpha_1,\dots,\alpha_k}$ by

$$\pi_{\alpha_1}\dots\pi_{\alpha_k} = \sum_\beta c^\beta_{\alpha_1,\dots,\alpha_k}\pi_\beta.$$

For two partitions α and γ, we define skew-Schur polynomial $\pi_{\gamma/\alpha}$ by

$$\pi_{\gamma/\alpha} = \sum_\beta c^\gamma_{\alpha,\beta}\pi_\beta.$$

It can be non-zero only when $\alpha \subset \gamma$.

If $\gamma = (\gamma_1,\dots,\gamma_a) \subset K_{a,b}$, then by $K_{a,b} - \gamma$ we denote the partition $(b-\gamma_a,\dots,b-\gamma_1)$. For a partition $\nu \in P(a)$, by $\nu + K_{a,b}$ we denote the partition $(\nu_1+b,\dots,\nu_a+b)$. Furthermore, for every two partitions $\psi \in P(a)$ and $\gamma \in P(a,b)$, we have that $c^{\nu+K_{a,b}}_{\gamma,\psi} = c^\psi_{\nu,K_{a,b}-\gamma}$. In particular, if $\nu = \emptyset$, one has $c^{K_{a,b}}_{\gamma,\psi} = c^\psi_{\emptyset,K_{a,b}-\gamma} = \delta_{\psi,K_{a,b}-\gamma}$, and so $\pi_{K_{a,b}/\gamma} = \pi_{K_{a,b}-\gamma}$.

The elementary symmetric polynomials $\varepsilon_m(x_1,\dots,x_a)$, for $m=0,\dots,a$ are special Schur polynomials: $\varepsilon_m(x_1,\dots,x_a)=\pi_{(\underbrace{1,1,\dots,1}_{m})}(x_1,\dots,x_a)$. For $m<0$ or $m>a$, we have $\varepsilon_m(x_1,\dots,x_a)=0$.

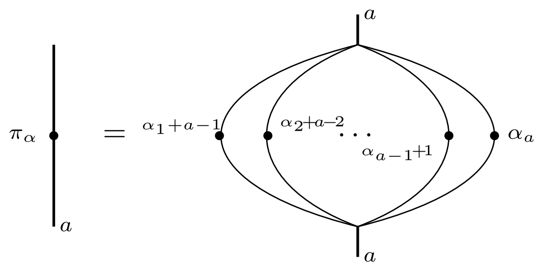
The Schur polynomials can be conveniently expressed as a determinant of a matrix whose entries are the elementary symmetric polynomials, by the following Giambelli formula: for a partition $\alpha=(\alpha_1,\dots,\alpha_a)$, we have

$$\pi_{\bar{\alpha}}=\det[\varepsilon_{\alpha_i+j-i}]_{i,j=1}^{a}. \tag{21}$$

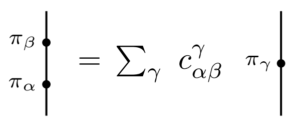
3.3.2. Decorated thick edges. Here, we recall some of the basic properties of decorated thick lines that we shall need in this paper. For more details, see, [6].

A thick line of thickness a can be decorated with any Schur polynomial π_α. For $\alpha\notin P(a)$, the resulting morphism is zero. For $\alpha\in P(a)$ of the form $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_a)$, one can express the decoration of a thick line in terms of thin lines and dots as follows:

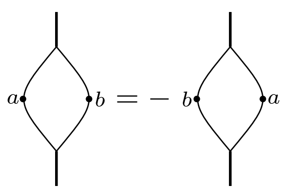
(22)



From the definition of the Littlewood–Richardson coefficient, we have

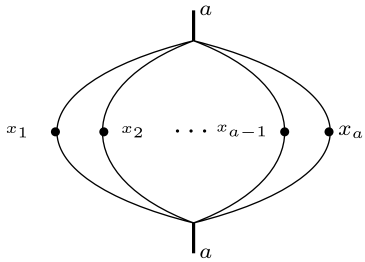


By "exploding" a thick edge into thin edges, we obtain diagrams that are antisymmetric with respect to the exchange of dots on two neighboring strands



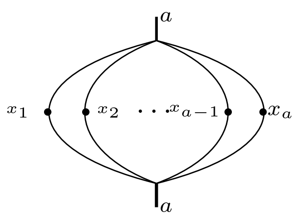
This antisymmetry implies the following.

Lemma 1.



$\neq 0 \quad \Rightarrow \quad x_r \neq x_s,$ *for all* $r \neq s$.

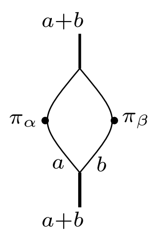
Moreover, if $\max_i\{x_i\} = a-1$*, then the diagram from above can be non-zero if and only if there exists a permutation* σ *of* $\{0, 1, \ldots, a-1\}$*, such that* $x_{a-i} = \sigma_i$*,* $i = 0, \ldots, a-1$*, in which case*



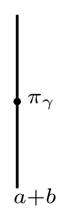
$= \operatorname{sgn} \sigma$

The above lemma implies the following.

