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On the Yoneda algebras of piecewise-Koszul algebras

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Let $A = \bigoplus_{i\geq 0} A_i$ be a piecewise-Koszul algebra with cohomology degree function δ_p^d such that $d > p \geqslant 2$ and $E(A) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ its Yoneda algebra. We introduce a new grading on $E(A)$:

$$\begin{aligned}\widehat{E}(A) &= \bigoplus_{i\geq 0} \widehat{E}^{\,i}(A) \text{ with } \widehat{E}^{\,i}(A)\\ &= \begin{cases}\mathrm{Ext}_A^0(A_0, A_0), & i = 0;\\ (\mathrm{Ext}_A^1(A_0, A_0) \oplus \mathrm{Ext}_A^p(A_0, A_0))^i, & i \geq 1.\end{cases}\end{aligned}$$

We use "$\widehat{E}(A)$" to replace "$E(A)$" to suggest the new grading. In the paper, we mainly prove that $\widehat{E}(A)$ is a quadratic algebra and $\widehat{E}(M)$ is a quadratic module over $\widehat{E}(A)$, where M is a piecewise-Koszul A-module with the same function δ_p^d. Moveover, we provide a concrete example to show that $\widehat{E}(A)$ is not a Koszul algebra in general, which is different from the Koszul and d-Koszul cases.

1. Introduction

Koszul algebra, a class of quadratic algebras possessing many nice homological properties, was originally introduced by Priddy in 1970 (cf. [15]). We refer to [3, 5, 14] for more details on the history, properties, and applications of Koszul algebras. The most fascinating results for such algebras may be:

- Let A be a Koszul algebra and $E(A)$ its Yoneda algebra. Then $E(A)$ is a Koszul algebra, and $A \cong E(E(A))$ as graded algebras (cf. [15]).

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Note that if A is a Koszul algebra, then its Yoneda algebra $E(A)$ can be generated in ext-degrees 0 and 1. And moreover $E(A)$ is Koszul and of course quadratic.

Motivated by the classification of the cubic Artin–Schelter regular algebras (cf. [1]), Berger first generalized Koszul algebras to higher homogeneous graded algebras and introduced the notion of *non-quadratic Koszul algebra* in 2001 (cf. [4]). These algebras came to be called d-Koszul algebras later (cf. [2, 7, 8], etc.), where $d \geq 2$ a fixed integer. Inspired by quiver theory, Green et al. extended this class of algebras to the non-local case in 2004 (cf. [7]). It should be noted that d-Koszul algebras share some properties with Koszul algebras (for example, the trivial module of a d-Koszul algebra has a pure resolution and it can be also characterized by its Yoneda algebra), but this class of algebras does have some essential differences from Koszul algebras (for example, the A_∞-structures on their Yoneda algebras are different (cf. [8])). Green et al. proved the following result:

- Let A be a d-Koszul algebra with $d \geq 3$ and $E(A)$ its Yoneda algebra. Regrade $E(A)$ by

$$\begin{aligned}\overline{E}(A) &= \bigoplus_{i\geq 0} \overline{E}^{\,i}(A) \text{ with } \overline{E}^{\,i}(A)\\ &= \begin{cases} \mathrm{Ext}_A^0(A_0, A_0), & i = 0;\\ \mathrm{Ext}_A^{2i-1}(A_0, A_0) \oplus \mathrm{Ext}_A^{2i}(A_0, A_0), & i \geq 1.\end{cases}\end{aligned}$$

  Then $\overline{E}(A)$ is a Koszul algebra (cf. [7]).

Note that if A is a d-Koszul algebra with $d \geq 3$, then its Yoneda algebra is generated in ext-degrees 0, 1, and 2 (cf. [7]). Thus, if we put

$$\begin{aligned}\widehat{E}(A) &= \bigoplus_{i\geq 0} \widehat{E}^{\,i}(A) \text{ with } \widehat{E}^{\,i}(A)\\ &= \begin{cases} \mathrm{Ext}_A^0(A_0, A_0), & i = 0;\\ (\mathrm{Ext}_A^{1}(A_0, A_0) \oplus \mathrm{Ext}_A^{2}(A_0, A_0))^i, & i \geq 1,\end{cases}\end{aligned}$$

then $\overline{E}(A) = \widehat{E}(A)$ is Koszul and quadratic.

In order to unify the notions of Koszul and d-Koszul algebras, Lu et al. defined the so-called *piecewise-Koszul algebra* in 2007 (cf. [11]). This new class of algebras includes Koszul and d-Koszul algebras, as well as many non-Koszul quadratic algebras (cf. [11, 13]). In particular, piecewise-Koszul algebras provide a negative answer to a question posed by Green and Marcos in [6] in 2005 (cf. [9, 10, 12, 13]).

From [11], we know that the Yoneda algebra of a piecewise-Koszul algebra with period p can be generated in ext-degrees 0, 1, and p. Therefore, motivated by the Koszul and d-Koszul algebras, one may ask the following natural question:

- Is the Yoneda algebra of a piecewise-Koszul algebra Koszul under a suitable new grading?

It is a pity that the answer is negative in general, which is easy to see from the counterexample in the last section. However, we can obtain the following result, which is the main result of the paper. We will give a detailed proof only for the case $p = 3$ since all other ones can be proved similarly.

Main result. Let $A = \bigoplus_{i\geq 0} A_i$ be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module with the same cohomology degree function δ_p^d ($d > p \geqslant 2$). Let $E(A) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ be the Yoneda algebra of A and $E(M) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ the Ext module of M. Put

$$\begin{aligned}\widehat{E}(A) &= \bigoplus_{i\geq 0} \widehat{E}^{\,i}(A) \text{ with } \widehat{E}^{\,i}(A) \\ &= \begin{cases} \mathrm{Ext}_A^0(A_0, A_0), & i = 0; \\ (\mathrm{Ext}_A^1(A_0, A_0) \oplus \mathrm{Ext}_A^p(A_0, A_0))^i, & i \geq 1 \end{cases}\end{aligned}$$

and

$$\widehat{E}(M) = \bigoplus_{i\geq 0} \widehat{E}^{\,i}(M) \text{ with } \widehat{E}^{\,i}(M) = \widehat{E}^{\,i}(A) \cdot \mathrm{Ext}_A^0(M, A_0).$$

Then, we have the following statements:

1) $\widehat{E}(A)$ is a quadratic algebra and $\widehat{E}(M)$ is a quadratic module over $\widehat{E}(A)$,

2) $\widehat{E}(A)$ has the degree distribution

$$\{\Delta(n) : n \leq \Delta(n) \leq \delta(n)\}$$

where

$$\delta(n) = \begin{cases} n, & \text{if } n \leq t+1, \\ d + (p-1)(n-t-2), & \text{else} \end{cases}$$

and $d = p + t$ ($t > 0$),

3) Proj.dim $_{\widehat{E}(A)}E^0(A) \geq d-p+2$, where "Proj.dim" stands for the projective dimension,

4) The $(d-p+2)$th syzygy Ω^{d-p+2} of $E^0(A)$ over $\widehat{E}(A)$ is minimally generated in hat-degrees $d-p+2$ and d.

2. Preliminaries

Throughout, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ denotes a graded $\Bbbk$-algebra with A_0 a semi-simple artin algebra, $\dim_{\Bbbk} A_i < \infty$ and $A_i \cdot A_j = A_{i+j}$ for all $i, j \geq 0$, where $\Bbbk$ is a base field. The graded Jacobson radical of A is $r = A_1 \oplus A_2 \oplus \cdots$. Assume that the trivial left A-module A_0 has a minimal graded projective resolution

$$\mathcal{P}: \quad \cdots \to P_n \to \cdots \to P_1 \to P_0 \to A_0 \to 0.$$

For a graded left A-module M, we denote the nth *shift* of M is $M[n]$ by $M[n]_i = M_{i-n}$. We say M is supported in χ, if $M_i = 0$ for $i \notin \chi$, where χ is a subset of the set of integers.

Assume that M over A has a minimal graded projective resolution

$$\mathcal{Q}: \quad \cdots \to Q_n \xrightarrow{d_n} \cdots \to Q_1 \to Q_0 \to M \to 0.$$

Denote the nth *syzygy* by $\Omega^n(M) := \ker\, d_{n-1}$ for any $n \geq 1$. We also use Ω_M to stand for the kernel of the projective cover of a graded A-module M.

Under the Yoneda product, $E(A) := \mathrm{Ext}_A^*(A_0, A_0) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ is a bi-graded $\Bbbk$-algebra and $E(M) := \mathrm{Ext}_A^*(M, A_0) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ is a bi-graded $E(A)$-module, and we usually denote $E_j^n(M) = \mathrm{Ext}_A^n(M, A_0)_{-j}$ for simplicity.

If for any $n \geq 0$, $\{\varpi(n)\}$ is defined to be the set of internal degrees in which $E^n(M)$ is supported, then the set $\varpi := \{\{\varpi(0)\}, \{\varpi(1)\}, \{\varpi(2)\}, \ldots\}$ is called the *degree distribution* of M. In particular, the degree distribution of A means that $M = A_0$.

Lemma 2.1 (cf. [7]). *Let A be a graded algebra and M a finitely generated graded left A-module. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are minimal graded projective resolutions of A_0 and M, respectively.*

1) *Let M be supported in $\{j \geq 0\}$. For all $n \geq 1$, if P_n is supported in $\{j | j \geq s\}$, then Q_n is supported in $\{j | j \geq s\}$,*

2) *Assume that P_n and Q_n are both generated in degree s, then*

$$\mathrm{Ext}^n_A(A_0, A_0)\,\mathrm{Hom}_A(M, A_0) = \mathrm{Ext}^n_A(M, A_0).$$

Lemma 2.2. *Assume that $E^i(A)$ is supported in $\{\Delta(i)\}$ for any $i \geq 0$. Assume that $W = W_n \oplus W_{n+1}$ is a graded left A-module which is generated in degree n for some $n \geq 0$. Then $E^i(W)$ is supported in $\{\Delta(i)+n, \Delta(i)+n+1\}$ for $i > 0$ and in degree n if $i = 0$.*

Proof. By the short exact sequence over A

$$0 \to W_{n+1} \to W \to W/W_{n+1} \to 0$$

we get the long exact sequence

$$\cdots \to E^i(W/W_{n+1}) \to E^i(W) \to E^i(W_{n+1}) \to \cdots .$$

We know that both W/W_{n+1} and W_{n+1} are semi-simple modules over A. So $E^i(W/W_{n+1})$ and $E^i(W_{n+1})$ are supported in $\Delta(i)+n$ and $\Delta(i)+n+1$, respectively. We get the result. □

Remark 2.3. If we replace $W = W_n \oplus W_{n+1}$ with $W = W_n \oplus W_{n+1} \oplus \cdots W_{n+p-2} (p > 3)$ which is also generated in degree n for some $n \geq 0$, then $E^i(W)$ is supported in $\{\Delta(i)+n, \Delta(i)+n+1, \ldots, \Delta(i)+n+p-2\}$ for $i > 0$ and in degree n if $i = 0$.

Lemma 2.4. *Assume that there is a short exact sequence $0 \to X \xrightarrow{f} Y \to Z \to 0$ as graded A-modules with $X, Y,$ and Z being all generated in degree s. Then we have the following commutative diagram:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_X & \to & \Omega_Y & \to & \Omega_Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_X & \to & P_Y & \to & P_Z & \to & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \xrightarrow{f} & Y & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $P_X \to X \to 0$, $P_Y \to Y \to 0$, and $P_Z \to Z \to 0$ are all graded projective covers.

Proof. Note that X, Y and Z are generated in the same degree, so $f(rX) = f(X) \cap rY$. Now the desired commutative diagram is obvious. $\square$

We say a graded module M over A has a *linear presentation* if there is an exact sequence over A, $P_1 \to P_0 \to M \to 0$, such that P_i is a graded projective A-module and generated in degree i, where $i = 0, 1$. Denote by $\mathscr{L}(A)$ the category consisting of graded A-modules with linear presentation and degree 0 maps.

Lemma 2.5. *The category $\mathscr{L}(A)$ is closed under extension, i.e., if $0 \to X \to Y \to Z \to 0$ is an exact sequence with $X, Z \in \mathscr{L}(A)$, then $Y \in \mathscr{L}(A)$.*

The double extension of modules with linear presentation also has a linear presentation.

Lemma 2.6. *Assume that $0 \to X \to Y \to Z \to 0$, $0 \to X_1 \to X \to X_2 \to 0$ and $0 \to Z_1 \to Z \to Z_2 \to 0$ are all exact sequences as graded A-modules. If $X_1, X_2, Z_1, Z_2 \in \mathscr{L}(A)$, then $Y \in \mathscr{L}(A)$. Moreover, there is an exact sequence $0 \to \Omega_X \to \Omega_Y \to \Omega_Z \to 0$ with the extensions $0 \to \Omega_{X_1} \to \Omega_X \to \Omega_{X_2} \to 0$, $0 \to \Omega_{Z_1} \to \Omega_Z \to \Omega_{Z_2} \to 0$.*

If Y satisfies the assumption in Lemma 2.5, we denote $Y \in \Gamma_1(A)$. And if Y satisfies the assumption in Lemma 2.6, we denote $Y \in \Gamma_2(A)$, that is, there are exact sequences $0 \to X \to Y \to Z \to 0$ such that $X, Z \in \Gamma_1(A)$. We define $\Gamma_n(A)$ $(n \geqslant 2)$ as follows.

$Y \in \Gamma_n(A)$ means that there is an exact sequence $0 \to X \to Y \to Z \to 0$ such that $X, Z \in \Gamma_{n-1}(A)$.

Lemma 2.7. *As sets, we have*

$$\cdots \subseteq \Gamma_n(A) \subseteq \Gamma_{n-1}(A) \subseteq \cdots \subseteq \Gamma_2(A) \subseteq \Gamma_1(A) \subseteq \mathscr{L}(A).$$

Now, we recall the definition of piecewise-Koszul object.

Given a pair of integers d and p ($d \geq p \geq 2$), we introduce a function $\delta_p^d : \mathbb{N} \to \mathbb{N}$ by

$$\delta_p^d(n) = \begin{cases} \dfrac{nd}{p}, & n \equiv 0 (\operatorname{mod} p), \\ \dfrac{(n-1)d}{p} + 1, & n \equiv 1 (\operatorname{mod} p), \\ \cdots & \cdots \\ \dfrac{(n-p+1)d}{p} + p - 1, & n \equiv p-1 (\operatorname{mod} p). \end{cases}$$

Definition 2.8 (cf. [11]). Let A be a graded algebra and M a finitely generated graded left A-module. Suppose that $\mathcal{Q}$ is a minimal graded projective resolution of M over A. Then M is called a *piecewise-Koszul module* if Q_n is generated in degree $\delta_p^d(n)$ for all $n \geq 0$. In particular, we say that A is a *piecewise-Koszul algebra* provided that A_0 is a piecewise-Koszul module.

Thus, it is easy to see that piecewise-Koszul algebras/modules are Koszul algebras/modules in the case of $p = d$, and are d-Koszul algebras/modules if $p = 2$.