Lemma 2 ([6, Proposition 2.4.1]). *Let* $\alpha \in P(a, x)$ *and* $\beta \in P(b, y)$ *be partitions. Then, we have that*

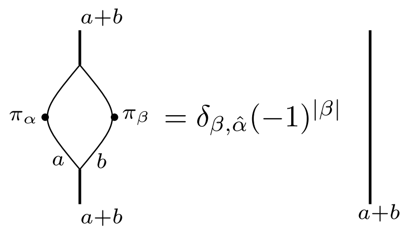


$= s$



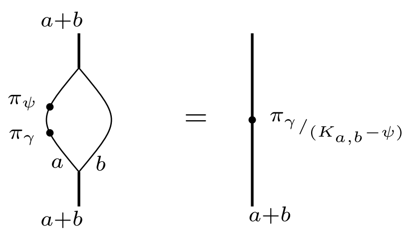
for some partition $\gamma \in P(a+b, \max\{x-b, y-a\})$ *and* $s \in \{-1, 0, 1\}$*. If* $s \neq 0$*, then* $|\gamma| = |\alpha| + |\beta| - ab$*.*

Moreover, if $\alpha \in P(a,b)$ *and* $\beta \in P(b,a)$*, then we have*

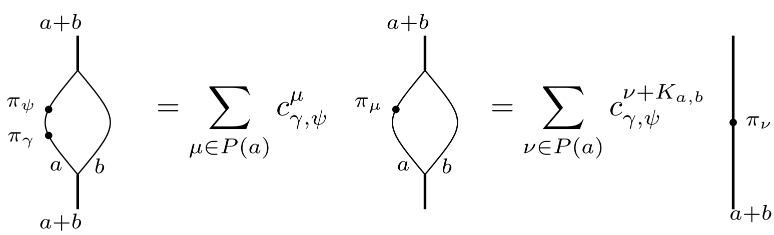


In particular, the left-hand side can be non-zero only when $|\alpha| + |\beta| = ab$*.*

Lemma 3. *Let* $\gamma \in P(a)$ *and* $\psi \in P(a,b)$ *be partitions. Then*



Proof.

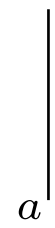


Here, in the second equality, we have used the defining relation (22) and Proposition 3 — Associativity of splitters (or, alternatively one can use [6, Proposition 2.4.1]).

Since $c^{\nu+K_{a,b}}_{\gamma,\psi} = c^{\gamma}_{\nu,K_{a,b}-\psi}$, we have $\pi_{\gamma/(K_{a,b}-\psi)} = \sum_\nu c^{\gamma}_{\nu,K_{a,b}-\psi}\pi_\nu = \sum_\nu c^{\nu+K_{a,b}}_{\gamma,\psi}\pi_\nu$, which gives the above lemma. □

So far we have been examining the diagrams of a single color. We use the following convention when drawing the diagrams involving two adjacent colors.

Notation convention: For two colors (indices) that satisfy $i \cdot j = -1$, we shall draw strands colored i as straight lines, and strands colored j as curly lines

$\mathrm{Id}_{\mathcal{E}_i^{(a)}}$: 

$\mathrm{Id}_{\mathcal{E}_j^{(b)}}$: 

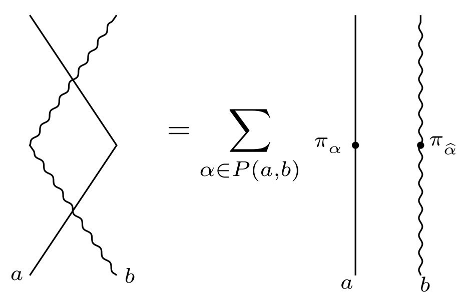
Thus, from now on, each line carries one label, and that label represents the thickness of a line.

The first "thick" property that we shall frequently use is about sliding the thick dots past crossings which involve strands of different colors. It follows straightforward from the analogous property for thin strands (17), the definition of thick dot (22) and associativity of splitters.

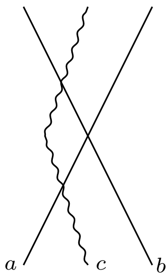
Proposition 6 (Dot slide). *The thick dots can be freely moved through the thick crossing of the two thick strands with different colors, i.e.,*

The following two propositions are extensions of the thin R2 and R3 relations (16) and (18). They are proved in [11], by using the defining relation of the thick edge in terms of thin edges ([6] and (22)), and then by using thin relations together with careful manipulation and simplification of the obtained diagrams.

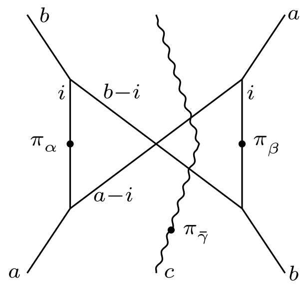
Proposition 7 (Thick R2 move [11]). *We have*



Proposition 8 (Thick R3 move [11]). *We have*



$$= \sum_{i=0}^{\min(a,b,c)} \sum_{\alpha,\beta,\gamma\in P(i,c-i)} c^{K_i}_{\alpha\beta\gamma}$$



where $K_i = (\underbrace{c-i, c-i, \ldots, c-i}_{i})$*, for* $i > 0$*, and* $K_0 = 0$.

4. Indecomposables in $\dot{\mathcal{U}}_3^+$

In this section, we compute the indecomposable objects of $\dot{\mathcal{U}}_3^+$. We show that they categorify the canonical basis B. Furthermore, we decompose an arbitrary object as a direct sum of these indecomposables, by categorifying (11).

Theorem 2. *The objects* $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ *and* $\mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)}\{t\}$*, for non-negative* a, b, c*, and* $t \in \mathbb{Z}$*, with* $b \geq a + c$*, are indecomposable in* $\dot{\mathcal{U}}_3^+$*. No two of them are isomorphic, except* $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}\{t\} \cong \mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}\{t\}$*, for* $a, c \geq 0$*,* $t \in \mathbb{Z}$*.*

The list of the objects from Theorem 2 we denote by $\mathcal{B}$.

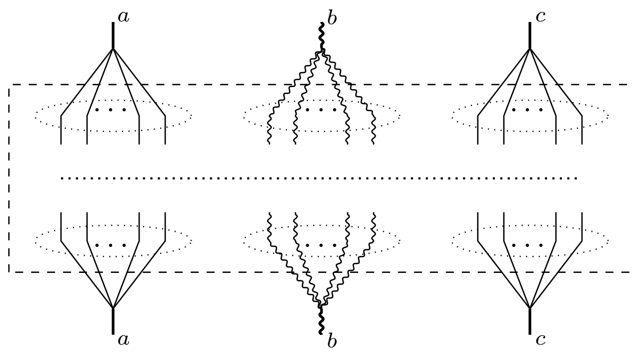
We will prove the theorem (apart from the last statement on the isomorphism of certain objects) by computing graded dimensions of certain Hom spaces. We note that these computations can be done by using Khovanov and Lauda's Hom formula (see, [3] — formula (2.45) and page 44). Nonetheless, we give here a direct, purely diagrammatic proof of the theorem.

Proof. We will show that $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$, with $b \geq a + c$, are indecomposable by showing that the graded ranks of their endomorphism rings satisfy

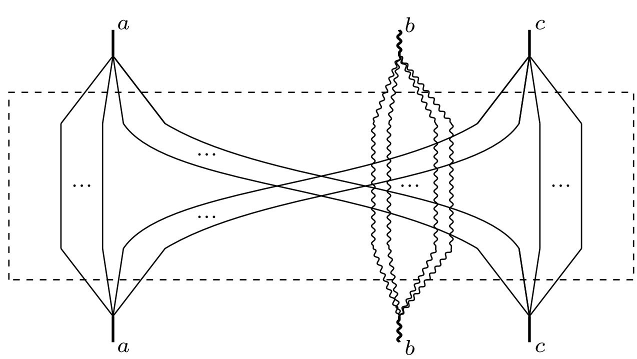
$$\text{(23)} \qquad \mathrm{rk}_q \,\mathrm{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}, \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}) \in 1 + q\mathbb{N}[q].$$

Here $\dot{\mathcal{U}}^*$ is the category with the same objects as $\dot{\mathcal{U}}_3^+$, while morphism between two objects can have arbitary degree. More precisely, for any two objects $x, y \in \dot{\mathcal{U}}^*$, we have $\mathrm{Hom}_{\dot{\mathcal{U}}^*}(x, y) = \oplus_{s \in \mathbb{Z}} \mathrm{Hom}_{\dot{\mathcal{U}}_3^+}(x\{s\}, y)$ (see, e.g., [7]).

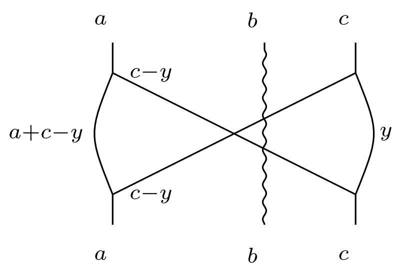
Any diagram from $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$ to itself can be represented as a linear combination of diagrams in a "standard" form. First, we use the defining relation for the thick edges ([6] — Section 4) to split them into thin ones



Here in the region within the dashed rectangle we have only thin strands. It is well known (see, e.g., [3] — Section 2.3) that any diagram consisting only of thin lines can be reduced to a diagram where any two thin lines intersect at most once, and all dots from one line are only at one segment. This is obtained by repeated use of local relations for the morphisms (16) and (18) and Proposition 2 (dot migration). In addition, due to a presence of the splitters of thick edges into thin ones, if any two thin strands from any of the six groups of thin edges (denoted by dotted ellipsis) intersect, than it can be written as linear combination of diagrams where no two such strands intersect. This follows from the fact that the thick strands and splitters are defined by "totally antisymmetrizing" thin strands ([6, pp. 15,16, and (2.25)]). Therefore, every diagram from $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$ to itself can be expressed as a linear combination of diagrams of the form



possibly with some dots on it. By regrouping the thin edges back into the thick ones, the only non-zero diagrams are of the form



possibly with dots. Thus, the lowest degree diagrams are the dotless ones of the form above, where $y \leq c$ varies.

For $y = c$ we have the identity — the degree zero map. If $y < c$, the degree of this dotless diagram is

$$-2(c-y)(a-c+y) - 2(c-y)y - 2(c-y)^2 + 2b(c-y)$$
$$= 2(c-y)(b-a-y) > 0$$

for $b \geq a+c$, and so we have proved (23). Thus, we have that $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ is indecomposable for $b \geq a+c$. In the same way, we have that $\mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)}\{t\}$ is indecomposable for $b \geq a+c$.

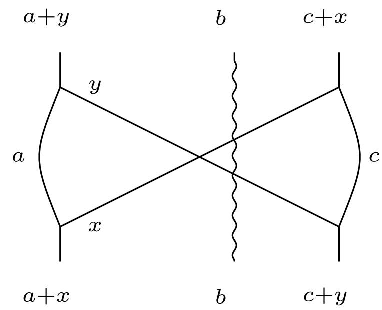
In order to prove that two indecomposables of the form $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ and $\mathcal{E}_1^{(p)}\mathcal{E}_2^{(k)}\mathcal{E}_1^{(r)}\{t'\}$, with $(a,b,c) \neq (p,k,r)$ are not isomorphic, we will show that

$$\mathrm{rk}_q \operatorname{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}, \mathcal{E}_1^{(p)}\mathcal{E}_2^{(k)}\mathcal{E}_1^{(r)}) \in q\mathbb{N}[q].$$

This Hom-space can be non-empty only when $k = b$ and $p + r = a + c$, and so we are left with proving

$$\mathrm{rk}_q \operatorname{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a+x)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c+y)}, \mathcal{E}_1^{(a+y)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c+x)}) \in q\mathbb{N}[q], \tag{24}$$

when at least one of x and y is non-zero, with $b \geq a+c+x+y$. Again, a general (dotless) diagram from the last Hom-space has the following form:



The degree of such dotless diagram is equal to

$$\begin{aligned}&-ax-ay-cx-cy-2xy+b(x+y)\\&\qquad\geq-(a+c)(x+y)-2xy+(a+c)(x+y)+(x+y)^2=x^2+y^2>0\end{aligned}$$

as desired.