Proposition 2.9 (cf. [11]). Let A be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module both with cohomology degree function δ_p^d ($d > p \geqslant 2$). Then

(1) $E(A)$ is generated in the ext-degrees $0, 1$ and p, especially, for any $n \geq 0$,

$$\begin{aligned} E^{pn+p}(A) &= E^{pn}(A)E^p(A), \\ E^{pn+i}(A) &= E^{pn}(A)\,\underbrace{E^1(A)\cdots E^1(A)}_{i} \\ &= \underbrace{E^1(A)\cdots E^1(A)}_{i}\,E^{pn}(A), 0 < i < p, \end{aligned}$$

(2) $E^{pl+m}(A)E^{pk+n}(A) = 0$ for any $l, k \geq 0$, $0 < m, n < p$ and $m + n = p$,

(3) $E(M)$ is generated in degree 0 as an $E(A)$-module.

From [11], it is easy to obtain.

Proposition 2.10. Let A be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module both with cohomology degree function δ_p^d ($d > p \geqslant 2$). Then rM has the following properties:

(1) $\Omega^{p-1}(rM)[-d]$ is a piecewise-Koszul A-module and then $\bigoplus_{n\geq p-1} E^n(rM)$ is generated by $E^{p-1}(rM)$ as an $E(A)$-module,

(2) for any $l \geq 0$,

$$E^{pl+i}(A)E^0(rM) = \begin{cases} E^{pl+i}(rM), & 0 \leq i < p-1, \\ 0, & i = p-1, \end{cases}$$

(3) for any $n \geq 0$, there is a short exact sequence

$$0 \to E^{n-1}(rM) \to E^n(M/rM) \to E^n(M) \to 0$$

with $E^{-1}(rM) = 0$.

3. Hat-degree

Let $A = \bigoplus_{i\geq 0} A_i$ be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module with the same cohomology degree function δ_p^d ($d > p \geqslant 2$). Let $E(A) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(A_0, A_0)$ be the Yoneda algebra of A and $E(M) = \bigoplus_{i\geq 0} \mathrm{Ext}_A^i(M, A_0)$ the Ext module of M. We now give a new grading on $E(A)$ and $E(M)$ as follows:

$$\begin{aligned}\widehat{E}(A) &= \bigoplus_{i\geq 0} \widehat{E}^{\,i}(A) \text{ with } \widehat{E}^{\,i}(A) \\ &= \begin{cases} \mathrm{Ext}_A^0(A_0, A_0), & i = 0; \\ (\mathrm{Ext}_A^1(A_0, A_0) \oplus \mathrm{Ext}_A^p(A_0, A_0))^i, & i \geq 1 \end{cases}\end{aligned}$$

and

$$\widehat{E}(M) = \bigoplus_{i\geq 0} \widehat{E}^{\,i}(M) \text{ with } \widehat{E}^{\,i}(M) = \widehat{E}^{\,i}(A) \cdot \mathrm{Ext}_A^0(M, A_0).$$

The new degree is called *hat-degree*. Note that if we neglect the grading on $\widehat{E}(A)$ and $\widehat{E}(M)$, as vector spaces it is obvious that $\widehat{E}(A) = E(A)$ and $\widehat{E}(M) = E(M)$.

Clearly, $\widehat{E}(A)$ is a graded algebra generated in hat-degrees $0, 1$ and $\widehat{E}(M)$ is generated in hat-degree 0 as a graded $\widehat{E}(A)$-module.

Now for simplicity, we concentrate on the piecewise-Koszul algebra A with cohomology degree function δ_3^d ($d > 3$). We know that for $\widehat{E}(A)$, $\widehat{E}^{\,0}(A) = E^0(A)$ and

$$\widehat{E}^{\,i}(A) = (E^1(A) \oplus E^3(A))^i, i \geq 1.$$

We write the first several terms:

$$\begin{aligned}
&\widehat{E}^{\,0}(A) = E^0(A)\\
&\widehat{E}^{\,1}(A) = E^1(A) \oplus E^3(A)\\
&\widehat{E}^{\,2}(A) = E^2(A) \oplus E^4(A) \oplus E^6(A)\\
&\widehat{E}^{\,3}(A) = E^5(A) \oplus E^7(A) \oplus E^9(A)\\
&\qquad\qquad\vdots
\end{aligned}$$

Conclude that for $n \geq 2$,

$$\begin{aligned}
\widehat{E}^{\,n}(A) &= (E^1(A) \oplus E^3(A))^n\\
&= E^{3n-4}(A) \oplus E^{3n-2}(A) \oplus E^{3n}(A)\\
&= E^{3(n-2)+2}(A) \oplus E^{3(n-1)+1}(A) \oplus E^{3n}(A).
\end{aligned}$$

We denote $\widehat{E}(N[n])[m]$ where the internal shift n is as a graded A-module such that $N[n]$ is a piecewise-Koszul A-module and the external shift m is in the hat-grading.

Proposition 3.1. *Let A be a piecewise-Koszul algebra with cohomology degree function δ_3^d ($d > 3$). Then $E^0(A) \in \mathscr{L}(\widehat{E}(A))$ and the first syzygy $\Omega_{E^0(A)}$ over $\widehat{E}(A)$ can be seen as an extension of all terms generated in hat-degree 1,*

$$0 \to \widehat{E}(K)[1] \to \Omega_{E^0(A)} \to G \to 0,$$

where

1) $G \cong E^1(A) \oplus E^2(A) = E^0(r) \oplus E^1(r)$ *as vector spaces,*
2) $K = \Omega^3(A_0)[-d]$ *is a piecewise-Koszul left A-module.*

Proof. The first term in the minimal resolution of $E^0(A)$ over $\widehat{E}(A)$ is

$$
\begin{array}{ccccccccc}
 & & 0 & \to & E^0(A) & \to & E^0(A) & \to & 0 \\
0 & \to & E^1(A) & \to & E^1(A) & \to & 0 & & \\
0 & \to & E^2(A) & \to & E^2(A) & \to & 0 & & \\
0 & \to & E^3(A) & \to & E^3(A) & \to & 0 & & \\
0 & \to & E^4(A) & \to & E^4(A) & \to & 0 & & \\
0 & \to & E^5(A) & \to & E^5(A) & \to & 0 & & \\
0 & \to & E^6(A) & \to & E^6(A) & \to & 0 & & \\
0 & \to & E^7(A) & \to & E^7(A) & \to & 0 & & \\
0 & \to & E^8(A) & \to & E^8(A) & \to & 0 & & \\
0 & \to & E^9(A) & \to & E^9(A) & \to & 0 & & \\
 & & & \vdots & & & & &
\end{array}
$$

Changing the order, we get

$$
\begin{array}{ccccccccc}
 & & 0 & \to & E^0(A) & \to & E^0(A) & \to & 0 \\
0 & \to & E^1(A) & \to & E^1(A) & \to & 0 & & \\
0 & \to & E^3(A) & \to & E^3(A) & \to & 0 & & \\
0 & \to & E^2(A) & \to & E^2(A) & \to & 0 & & \\
0 & \to & E^4(A) & \to & E^4(A) & \to & 0 & & \\
0 & \to & E^6(A) & \to & E^6(A) & \to & 0 & & \\
0 & \to & E^5(A) & \to & E^5(A) & \to & 0 & & \\
0 & \to & E^7(A) & \to & E^7(A) & \to & 0 & & \\
0 & \to & E^9(A) & \to & E^9(A) & \to & 0 & & \\
0 & \to & E^8(A) & \to & E^8(A) & \to & 0 & & \\
 & & & \vdots & & & & &
\end{array}
$$

For the projective cover of $E^0(A)$ over $\widehat{E}(A)$, we have the syzygy

$$
\begin{aligned}
\Omega_{E^0(A)} = & \underbrace{E^1(A) \oplus E^3(A)}_{1} \oplus \underbrace{E^2(A) \oplus E^4(A) \oplus E^6(A)}_{2} \\
& \oplus \underbrace{E^5(A) \oplus E^7(A) \oplus E^9(A)}_{3} \oplus \cdots ,
\end{aligned}
$$

which is generated by $E^1(A) \oplus E^3(A)$. Thus, $\Omega_{E^0(A)}$ is generated in hat-degree 1.