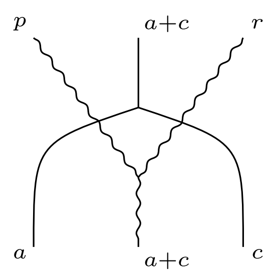
Finally, to prove that two indecomposables of the form $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ and $\mathcal{E}_2^{(p)}\mathcal{E}_1^{(k)}\mathcal{E}_2^{(r)}\{t'\}$ are not isomorphic, again we compute

$$\operatorname{rk}_q \operatorname{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)},\mathcal{E}_2^{(p)}\mathcal{E}_1^{(k)}\mathcal{E}_2^{(r)})$$

for $b\geq a+c$ and $k\geq p+r$. In order for this Hom-space to be non-empty, we must have $b=p+r$ and $k=a+c$, and so $b=k=p+r=a+c$. Hence, we are left with computing

$$\operatorname{rk}_q \operatorname{Hom}_{\dot{\mathcal{U}}^*}(\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)},\mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}) \tag{25}$$

with $p+r=a+c$. Again, a general dotless morphism from this Hom-space has the following form:



The degree of the diagram from above is equal to

$$-ac-pr+ap+cr=(a-r)(p-c)=(p-c)^2>0$$

for $p \neq c$. Thus, for $p \neq c$, the graded rank (25) is in $q\mathbb{N}[q]$, and so we have that the two indecomposable objects $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\}$ and $\mathcal{E}_2^{(p)}\mathcal{E}_1^{(k)}\mathcal{E}_2^{(r)}\{t'\}$ are not isomorphic.

The only possible isomorphism remaining between two objects in $\mathcal{B}$ would be an isomorphism between $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}\{t\}$ and $\mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}\{t\}$, and below (Theorem 3) we show that they are indeed isomorphic. $\square$

4.1. Decomposition of an arbitrary object of $\dot{\mathcal{U}}_3^+$

In this section, we decompose an arbitrary object from $\dot{\mathcal{U}}_3^+$ as a direct sum of objects in $\mathcal{B}$. We do this by categorifying the Proposition 1, i.e., by proving the following:

Theorem 3. *For $b \leq a+c$, we have the following canonical decomposition:*

$$\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \cong \bigoplus_{\substack{p+r=b\\ p\leq c\\ r\leq a}} \bigoplus_{\alpha\in P(c-p,a-r)} \mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}\{2|\alpha|-(c-p)(a-r)\}. \tag{26}$$

Note that this indeed categorifies (11), since by (19)

$$\begin{bmatrix} a+c-b \\ c-p \end{bmatrix} = \begin{bmatrix} a-r+c-p \\ c-p \end{bmatrix} = \sum_{\alpha\in P(c-p,a-r)} q^{2|\alpha|-(c-p)(a-r)}.$$

In the same way as in Theorem 1 (by changing sums with direct sums), the decomposition (26), together with the decomposition (see, [6, Theorem 5.1])

$$\mathcal{E}_i^{(a)}\mathcal{E}_i^{(b)} \cong \bigoplus_{\alpha\in P(a,b)} \mathcal{E}_i^{(a+b)}\{2|\alpha|-ab\}$$

that categorifies (7), implies the decomposition of an arbitrary object as a direct sum of objects in $\mathcal{B}$. Thus, the decomposition (26) gives:

Theorem 4. *The set of indecomposable objects of $\dot{\mathcal{U}}_3^+$ is the set $\mathcal{B}$:*

$$\mathcal{B} = \{\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}\{t\},\ \mathcal{E}_2^{(a)}\mathcal{E}_1^{(b)}\mathcal{E}_2^{(c)}\{t\} \mid \quad b\geq a+c,\ a,b,c\geq 0,\ t\in\mathbb{Z}\}. \tag{27}$$

No two elements from $\mathcal{B}$ are isomorphic, except

$$\mathcal{E}_1^{(a)}\mathcal{E}_2^{(a+c)}\mathcal{E}_1^{(c)}\{t\} \cong \mathcal{E}_2^{(c)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(a)}\{t\}, \quad a,c\geq 0,\ t\in\mathbb{Z}.$$

An arbitrary object of $\dot{\mathcal{U}}_3^+$ can be decomposed as a direct sum of the elements in $\mathcal{B}$.

In this way, we have obtained a bijection between the canonical basis B of $U_q^+(\mathfrak{sl}_3)$ and the indecomposable objects from $\mathcal{B}$ with no shifts.

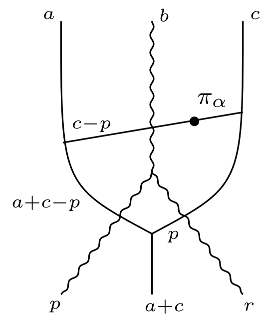
So, we are left with proving the decomposition (26) (i.e., Theorem 3) and that is done in the rest of this section.

4.2. Proof of Theorem 3

The conditions $p+r=b$, $p \le c$ and $r \le a$ together, are equivalent to $\max(0, b-a) \le p \le \min(b,c)$, with $r = b-p$. For every non-negative integer p with $\max(0,b-a) \le p \le \min(b,c)$, and partition $\alpha \in P(c-p, a-r)$, we define the following 2-morphisms:

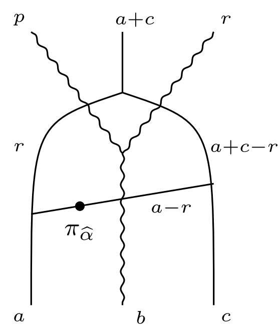
$$\lambda_\alpha^p : \mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}\{2|\alpha| - (c-p)(a-r)\} \longrightarrow \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}, \tag{28}$$

$$\lambda_\alpha^p := (-1)^{r(a+c-r)+|\alpha|}$$



$$\sigma_\alpha^p : \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \longrightarrow \mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}\{2|\alpha| - (c-p)(a-r)\}, \tag{29}$$

$$\sigma_\alpha^p :=$$



$$e_\alpha^p := \lambda_\alpha^p \sigma_\alpha^p : \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \longrightarrow \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}. \tag{30}$$

The following lemma is the key result.