The syzygy $\Omega_{E^0(A)}$ can be seen as a short exact sequence of $\widehat{E}(A)$-modules all generated in hat-degree 1 as

$$0 \to \widehat{E}(K)[1] \to \Omega_{E^0(A)} \to G \to 0,$$

where $G \cong E^1(A) \oplus E^2(A) = E^0(r) \oplus E^1(r)$, $K = \Omega^3(A_0)[-d]$ is a piecewise-Koszul A-module and

$$\begin{aligned}\widehat{E}(K)[1] &= \underbrace{E^3(A)}_{1} \oplus \underbrace{E^4(A) \oplus E^6(A)}_{2} \oplus \underbrace{E^5(A) \oplus E^7(A) \oplus E^9(A)}_{3} \oplus \cdots \\ &= \underbrace{E^0(\Omega^3(A_0))}_{1} \oplus \underbrace{E^1(\Omega^3(A_0)) \oplus E^3(\Omega^3(A_0))}_{2} \\ &\oplus \underbrace{E^2(\Omega^3(A_0)) \oplus E^4(\Omega^3(A_0)) \oplus E^6(\Omega^3(A_0))}_{3} \oplus \cdots .\end{aligned}$$

We finishes the proof. □

Proposition 3.2. *Let A be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module both with cohomology degree function δ_3^d ($d > 3$). Then $\widehat{E}(M) \in \mathscr{L}(\widehat{E}(A))$ and the first syzygy $\Omega_{\widehat{E}(M)}$ over $\widehat{E}(A)$ can be viewed as an extension of all terms generated in hat-degree 1,*

$$0 \to \widehat{E}(T)[1] \to \Omega_{\widehat{E}(M)} \to R \to 0,$$

where

1) *$R \cong E^0(rM) \oplus E^1(rM)$ as vector spaces,*
2) *$T = \Omega^2(rM)[-d]$ is a piecewise-Koszul left A-module.*

Proof. By Proposition 2.10, we characterize the first term in the minimal resolution of $\widehat{E}(M)$ over $\widehat{E}(A)$

$$\begin{array}{ccccccccc} & & 0 & \to & E^0(M/rM) & \to & E^0(M) & \to & 0 \\ 0 & \to & E^0(rM) & \to & E^1(M/rM) & \to & E^1(M) & \to & 0 \\ 0 & \to & E^1(rM) & \to & E^2(M/rM) & \to & E^2(M) & \to & 0 \\ 0 & \to & E^2(rM) & \to & E^3(M/rM) & \to & E^3(M) & \to & 0 \end{array}$$

$$\begin{array}{ccccccccc}
0 & \to & E^3(rM) & \to & E^4(M/rM) & \to & E^4(M) & \to & 0 \\
0 & \to & E^4(rM) & \to & E^5(M/rM) & \to & E^5(M) & \to & 0 \\
0 & \to & E^5(rM) & \to & E^6(M/rM) & \to & E^6(M) & \to & 0 \\
0 & \to & E^6(rM) & \to & E^7(M/rM) & \to & E^7(M) & \to & 0 \\
0 & \to & E^7(rM) & \to & E^8(M/rM) & \to & E^8(M) & \to & 0 \\
0 & \to & E^8(rM) & \to & E^9(M/rM) & \to & E^9(M) & \to & 0 \\
 & & & & & \vdots & & &
\end{array}$$

Changing the order, we get

$$\begin{array}{ccccccccc}
 & & 0 & \to & E^0(M/rM) & \to & E^0(M) & \to & 0 \\
0 & \to & E^0(rM) & \to & E^1(M/rM) & \to & E^1(M) & \to & 0 \\
0 & \to & E^2(rM) & \to & E^3(M/rM) & \to & E^3(M) & \to & 0 \\
0 & \to & E^1(rM) & \to & E^2(M/rM) & \to & E^2(M) & \to & 0 \\
0 & \to & E^3(rM) & \to & E^4(M/rM) & \to & E^4(M) & \to & 0 \\
0 & \to & E^5(rM) & \to & E^6(M/rM) & \to & E^6(M) & \to & 0 \\
0 & \to & E^4(rM) & \to & E^5(M/rM) & \to & E^5(M) & \to & 0 \\
0 & \to & E^6(rM) & \to & E^7(M/rM) & \to & E^7(M) & \to & 0 \\
0 & \to & E^8(rM) & \to & E^9(M/rM) & \to & E^9(M) & \to & 0 \\
0 & \to & E^7(rM) & \to & E^8(M/rM) & \to & E^8(M) & \to & 0 \\
 & & & & & \vdots & & &
\end{array}$$

For the projective cover of $\widehat{E}(M)$ over $\widehat{E}(A)$, we have the syzygy

$$\begin{aligned}
\Omega_{\widehat{E}(M)} = {} & \underbrace{E^0(rM) \oplus E^2(rM)}_{1} \oplus \underbrace{E^1(rM) \oplus E^3(rM) \oplus E^5(rM)}_{2} \\
& \oplus \underbrace{E^4(rM) \oplus E^6(rM) \oplus E^8(rM)}_{3} \oplus \cdots,
\end{aligned}$$

where $E^n(A)E^2(rM) = E^{n+2}(rM)$ for any $n \geq 0$ and $E^1(A)E^0(rM) = E^1(rM)$ by Proposition 2.10. Thus, $\Omega_{\widehat{E}(M)}$ is generated in hat-degree 1.

The syzygy $\Omega_{\widehat{E}(M)}$ can be seen as a short exact sequence of $\widehat{E}(A)$-modules all generated in hat-degree 1 as

$$0 \to \widehat{E}(T)[1] \to \Omega_{\widehat{E}(M)} \to R \to 0,$$

where $R \cong E^0(rM) \oplus E^1(rM)$, $T = \Omega^2(rM)[-d]$ is a piecewise-Koszul A-module and

$$\begin{aligned}\widehat{E}(T)[1] &= \underbrace{E^2(rM)}_{1} \oplus \underbrace{E^3(rM) \oplus E^5(rM)}_{2} \oplus \underbrace{E^4(rM) \oplus E^6(rM) \oplus E^8(rM)}_{3} \oplus \cdots \\ &= \underbrace{E^0(\Omega^2(rM))}_{1} \oplus \underbrace{E^1(\Omega^2(rM)) \oplus E^3(\Omega^2(rM))}_{2} \\ &\oplus \underbrace{E^2(\Omega^2(rM)) \oplus E^4(\Omega^2(rM)) \oplus E^6(\Omega^2(rM))}_{3} \oplus \cdots .\end{aligned}$$

We get the result. $\square$

Theorem 3.3. *Let A be a piecewise-Koszul algebra with cohomology degree function δ_3^d $(d > 3)$. Then $\widehat{E}(A)$ has the degree distribution*

$$\{\Delta(n) : n \leq \Delta(n) \leq \delta(n)\}$$

where

$$\delta(n) = \begin{cases} 0, & n = 0, \\ 2n - 1, & n \geq 1. \end{cases}$$

Proof. By Proposition 3.1, the projective cover and the first syzygy of $E^0(A)$ over $\widehat{E}(A)$ are

$$0 \to \Omega^1 \to \widehat{E}(A) \to E^0(A) \to 0.$$

The syzygy Ω^1 can be seen as an extension of $\widehat{E}(A)$-modules all generated in hat-degree 1

$$0 \to L^1 \to \Omega^1 \to G^1 \to 0,$$

where $G^1 = G^1_1 \oplus G^1_2 \cong E^1(A) \oplus E^2(A)$ and $L^1 = \widehat{E}(K^1)[1]$ with $K^1 = \Omega^3(A_0)[-d]$ is a piecewise-Koszul A-module.

Since $K^1 = \Omega^3(A_0)[-d]$ is a piecewise-Koszul A-module, by Proposition 3.2, the first syzygy $\Omega^1(L^1)$ of $L^1 = \widehat{E}(K^1)[1]$ over $\widehat{E}(A)$ can be seen as an extension of $\widehat{E}(A)$-modules all generated in hat-degree 2

$$0 \to L^2 \to \Omega^1(L^1) \to G^2 \to 0,$$

where $G^2 = G^2_2 \oplus G^2_3 \cong E^0(rK^1) \oplus E^1(rK^1)$ and $L^2 = \widehat{E}(K^2)[2]$ with $K^2 = \Omega^2(rK^1)[-d]$ is a piecewise-Koszul A-module.

We can get the following diagram as a part of the minimal resolution of $E^0(A)$ over $\widehat{E}(A)$

$$
\begin{tikzcd}
 & \cdots \arrow[d] & & & & & \\
0 \arrow[r] & L^3 \arrow[r] & \Omega^1(L^2) \arrow[r] \arrow[d] & G^3 \arrow[r] & 0 & & \\
 & & \widehat{E}(K^2/rK^2)[2] \arrow[d] & & & & \\
 & 0 \arrow[r] & L^2 \arrow[r] & \Omega^1(L^1) \arrow[r] \arrow[d] & G^2 \arrow[r] & 0 & \\
 & & & \widehat{E}(K^1/rK^1)[1] \arrow[d] & & & \\
 & & 0 \arrow[r] & L^1 \arrow[r] & \Omega^1 \arrow[r] \arrow[d] & G^1 \arrow[r] & 0, \\
 & & & & \widehat{E}(A) \arrow[d] & & \\
 & & & & E^0(A) & &
\end{tikzcd}
$$

where for any $n \geq 1$, $G^n = G_n^n \oplus G_{n+1}^n \cong E^0(rK^{n-1}) \oplus E^1(rK^{n-1})$, $L^n = \widehat{E}(K^n)[n]$, and $K^n = \Omega^2(rK^{n-1})[-d]$ is a piecewise-Koszul A-module. Set $K^0 = A$.

There are short exact sequences over $\widehat{E}(A)$

$$
\begin{gathered}
0 \to L^1 \to \Omega^1 \to G^1 \to 0 \\
0 \to L^2 \to \Omega^1(L^1) \to G^2 \to 0 \\
0 \to L^3 \to \Omega^1(L^2) \to G^3 \to 0 \\
\vdots
\end{gathered}
$$

Applying $\mathcal{E}(-) := \mathrm{Ext}^*_{\widehat{E}(A)}(-, E^0(A))$, we get a series of long exact sequences over $\widehat{E}(A)$

$$
\begin{gathered}
\cdots \to \mathcal{E}^n(G^1) \to \mathcal{E}^n(\Omega^1) \to \mathcal{E}^n(L^1) \to \cdots \\
\cdots \to \mathcal{E}^n(G^2) \to \mathcal{E}^n(\Omega^1(L^1)) \to \mathcal{E}^n(L^2) \to \cdots \\
\cdots \to \mathcal{E}^n(G^3) \to \mathcal{E}^n(\Omega^1(L^2)) \to \mathcal{E}^n(L^3) \to \cdots \\
\vdots
\end{gathered}
$$

Therefore, $\mathrm{Ext}^n_{\widehat{E}(A)}(E^0(A), E^0(A)) = \mathcal{E}^{n-1}(\Omega^1)$ is supported in $\Delta(n)$, which is the internal degree of

$$\mathcal{E}^{n-1}(G^1), \mathcal{E}^{n-2}(G^2), \mathcal{E}^{n-3}(G^3), \ldots, \mathcal{E}^1(G^{n-1}), \mathcal{E}^0(G^n), \mathcal{E}^0(L^n)$$

By Lemma 2.2, we know that $\Delta(n) = \{\Delta(n-1)+1, \Delta(n-1)+2, \Delta(n-2)+2, \Delta(n-2)+3, \ldots, \Delta(1)+n-1, \Delta(1)+n, n\}$. Thus the maximal degree in $\Delta(n)$ is $\max\{\Delta(n-1)\}+2$.

$\square$

We end this section with the general version of Theorem 3.3 without proof since it is similar to the case of $p = 3$.

Theorem 3.4. *Let A be a piecewise-Koszul algebra with cohomology degree function δ_p^d ($d > p \geqslant 2$). Then $\widehat{E}(A)$ has the degree distribution*

$$\{\Delta(n) : n \leq \Delta(n) \leq \delta(n)\},$$

where

$$\delta(n) = \begin{cases} 0, & n = 0, \\ p + (n-2)(p-1), & n \geq 1. \end{cases}$$

Remark 3.5. We know that if $p = 2$, then A is a d-Koszul algebra and $\widehat{E}(A)$ is a Koszul algebra.

4. The main result

Let A be a piecewise-Koszul algebra and M a piecewise-Koszul left A-module both with cohomology degree function δ_3^d ($d > 3$). Consider the short exact sequence

$$0 \to r^{k+1}M \to r^kM \to r^kM/r^{k+1}M \to 0$$

for any $k \geq 1$. Applying $\mathrm{Ext}^*_A(-, A_0)$, we obtain a long exact sequence

$$\begin{aligned} 0 &\to E^0(r^kM/r^{k+1}M) \to E^0(r^kM) \to E^0(r^{k+1}M) \\ &\to E^1(r^kM/r^{k+1}M) \to E^1(r^kM) \to E^1(r^{k+1}M) \\ &\to E^2(r^kM/r^{k+1}M) \to E^2(r^kM) \to E^2(r^{k+1}M) \\ &\to E^3(r^kM/r^{k+1}M) \to E^3(r^kM) \to E^3(r^{k+1}M) \\ &\qquad\vdots \end{aligned}$$

as an exact triangle of left $E(A)$-modules.

Since M is a left piecewise-Koszul module over A and $r^kM/r^{k+1}M$ is semisimple, we get that $E^i(r^kM/r^{k+1}M)$ is supported in degree $\delta_3^d(i)+k$ for any $i \geq 0$.

By Proposition 2.10, we get $E^0(rM), E^1(rM), E^2(rM)$ are supported in degrees $1, 2, d$, respectively. By induction, we know that $E^0(r^kM)$ is supported in degree k and $E^1(r^kM)$ is supported in degree $k+1$. So we get $E^1(r^kM) = E^1(A)E^0(r^kM)$ by Lemma 2.1.

The above long exact sequence can be decomposed into the following several exact sequences:

1) $0 \to E^0(r^kM/r^{k+1}M) \to E^0(r^kM) \to 0,$

2) $0 \to E^0(r^{k+1}M) \xrightarrow{\psi} E^1(r^kM/r^{k+1}M) \to E^1(r^kM) \to 0,$

3) $0 \to E^1(r^{k+1}M) \to E^2(r^kM/r^{k+1}M) \xrightarrow{\varphi} E^2(r^kM) \to E^2(r^{k+1}M)$
$\to E^3(r^kM/r^{k+1}M) \to \cdots .$

By induction, we get that $E^2(r^kM)$ is supported in $\{n \mid n \geq k+2\} \cap \{d, d+1, \ldots, d+k-1\}$.