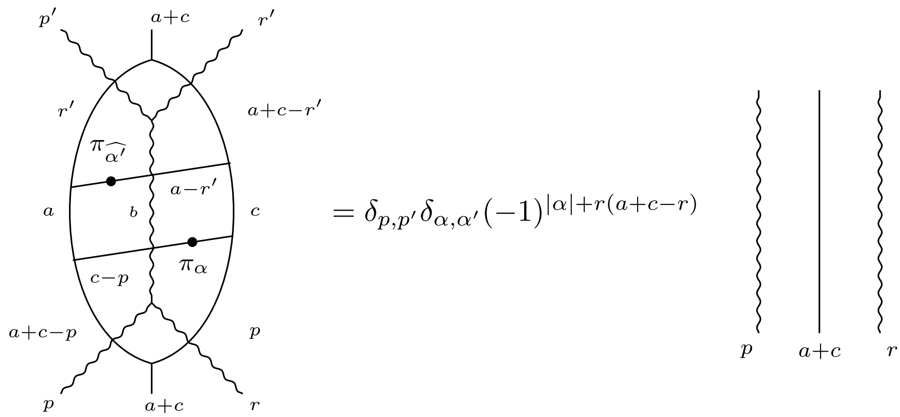
Lemma 4. *Let* $b \leq a + c$. *Let* $\max(0, b-a) \leq p, p' \leq \min(b, c)$, $\alpha \in P(c-p, a-r)$ *and* $\alpha' \in P(c-p', a-r')$, *where* $r = b - p$ *and* $r' = b - p'$. *Then*

$$\sigma_{\alpha'}^{p'} \lambda_{\alpha}^{p} = \delta_{p,p'} \delta_{\alpha,\alpha'} \mathrm{Id}_{\mathcal{E}_2^{(p)} \mathcal{E}_1^{(a+c)} \mathcal{E}_2^{(r)}}. \tag{31}$$

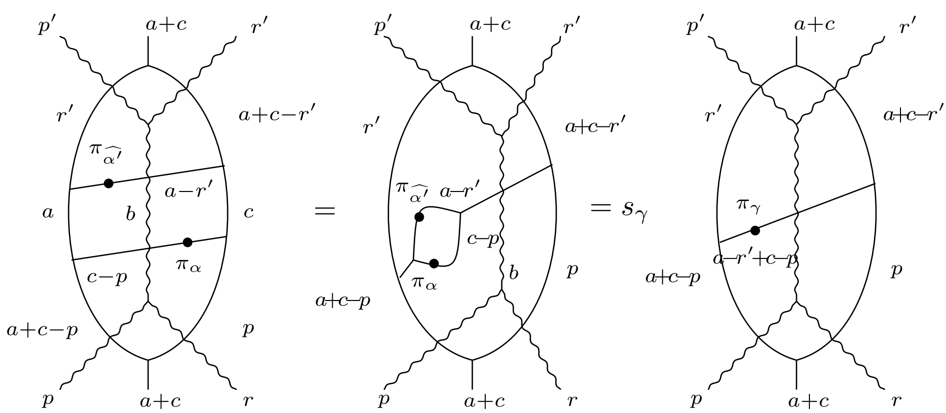
This lemma implies the theorem below, which is the main step in the proof of Theorem 3.

Theorem 5. *The collection* $\{e_{\alpha}^{p}\}$ *is a collection of mutually orthogonal idempotents.*

Proof. (of Lemma 4): In pictures, the statement of the lemma above is the following:

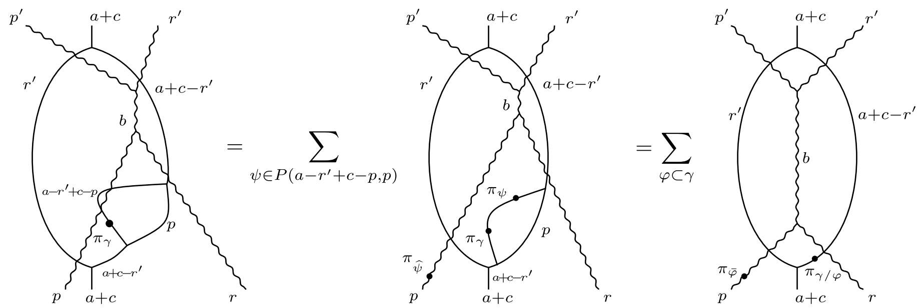


We shall prove the formula by simplifying the diagram on the left-hand side



The first equality holds because of the dot slide, associativity of splitters and the pitchfork lemma. The second equality follows from Lemma 2, and so we have $\gamma \in P(a-r'+c-p, r'-r)$ and $s_\gamma \in \{-1, 0, 1\}$. Thus, in order for the last diagram to be non-zero, we must have $r' \geq r$. Moreover, if $r' = r$, by the second part of Lemma 2, we must also have $\alpha = \alpha'$ and $s_\gamma = (-1)^{|\alpha|}$.

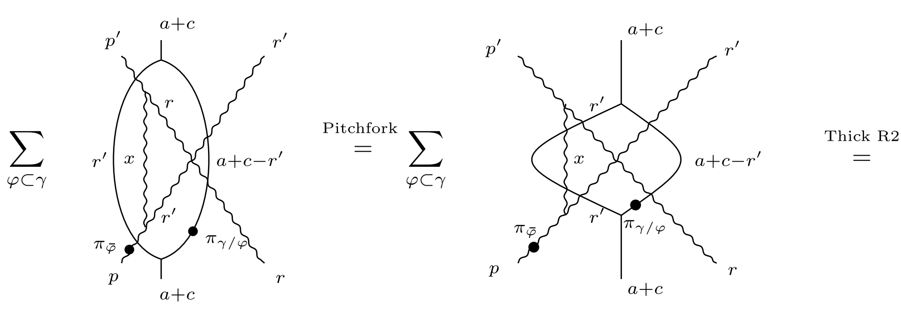
We simplify the last diagram (without the sign s_γ) by applying the Pitchfork lemma twice, the dot slide move (for π_γ), and thus get the first diagram below. Then we apply a thick R2 move, followed by associativity of splitters and a dot slide move (for $\pi_{\widehat{\psi}}$), which gives the first equality:



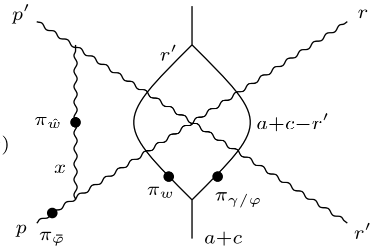
The second equality follows by Lemma 3, and we denote $\varphi = K_{a-r'+c-p,p} - \psi$.

On the last diagram, we shall apply opening of a thick edge, for the curly line of thickness b. We have two possibilities: either $p \geq r'$ or $p < r'$. We shall assume that the first one is satisfied — the other case is done completely analogously.

Let $p = r' + x$, for some $x \geq 0$. Note that then also $p' = r + x$. Using Proposition 5 (opening of a thick edge), the last diagram becomes

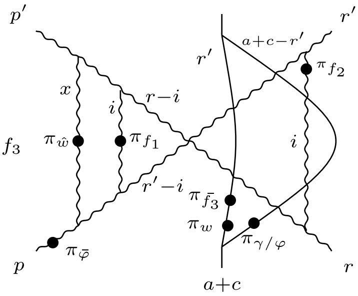


$$= \sum_{\varphi \subset \gamma} \sum_{w \in P(r',x)}$$



Thick R3 + Pitchfork $=$

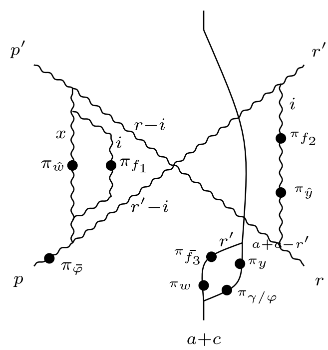
$$= \sum_{\varphi \subset \gamma} \sum_{w \in P(r',x)} \sum_{i=0}^{r} \sum_{f_1,f_2,f_3 \in P(i,r'-i)} c^{K_i}_{f_1 f_2 f_3}$$



Thick R2 $=$

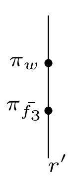
(32)

$$= \sum_{\varphi \subset \gamma} \sum_{w \in P(r',x)} \sum_{i=0}^{r} \sum_{f_1,f_2,f_3 \in P(i,r'-i)} \sum_{y \in P(a+c-r',i)} c^{K_i}_{f_1 f_2 f_3}$$

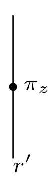


where $K_i = K_{i,r'-i}$.

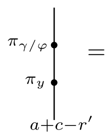
Although the last expression has many sums in it, very few summands can be non-zero. The limits on the sizes of partitions in the sums below come from the basic property of the non-zero Littlewood–Richardson coefficients (20). First of all, we have that

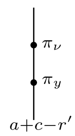


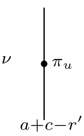
$$= \sum_{z \in P(r',x+i)} c^{z}_{w,\bar{f}_3}$$



and

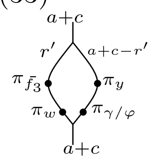


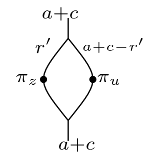
$= \sum_{\nu\in P(a+c-r',r'-r)} c^{\gamma}_{\varphi,\nu}$


$= \sum_{u\in P(a+c-r',r'-r+i)} \sum_{\nu\in P(a+c-r',r'-r)} c^{\gamma}_{\varphi,\nu} c^{u}_{y,\nu}$


and so the digon on thick line can be written as

(33)

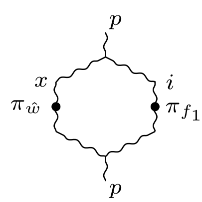


$= \sum_{z\in P(r',x+i)} \sum_{u\in P(a+c-r',r'-r+i)} \sum_{\nu\in P(a+c-r',r'-r)} c^{\gamma}_{\varphi,\nu} c^{u}_{y,\nu} c^{z}_{w,\bar{f}_3}$


Since $r' \geq r \geq i$ and by assumption $b \leq a+c$, we have that $x+i = p-r'+i \leq p-r'+r = b-r' \leq a+c-r'$ and $r'-r+i \leq r'$, and so $z \in P(r', a+c-r')$ and $u \in P(a+c-r', r')$. Thus, by Lemma 2, we have that the last diagram can be non-zero only when $|z|+|u| = r'(a+c-r')$, i.e., we must have

$$|w| + |f_3| + |y| + |\gamma| - |\varphi| = r'(a+c-r'). \tag{34}$$

As for the digon on curly lines



again by Lemma 2 it can be non-zero only when $|\hat{w}| + |f_1| \geq xi$, i.e.,

$$r'x - |w| + |f_1| \geq xi. \tag{35}$$

From (34) and (35), we have

$$r'(a+c-r') \leq x(r'-i) + |f_1| + |f_3| + |y| + |\gamma| - |\varphi|.$$

From $|f_1| + |f_3| \leq |f_1| + |f_2| + |f_3| = i(r' - i)$ and since $y \in P(a + c - r', i)$, $\gamma \in P(a + c - r' - p, r' - r)$ and $x = p - r'$, we obtain

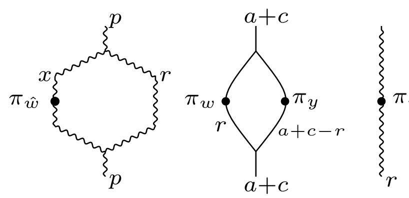
$$r'(a + c - r') \leq (p - r')(r' - i) + i(r' - i) + (a + c - r')i \\ + (a + c - r' - p)(r' - r) - |\varphi|.$$