We get the projective cover of $W = E^0(r^kM) \oplus E^1(r^kM)$ as

$$\begin{array}{ccccccccc}
 & & 0 & \to & E^0(r^kM/r^{k+1}M) & \to & E^0(r^kM) & \to & 0\\
0 & \to & E^0(r^{k+1}M) & \xrightarrow{\psi} & E^1(r^kM/r^{k+1}M) & \to & E^1(r^kM) & \to & 0\\
0 & \to & E^2(r^kM/r^{k+1}M) & \to & E^2(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^3(r^kM/r^{k+1}M) & \to & E^3(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^4(r^kM/r^{k+1}M) & \to & E^4(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^5(r^kM/r^{k+1}M) & \to & E^5(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^6(r^kM/r^{k+1}M) & \to & E^6(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^7(r^kM/r^{k+1}M) & \to & E^7(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^8(r^kM/r^{k+1}M) & \to & E^8(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^9(r^kM/r^{k+1}M) & \to & E^9(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
 & & & & & \vdots & & &
\end{array}$$

Changing the order, we get

$$\begin{array}{ccccccccc}
 & & 0 & \to & E^0(r^kM/r^{k+1}M) & \to & E^0(r^kM) & \to & 0\\
0 & \to & E^0(r^{k+1}M) & \to & E^1(r^kM/r^{k+1}M) & \to & E^1(r^kM) & \to & 0\\
0 & \to & E^3(r^kM/r^{k+1}M) & \to & E^3(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^2(r^kM/r^{k+1}M) & \to & E^2(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^4(r^kM/r^{k+1}M) & \to & E^4(r^kM/r^{k+1}M) & \to & 0 & \to & 0\\
0 & \to & E^6(r^kM/r^{k+1}M) & \to & E^6(r^kM/r^{k+1}M) & \to & 0 & \to & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \to & E^5(r^kM/r^{k+1}M) & \to & E^5(r^kM/r^{k+1}M) & \to & 0 & \to & 0 \\
0 & \to & E^7(r^kM/r^{k+1}M) & \to & E^7(r^kM/r^{k+1}M) & \to & 0 & \to & 0 \\
0 & \to & E^9(r^kM/r^{k+1}M) & \to & E^9(r^kM/r^{k+1}M) & \to & 0 & \to & 0 \\
0 & \to & E^8(r^kM/r^{k+1}M) & \to & E^8(r^kM/r^{k+1}M) & \to & 0 & \to & 0 \\
 & & & & & & \vdots & &
\end{array}$$

Viewing W as an $\widehat{E}(A)$-module generated in hat-degree 0, the first syzygy of W is

$$\Omega_W = \underbrace{E^0(r^{k+1}M) \oplus E^3(r^kM/r^{k+1}M)}_{1}$$
$$\oplus \underbrace{E^2(r^kM/r^{k+1}M) \oplus E^4(r^kM/r^{k+1}M) \oplus E^6(r^kM/r^{k+1}M)}_{2} \oplus \cdots,$$

where

$$\bigoplus_{n\geq 3} E^n(r^kM/r^{k+1}M) = \bigoplus_{n\geq 0} E^n(\Omega^3(r^kM/r^{k+1}M))$$

with $\Omega^3(r^kM/r^{k+1}M)[-d-k]$ being a piecewise-Koszul module over A. Here, the action of $\widehat{E}(A)$ on $E^0(r^{k+1}M)$ in Ω_W is defined as $a * f = a \cdot \psi(f)$ with $a \in \widehat{E}(A)$ and $f \in E^0(r^{k+1}M)$.

Lemma 4.1. *The following statements are equivalent:*

1) Ω_W *is generated in hat-degree* 1*,*
2) $E^2(r^kM/r^{k+1}M) \cong E^1(A)E^0(r^{k+1}M)$ *as vector spaces,*
3) $\varphi = 0$.

We know that both $E^1(r^{k+1}M)$ and $E^2(r^kM/r^{k+1}M)$ are supported in $k+2$. Note that $E^2(r^kM)$ is supported in $\{n \mid n \geq k+2\} \cap \{d, d+1, \ldots, d+k-1\}$ and $E^2(r^{k+1}M)$ is supported in $\{n \mid n \geq k+3\} \cap \{d, d+1, \ldots, d+k\}$.

Comparing the internal degrees in the following exact sequence:

$$0 \to E^1(r^{k+1}M) \to E^2(r^kM/r^{k+1}M) \xrightarrow{\varphi} E^2(r^kM) \to E^2(r^{k+1}M) \to \cdots$$

we get $E^2(rM) = E^2_d(rM) \cong E^2_d(r^2M) \cong \cdots \cong E^2_d(r^{d-2}M)$ as vector spaces.

Lemma 4.2. *Assume that* $E^2(rM) \neq 0$*. Let* $k \geq 1$*.*

1) *If* $k < d-1$*, then* $E^2_d(r^kM) \neq 0$*,*

2) *If* $k < d-2$*, then* $\varphi = 0$ *and* Ω_W *is generated in hat-degree* 1*,*

3) *If* $k = d-2$*, then* $\varphi \neq 0$*, so* Ω_W *can not be generated in hat-degree 1, but is generated in hat-degrees* $1, 2$*.*

If $k < d-2$, $W \in \mathscr{L}(\widehat{E}(A))$ and the first syzygy Ω_W of W over $\widehat{E}(A)$ can be seen as an extension of all terms generated in hat-degree 1

$$0 \to \widehat{E}(U)[1] \to \Omega_W \to V \to 0,$$

where

1) $V \cong E^0(r^{k+1}M) \oplus E^1(r^{k+1}M)$ as vector spaces,

2) $U = \Omega^3(r^k M/r^{k+1}M)[-d-k]$ is a piecewise-Koszul A-module.

Proposition 4.3. *Let* A *be a piecewise-Koszul algebra and* M *a piecewise-Koszul module both with cohomology degree function* δ_3^d $(d > 3)$*. Denote* $W = E^0(rM) \oplus E^1(rM)$*. Then* $W \in \mathscr{L}(\widehat{E}(A))$ *and the first syzygy* Ω_W *of* W *over* $\widehat{E}(A)$ *can be seen as an extension of all terms generated in hat-degree 1*

$$0 \to \widehat{E}(U)[1] \to \Omega_W \to V \to 0,$$

where

1) $V \cong E^0(r^2M) \oplus E^1(r^2M)$ *as vector spaces,*

2) $U = \Omega^3(rM/r^2M)[-d-1]$ *is a piecewise-Koszul* A*-module.*

Proof. It is immediate from Lemmas 4.1 and 4.2. □

Theorem 4.4. *Let* A *be a piecewise-Koszul algebra and* M *a piecewise-Koszul module both with cohomology degree function* δ_3^d $(d > 3)$*. Then* $\widehat{E}(A)$ *is a quadratic algebra and* $\widehat{E}(M)$ *is a quadratic module over* $\widehat{E}(A)$*.*

Proof. By Proposition 3.1, the first syzygy $\Omega_{E^0(A)}$ of $E^0(A)$ over $\widehat{E}(A)$ can be seen as an extension of all terms generated in hat-degree 1

$$0 \to \widehat{E}(K)[1] \to \Omega_{E^0(A)} \to G \to 0,$$

where $G \cong E^1(A) \oplus E^2(A) = E^0(r) \oplus E^1(r)$ as vector spaces and $K = \Omega^3$ $(A_0)[-d]$ is a piecewise-Koszul A-module. By Propositions 3.2 and 4.3, we have $\widehat{E}(K), G[-1] \in \mathscr{L}(\widehat{E}(A))$. So $\Omega_{E^0(A)}[-1] \in \Gamma_1(\widehat{E}(A)) \subseteq \mathscr{L}(\widehat{E}(A))$ by Lemma 2.7. Here, the shift -1 is in the hat-grading. Therefore, $\widehat{E}(A)$ is a quadratic algebra. Similarly, we get that $\widehat{E}(M)$ is a quadratic module over $\widehat{E}(A)$. $\square$