The last can be rewritten as

$$|\varphi| + (a + c - i - p)(r - i) + (r' - i)(r' - r) \leq 0.$$

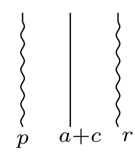
Since $r' \geq r \geq i$ and $a + c \geq b = p + r \geq p + i$, all terms on the left-hand side must be equal to zero, i.e., we must have $r' = r = i$, and so $f_1 = f_2 = f_3 = 0$, and $\varphi = 0$. Moreover, since $r' = r$, we also have $\gamma = 0$, and so by Lemma 2 we have that $\alpha' = \alpha$ and $s_\gamma = (-1)^{|\alpha|}$. By replacing all this in (32), it becomes

$$\delta_{r,r'}\delta_{\alpha,\alpha'}(-1)^{|\alpha|} \sum_{w \in P(r,x)} \sum_{y \in P(a+c-r,r)}$$



Again, by Lemma 2, the last is non-zero only when $w = 0$ and $y = K_{a+c-r,r}$ and thus (32) reduces to

$$\delta_{r,r'}\delta_{\alpha,\alpha'}(-1)^{|\alpha|}(-1)^{r(a+c-r)}$$



as desired. □

Proof. (of Theorem 3): Lemma 4 gives a collection of mutually orthogonal idempotents

$$e^p_\alpha : \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)} \longrightarrow \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$$

for every integer p with $\max(0, b - a) \leq p \leq \min(b, c)$ and partition $\alpha \in P(c - p, a - r)$, which are projections onto direct summands isomorphic to $\mathcal{E}_2^{(p)}\mathcal{E}_1^{(a+c)}\mathcal{E}_2^{(r)}\{2|\alpha| - (c - p)(a - r)\}$.

In [3] it was shown that when the base field is $\mathbb{Q}$ then $\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}$ is isomorphic to the direct sum of these summands, over all values of p and α. Hence, the identity decomposition

$$\sum_{p,\alpha} e_\alpha^p = \mathrm{Id}_{\mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(c)}}$$

holds over $\mathbb{Q}$, and consequently, over $\mathbb{Z}$, since all coefficients in the decomposition are integers. This gives the desired decomposition (26). □

5. A categorification of the higher quantum Serre relations

In this section, we give a direct categorification of the higher quantum Serre relations for type A. This will be done in the homotopy category of $\dot{\mathcal{U}}_3^+$. The higher quantum Serre relations

$$\sum_{i=0}^{a}(-1)^i q^{+(a-b-1)i}\ E_1^{(a-i)}E_2^{(b)}E_1^{(i)} = 0, \quad a > b > 0, \tag{36}$$

$$\sum_{i=0}^{a}(-1)^i q^{-(a-b-1)i}\ E_1^{(a-i)}E_2^{(b)}E_1^{(i)} = 0, \quad a > b > 0, \tag{37}$$

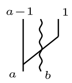
state that certain alternating sum of monomials in $U_q^+(\mathfrak{sl}_3)$ is equal to zero. Thus, it is natural to categorify it by building a complex of objects of $\dot{\mathcal{U}}_3^+$ lifting those monomials, which is homotopic to zero.

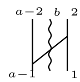
For a categorification of the relation (36) our goal is to define a complex of the form

$$\begin{gathered}0 \to \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)} \to \mathcal{E}_1^{(a-1)}\mathcal{E}_2^{(b)}\mathcal{E}_1\{a-b-1\} \to \mathcal{E}_1^{(a-2)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(2)}\{2(a-b-1)\} \to \cdots \\ \cdots \to \mathcal{E}_1\mathcal{E}_2^{(b)}\mathcal{E}_1^{(a-1)}\{(a-1)(a-b-1)\} \to \mathcal{E}_2^{(b)}\mathcal{E}_1^{(a)}\{a(a-b-1)\} \to 0,\end{gathered} \tag{38}$$

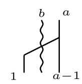
that is homotopic to zero.

Theorem 6. *The following complex*

$$0 \longrightarrow \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)} \xrightarrow{\quad}$$


$$\mathcal{E}_1^{(a-1)}\mathcal{E}_2^{(b)}\mathcal{E}_1\{a-b-1\} \xrightarrow{\quad}$$


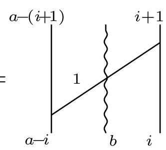
$$\mathcal{E}_1^{(a-2)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(2)}\{2(a-b-1)\} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{E}_1\mathcal{E}_2^{(b)}\mathcal{E}_1^{(a-1)}\{(a-1)(a-b-1)\} \xrightarrow{\quad}$$


$$\mathcal{E}_2^{(b)}\mathcal{E}_1^{(a)}\{a(a-b-1)\} \longrightarrow 0$$

is homotopic to zero.

Proof. Denote $C_i := \mathcal{E}_1^{(a-i)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(i)}\{i(a-b-1)\}$, and

$$d_i :=$$


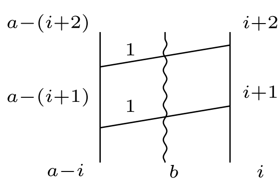
$$: \quad C_i \longrightarrow C_{i+1}, \quad i = 0, \dots, a-1.$$

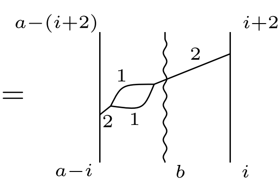
Thus, we have to show that the complex

$$\mathcal{C}: \quad 0 \to C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{a-1}} C_a \to 0$$

is homotopic to zero.