Theorem 4.5. *Let A be a non-trivial piecewise-Koszul algebra with δ_3^d $(d > 3)$. Then $\widehat{E}(A)$ has the degree distribution*

$$\{\Delta(n) : n \leq \Delta(n) \leq \delta(n)\},$$

where

$$\delta(n) = \begin{cases} n, & n \leq d-2, \\ 2(n+1)-d, & n > d-2. \end{cases}$$

Moreover, $\mathrm{Proj.dim}_{\widehat{E}(A)} E^0(A) \geq d-1$ *and the* $(d-1)$th *syzygy* Ω^{d-1} *of* $E^0(A)$ *over* $\widehat{E}(A)$ *can not be generated in hat-degree $d-1$, but is generated in hat-degrees $d-1$ and d.*

Proof. Denote the nth syzygy in the minimal projective resolution of $E^0(A)$ over $\widehat{E}(A)$ by Ω^n. We have the minimal projective resolution of $E^0(A)$ over $\widehat{E}(A)$

$$\begin{array}{ccccccccc}
\cdots & & & & & & & & \\
L^{21} & \to & * & \to & G^{21} & L^{22} & \to & * & \to & G^{22} \\
 & & \downarrow & & & & & \downarrow & & \\
 & & \widehat{E}(\Omega^3(A_0)/r\Omega^3(A_0)[-d])[1] & & & & & \widehat{E}(r/r^2[-1])[1] & & \\
 & & \downarrow & & & & & \downarrow & & \\
 & & \widehat{E}(\Omega^3(A_0)[-d])[1] & & \to & \Omega^1 & \to & E^0(r) \oplus E^1(r), & & \\
 & & & & & \downarrow & & & & \\
 & & & & & \widehat{E}(A) & & & & \\
 & & & & & \downarrow & & & & \\
 & & & & & E^0(A) & & & &
\end{array}$$

where

$$\begin{aligned}
L^{21} &= \widehat{E}(\Omega^2(r\Omega^3(A_0))[-2d])[2],\ G^{21} \cong E^0(r\Omega^3(A_0)) \oplus E^1(r\Omega^3(A_0)), \\
L^{22} &= \widehat{E}(\Omega^3(r/r^2)[-d-1])[2],\ \ G^{22} \cong E^0(r^2) \oplus E^1(r^2).
\end{aligned}$$

We can continue the fashion until Ω^{d-3} and obtain that

$$\Omega^1[-1] \in \Gamma_1(\widehat{E}(A)), \Omega^2[-2] \in \Gamma_2(\widehat{E}(A)), \ldots, \Omega^{d-3}[-d+3] \in \Gamma_{d-3}(\widehat{E}(A)).$$

Note that all shifts here are in the hat-grading. Thus, for $0 \leq n \leq d-2$, $\delta(n) = n$.

We know that $\widehat{E}(A)$ has a linear resolution for the first $d-2$ terms and track the construction of the minimal resolution. We consider some pieces in the minimal resolution

$\cdots$

$\cdots$

$$\begin{array}{ccccccccccc}
 & & & & & & & & \downarrow & & \\
 & & 0 & \to & L^{d-2} & \to & \Omega^1(G^{d-3}) & \to & G^{d-2} & \to & 0 \\
 & & & & & & \downarrow & & & & \\
 & & & & & & \widehat{E}(r^{d-3}/r^{d-2}[-d+3])[d-3] & & & & \\
 & & & & & & \downarrow & & & & \\
0 & \to & L^{d-3} & \to & \Omega^1(G^{d-4}) & \to & G^{d-3} & \to & 0 & & \\
 & & & & \downarrow & & & & & &
\end{array}$$

$\cdots$

$\cdots$

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & \downarrow & & \\
 & & & & 0 & \to & L^3 & \to & \Omega^1(G^2) & \to & G^3 & \to & 0 \\
 & & & & & & & & \downarrow & & & & \\
 & & & & & & & & \widehat{E}(r^2/r^3[-2])[2] & & & & \\
 & & & & & & & & \downarrow & & & & \\
 & & 0 & \to & L^2 & \to & \Omega^1(G^1) & \to & G^2 & \to & 0 & & \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & \widehat{E}(r/r^2[-1])[1] & & & & & & \\
 & & & & & & \downarrow & & & & & & \\
0 & \to & L^1 & \to & \Omega^1 & \to & G^1 & \to & 0, & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & \widehat{E}(A) & & & & & & & & \\
 & & & & \downarrow & & & & & & & & \\
 & & & & E^0(A) & & & & & & & &
\end{array}$$

where for $1 \le i \le d-2$, $G^i = G^i_i \oplus G^i_{i+1} \cong E^0(r^i) \oplus E^1(r^i)$ as vector spaces and $L^i = \widehat{E}(\Omega^3(r^{i-1}/r^i)[-d-i+1])[i]$. Set $r^0 = A$.

We can show that G^i does not vanish for all $1 \le i \le d-2$. In fact, if there is certain $G^l = 0$ for some $1 \le l \le d-2$, then $E^0(r^l) \cong \mathrm{Hom}_A(r^l, A_0) \cong \mathrm{Hom}_{A_0}(A_l, A_0) = 0$, which implies that $r^l = 0$ and $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{l-1}$. Consider the minimal resolution of A_0 over A

$$\cdots \to P_3 \xrightarrow{\theta_3} P_2 \xrightarrow{\theta_2} P_1 \to P_0 \to A_0 \to 0.$$

We know that

$$P_2 = A \otimes P_2^2 = A_0 \otimes P_2^2 \oplus A_1 \otimes P_2^2 \oplus \cdots \oplus A_{l-1} \otimes P_2^2.$$

The maximal degree in P_2 is $l+1 \leq d-1$. But, P_3 is generated in degree d. So $\theta_3 = 0$ and $\mathrm{Proj.dim}_A A_0 \leq 2$. We get that A is a Koszul algebra, a contradiction. Hence $\mathrm{Proj.dim}_{\widehat{E}(A)} E^0(A) \geq d-2$.

Assume that $\mathrm{Proj.dim}_{\widehat{E}(A)} E^0(A) = d-2$, then $0 = \Omega^1(G^{d-2}) \cong E^0(r^{d-1}) \oplus E^{\geq 2}(r^{d-2}/r^{d-1})$ as vector spaces. So $E^0(r^{d-1}) = 0$ and $r^{d-1} = 0$. We get $0 = E^2(r^{d-2}/r^{d-1}) = E^2(r^{d-2})$, but $E_d^2(r^{d-2}) \neq 0$ by Lemma 4.2. So $\mathrm{Proj.dim}_{\widehat{E}(A)} E^0(A) \geq d-1$.

We get that $\Omega^{d-2}[-d+2]$ is not in $\Gamma_{d-2}(\widehat{E}(A))$ because G^{d-2} is not in $\mathcal{L}(\widehat{E}(A))$ by Lemma 4.2. So Ω^{d-1} over $\widehat{E}(A)$ cannot be generated in hat-degree $d-1$, but is generated in hat-degrees $d-1, d$. From the proof of Theorem 3.3, we know that $\max\{\Delta(\mathrm{n})\} = \max\{\Delta(\mathrm{n}-1)\} + 2$ for any $n \geq 2$. Thus, for $n > d-2$, $\delta(n) = d + 2[n-(d-1)] = 2(n+1) - d$. $\square$

We end this section with a general version of the main results we obtained without proofs since they are similar to the case of $p = 3$.