First of all, $\mathcal{C}$ is indeed a complex, since

$$d_{i+1}d_i =$$


$$=$$


$$= 0.$$

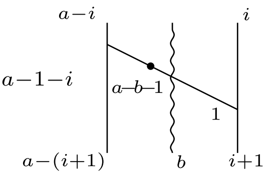
In order to prove that $\mathcal{C}$ is homotopic to zero, we are left with defining morphisms $h_i : C_{i+1} \to C_i$, for $i = 0, \dots, a-1$, such that

$$h_i d_i + d_{i-1}h_{i-1} = \mathrm{Id}_{C_i}, \qquad i = 1, \dots, a-1, \tag{39}$$

$$h_0 d_0 = \mathrm{Id}_{C_0}, \tag{40}$$

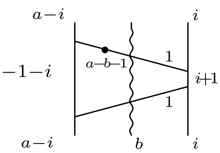
$$d_{a-1}h_{a-1} = \mathrm{Id}_{C_a}. \tag{41}$$

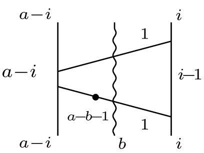
To that end, for every $i = 0, \ldots, a-1$, we define $h_i : C_{i+1} \to C_i$, as follows:

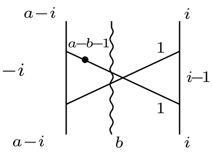
$$h_i := (-1)^{a-1-i}$$ 

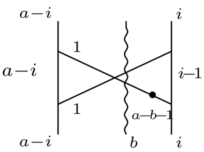
.

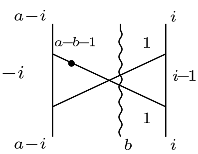
Then, for every $i = 1, \ldots, a-1$, we have

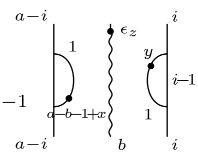
$$h_i d_i + d_{i-1} h_{i-1} = (-1)^{a-1-i}$$ 

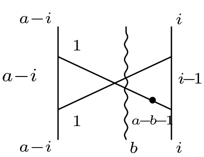
$+(-1)^{a-i}$ 

$= (-1)^{a-1-i}$ 

$+(-1)^{a-i}$ 

$= \quad (-1)^{a-1-i}$ 

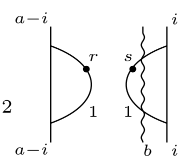
$+(-1)^{a-1-i} \sum_{x+y+z=b-1}$ 

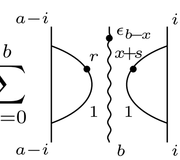
$+(-1)^{a-i}$ 

.

(42)

Here in the second equality we have used opening of a thick edge, while in the third equality we have applied the thick R3 move to the first term. By dot migration, the sum of the first and the third summand from above is equal to

$(-1)^{a-1-i} \sum_{r+s=a-b-2}$ 

$= (-1)^{a-1-i} \sum_{r+s=a-b-2} \sum_{x=0}^{b}$ 

(43)

where we have used the thick R2 move.

The last double sum is non-zero only when $r \geq a-i-1$ and $x+s \geq i-1$. Since $x \leq b$ and $r+s=a-b-2$, the inequalities must be equalities, i.e., $r=a-i-1$, $x=b$ and $s=i-b-1$. This is possible only when $i \geq b+1$, and so (43) is equal to $\delta_{\{i\geq b+1\}} \operatorname{Id}_{C_i}$.

Analogously, the remaining second term from (42) is non-zero, only when $a-b-1+x \geq a-i-1$, $y \geq i-1$ and $z \geq 0$. Since $x+y+z=b-1$, again we must have all equalities. Therefore, the second term is equal to Id_{C_i}. Since $x \geq 0$, we must have $i \leq b$, and so altogether we have

$$h_i d_i + d_{i-1}h_{i-1} = \delta_{\{i\geq b+1\}} \operatorname{Id}_{C_i} + \delta_{\{i \leq b\}} \operatorname{Id}_{C_i} = \operatorname{Id}_{C_i},$$

thus proving (39). The equalities (40) and (41) can be obtained completely analogously. Hence, $\mathcal{C}$ is homotopic to zero, as desired. □

As for the categorification of the higher quantum Serre relations (37), we obtain it by defining a complex of the form (38), but with the arrows pointing in the opposite direction. By exchanging the roles of d_i's and h_i's in the theorem above, we have:

Theorem 7. *The following complex*

$$0 \leftarrow \mathcal{E}_1^{(a)}\mathcal{E}_2^{(b)} \xleftarrow{h_1} \mathcal{E}_1^{(a-1)}\mathcal{E}_2^{(b)}\mathcal{E}_1\{-(a-b-1)\} \xleftarrow{h_2}$$
$$\mathcal{E}_1^{(a-2)}\mathcal{E}_2^{(b)}\mathcal{E}_1^{(2)}\{-2(a-b-1)\} \xleftarrow{h_3} \cdots \xleftarrow{h_{a-1}} \mathcal{E}_2^{(b)}\mathcal{E}_1^{(a)}\{-a(a-b-1)\} \leftarrow 0$$

is homotopic to zero.

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References

[1] J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), 1883–1942.

[2] W. Fulton, *Young Tableaux: with applications to representation theory and geometry*, Cambridge University Press, Cambridge, 1997.

[3] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups I*, Represent. Theory **13** (2009), 309–347.

[4] ______, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), 2685–2700.

[5] ______, *A diagrammatic approach to categorification of quantum groups III*, Quantum Topology, **1**(1) (2010), 1–92.

[6] M. Khovanov, A. Lauda, M. Mackaay and M. Stošić: *Extended graphical calculus for categorified quantum* $sl(2)$, Memoirs of the AMS, **219** (1029) (2012), 1–87.

[7] A. Lauda, *A categorification of quantum* $sl(2)$, Adv. Math., **225** (6) (2010), 3327–3424.

[8] G. Lusztig, *Introduction to quantum groups*, in 'Progress in mathematics', **110**, Birkhäuser, Boston, 1993.

[9] ______, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.

[10] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.

[11] M. Stošić, *Extended graphical calculus for categorified quantum* $sl(n)$, in preparation.

[12] M. Varagnolo and E. Vasserot, *Canonical basis and KLR-algebras*, J. Reine Angew. Math. **659** (2011), 67–100.

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