Theorem 4.6. *Let A be a non-trivial piecewise-Koszul algebra and M a piecewise-Koszul module both with cohomology degree function δ_p^d $(d > p \geqslant 2)$. Then we have the following statements:*

1) *$\widehat{E}(A)$ is a quadratic algebra and $\widehat{E}(M)$ is a quadratic module over $\widehat{E}(A)$,*
2) *$\widehat{E}(A)$ has the degree distribution*

$$\{\Delta(n) : n \leq \Delta(n) \leq \delta(n)\},$$

where

$$\delta(n) = \begin{cases} n, & \text{if } n \leq t+1, \\ d + (p-1)(n-t-2), & \text{else} \end{cases}$$

and $d = p + t$ $(t > 0)$.

3) $\mathrm{Proj.dim}_{\widehat{E}(A)} E^0(A) \geq d-p+2$,
4) *The $(d-p+2)$*th *syzygy Ω^{d-p+2} of $E^0(A)$ over $\widehat{E}(A)$ is minimally generated in hat-degrees $d-p+2$ and d.*

Remark 4.7. By the above theorem, A is a d-Koszul algebra, $\widehat{E}(A)$ is a Koszul algebra and $\widehat{E}(M)$ is a Koszul $\widehat{E}(A)$-module in the case of $p=2$, which coincides with the results in [7].

5. An example

In this section, we will give a concrete piecewise-Koszul algebra A such that $\widehat{E}(A)$ is not Koszul.

Example 5.1. Let $\Bbbk$ be a field and Γ be the quiver:

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, "\alpha_1"] & \overset{2}{\bullet} \arrow[r, bend left, "\alpha_2"] \arrow[r, bend right, "\alpha_3"'] & \overset{3}{\bullet} \arrow[r, bend left, "\alpha_4"] \arrow[r, "\alpha_5"] \arrow[r, bend right, "\alpha_6"'] & \overset{4}{\bullet} \arrow[r, "\alpha_7"] & \overset{5}{\bullet} \end{tikzcd}$$

Now let $A=\Bbbk\Gamma/R$, where $\Bbbk\Gamma$ is a path algebra and R is the ideal generated by the following relations:

$$\alpha_1\alpha_2-\alpha_1\alpha_3,\quad \alpha_4\alpha_7-\alpha_5\alpha_7,\quad \alpha_5\alpha_7-\alpha_6\alpha_7,\quad \alpha_2\alpha_4,\quad \alpha_3\alpha_6.$$

Then A is a piecewise-Koszul algebra (cf. [11]). We can compute out that A has the degree distribution $\{0,1,2,4,5\}$ and

$$\widehat{E}(A)=E^0(A)\oplus\underbrace{E^1(A)\oplus E^3(A)}_{1}\oplus\underbrace{E^2(A)\oplus E^4(A)}_{2}.$$

If M is a piecewise-Koszul module over A, we have $E^{>4}(M)=0$ and $E^{>3}(rM)=0$ by Lemma 2.10. For a semi-simple module S, we get $E^{>1}(\Omega^3(S))=0$.

We construct the minimal projective resolution of $E^0(A)$ over $\widehat{E}(A)$. The first syzygy Ω^1 is given as

$$0\to\Omega^1\to\widehat{E}(A)\to E^0(A)\to 0$$

with Ω^1 generated in hat-degree 1. We have a commutative diagram for the second syzygy Ω^2 in the minimal resolution of $E^0(A)$ over $\widehat{E}(A)$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^{21} & \to & \Omega^2 & \to & \Omega^{22} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widehat{E}(\Omega^3(A_0)/r\Omega^3(A_0)[-4])[1] & \to & P^1 & \to & \widehat{E}(r/r^2[-1])[1] & \to & 0
\end{array}$$

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{E}(\Omega^3(A_0)[-4])[1] & \to & \Omega^1 & \to & \Omega^{12} & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where all the three columns are projective covers.

By Propositions 3.2 and 4.3, we have that $\widehat{E}(\Omega^3(A_0)[-4]), \Omega^{12}[-1] \in \mathscr{L}(\widehat{E}(A))$, where the shift -1 is in the hat-grading. So Ω^2 is generated in hat-degree 2 and $\widehat{E}(A)$ is quadratic.

There is a commutative diagram in which the third syzygy Ω^3 in the minimal resolution of $E^0(A)$ over $\widehat{E}(A)$ is given

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^{31} & \to & \Omega^3 & \to & \Omega^{32} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P^{21} & \to & P^2 & \to & P^{22} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^{21} & \to & \Omega^2 & \to & \Omega^{22} & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the three columns are all projective covers. We also obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^{321} & \to & \Omega^{32} & \to & \Omega^{322} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{E}(\Omega^3(r/r^2)/r\Omega^3(r/r^2)[-5])[2] & \to & P^{22} & \to & \widehat{E}(r^2/r^3[-2])[2] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{E}(\Omega^3(r/r^2)[-5])[2] & \to & \Omega^{22} & \to & E^0(r^2)\oplus E^1(r^2) & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the three columns are all projective covers and

$$\Omega^{322} = \underbrace{E^0(r^3)\oplus E^3(r^2/r^3)}_{3} \oplus \underbrace{E^2(r^2/r^3)\oplus E^4(r^2/r^3)}_{4},$$

which is generated in hat-degrees 3 and 4, but not generated in hat-degree 3, since $E^3(A) = E^2(r) = E^2_4(r) \cong E^2_4(r^2) \neq 0$ and $0 \to E^1(r^3) \to E^2(r^2/r^3) \xrightarrow{\varphi}$

$E^2(r^2)$ is exact with $\varphi \neq 0$ by Lemma 4.2. Thus $\widehat{E}(A)$ is not a Koszul algebra.

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References

[1] M. Artin and W.F. Schelter, *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171–216.

[2] N. Bian, Y. Ye and P. Zhang, *Generalized d-Koszul modules*, Math. Res. Lett. **18** (2011), 965–972.

[3] A. Beilinson, V. Ginszburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), 473–525.

[4] R. Berger, *Koszulity for nonquadratic algebras*, J. Algebra **239** (2001), 705–734.

[5] R. Fröberg, *Koszul algebras*, in 'Advances in commutative ring theory', Fez, 1997, Lecture Notes Pure Appl. Mathematics, **205**, Dekker, New York, 1999, 337–50.

[6] E.L. Green and E.N. Marcos, *δ-Koszul algebras*, Comm. Algebra **33** (2005), 1753–1764.

[7] E.L. Green, E.N. Marcos, R. Martinez-Villa and P. Zhang, *D-Koszul algebras*, J. Pure Appl. Algebra **193** (2004), 141–162.

[8] J.-W. He and D.-M. Lu, *Higher Koszul Algebras and A-infinity Algebras*, J. Algebra **293** (2005), 335–362.

[9] J.-F. Lü, *On an example of δ-Koszul algebras*, Proc. Amer. Math. Soc. **138** (2010), 3777–3781.

[10] ———, *Notes on δ-Koszul algebras*, Appl. Categ. Structures **20** (2012), 143–159.

[11] J.-F. Lü, J.-W. He and D.-M. Lu, *Piecewise–Koszul algebras*, Sci. China Math. **50** (2007), 1785–1794.

[12] J.-F. Lü and X.-L. Yu, *A note on the bound of the generation degree for Ext-algebras*, Sci. China Math. **40** (2010), 223–226.

[13] J.-F. Lü and X.-L. Yu, *Notes on piecewise-Koszul algebras*, Acta Math. Sin., (Engl. Ser.) **27(5)** (2011), 943–952.

[14] A. Polishchuk and L. Positselski, *Quadratic algebras*, University Lecture Serise, **37**, American Mathematical Society, Providence, RI, 2005.

[15] S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

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