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More Reduced Obstruction Theories

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We first develop a general formalism for globally removing factors from a 1-perfect obstruction theory, analogous to Manetti's formalism for deformation functors. We then apply this formalism to give a construction of a reduced 1-perfect obstruction theory on the moduli space of morphisms from a curve to a surface $f : C \to S$ in class β such that $H^1(C, f^*T_S) \xrightarrow{-\cup\beta} H^2(S, \mathcal{O}_S)$ is surjective. This condition appears in recent work of Kool and Thomas.

1. Introduction

Due to the deformation invariance of Gromov–Witten invariants, smooth complex projective surfaces having deformations in a direction where the topological class $\beta \in H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$ does not stay of type $(1, 1)$ have no Gromov–Witten invariants.

To nevertheless study the rich enumerative geometry of such surfaces two solutions have been developed. The first, pioneered by Bryan and Leung in [4], uses analytic techniques and twistor spaces to perform a refined curve count. The second, purely algebraic approach, is given by removing a factor from the usual 1-perfect obstruction theory on the moduli of maps from a curve to the surface in question. This reduced obstruction theory was introduced in the case of $K3$ surfaces by Maulik and Pandharipande [13]. The key difficulty in the algebraic approach is to verify that the complex obtained after removing a factor from the standard obstruction theory indeed satisfies all requirements to again be an obstruction theory. This is closely related to the problem of showing that obstructions to infinitesimal deformations of the objects parametrized by the moduli space lie in the kernel of a semiregularity map as introduced by Bloch in [3].

In the context of formal deformation theory (such formal deformation problems are for instance given by localizing at a point of the moduli space) there is a powerful method for removing a factor from an obstruction space developed by Manetti (see [12] for an overview). We briefly recall the argument. Assume $F : \mathbf{Art}_k \to \mathbf{Set}$ is a deformation functor on the category of local Artinian k-algebras with residue field k. Furthermore, assume that F is equipped with a choice of an obstruction space T^2_F. Recall that an obstruction space for F is a k-vector space T^2_F such that for every small extension $I \to A \to B$ in **Art** there is an exact sequence

$$T^1_F \otimes I \to F(A) \longrightarrow F(B) \xrightarrow{\text{ob}} T^2_F \otimes I$$

which is functorial in small extensions. Here T^1_F is the tangent space to the deformation functor F. We would like to show that T^2_F is too large a choice for the obstruction space, and that in fact a smaller space suffices. Assume there exists another deformation functor G with chosen obstruction space T^2_G and a natural transformation $\nu : F \to G$. Furthermore, assume there is a linear map $\sigma : T^2_F \to T^2_G$ such that for all small extensions $I \to A \to B$ the diagram

$$\begin{tikzcd} F(B) \arrow[r, "\text{ob}"] \arrow[d, "\nu"'] & T^2_F \otimes I \arrow[d, "\sigma\otimes\text{id}"] \\ G(B) \arrow[r, "\text{ob}"'] & T^2_G \otimes I \end{tikzcd} \tag{1}$$

commutes. Then if G is unobstructed, the lower horizontal morphism in (1) is zero for all small extensions, and the obstruction map of F factors over the kernel of σ. Thus, the kernel of σ is a new obstruction space for the deformation functor F. In summary, to remove a factor from an obstruction space of a formal deformation problem, it suffices to find a morphism to an unobstructed deformation functor along with a compatible morphism on the obstruction spaces.

This, of course, leads to the question how to find compatible morphisms between the obstruction spaces of deformation functors. As explained by Manetti in [12], these data are automatic if the deformation functors are controlled by differential graded Lie algebras and if the natural transformation between the deformation functors comes from a morphism of differential graded Lie algebras.

The aim of this note is to develop a global version of Manetti's argument. The global counterpart of the formal deformation functor F equipped with an obstruction space T^2_F is given by a moduli space X equipped with a 1-perfect obstruction theory $\phi : E \to L_X$. To remove a factor from the obstruction theory, we want to find a morphism $f : X \to Y$ to a smooth space Y equipped with a 1-perfect obstruction theory $\psi : F \to L_Y$. The global role of σ is played by a morphism $\alpha : f^*F \to E$. It remains to find a global counterpart of the commutativity of the diagram (1). It is tempting to require the diagram

$$
\begin{tikzcd}
f^*F \arrow[r, "\psi"] \arrow[d, "\alpha"'] & f^*L_Y \arrow[d] \\
E \arrow[r, "\phi"'] & L_X
\end{tikzcd}
\tag{2}
$$

to commute in the derived category $D_{\mathrm{qcoh}}(X)$. The main point of this note is that commutativity of the above diagram in the derived category alone does not suffice to give a well-defined reduced obstruction theory on X. Instead, to globally carry out the above argument one has to explicitly know the homotopy making diagram (2) commutative. In other words, one requires commutativity of the diagram (2) not only in the derived category, but in some higher categorical model, as is for instance given by the ∞-category of quasi-coherent complexes $\mathrm{QCoh}(X)$ constructed by Lurie in [10]. With this amount of data, the necessary calculations can be carried out exactly as in the formal picture.

Again, this raises the question how such a strong compatibility between the obstruction theories including a canonical choice of homotopy in diagram (2) can be obtained. The global counterpart of deformation functors controlled by differential graded Lie algebras are given by derived moduli spaces as developed by Toën–Vezzosi and Lurie. We argue that if the 1-perfect obstruction spaces are induced by derived moduli spaces, and if the morphism between the obstruction theories is induced by a morphism between the derived moduli spaces, then all necessary data are automatically available.

The formalism developed here is applicable to a wide range of moduli spaces used in enumerative geometry. For example, in the case of Donaldson–Thomas invariants and Pandharipande–Thomas invariants, a trace-free Ext group has to be considered as obstruction space instead of a full Ext group. The necessary calculations to verify that the trace free Ext group defines an obstruction theory are long and involved, see [6, 16]. With a global version of Manetti's argument the same result can be obtained very quickly.

As an application, we give a purely algebraic construction of a reduced obstruction theory on the moduli of morphisms from a curve C to a surface S, where we require the curve to be of class $\beta \in H^{1,1}(S, \mathbb{C}) \cap H^2(S, \mathbb{Z})$ and further require the $H^1(C, f^*T_S) \stackrel{-\cup\beta}{\to} H^2(S, \mathcal{O}_S)$ is surjective. This reduced obstruction theory has been previously been constructed by Kool and Thomas in [9] using twistor methods.

Conventions

We have tried to adhere to the following conventions. We work throughout over an arbitrary base ring k, which in Section 4 becomes the field of complex numbers. We will denote the cotangent complex of a scheme, or more generally an Artin stack, over k by L_X instead of $L_{X/k}$. Contrary to what is common in algebraic geometry, we have used homological grading. Finally, we will denote by $\mathrm{QCoh}(X)$ the ∞-category of quasi-coherent complexes constructed by Lurie in [10]. The reason for employing this category instead of the derived category is that at certain points it is important to know why things are homotopic, and not only that they are homotopic. It also allows us to carry out proofs as if one was only dealing with modules, and not with complexes. Recall that a *cofiber sequence* in $\mathrm{QCoh}(X)$ consists of a sequence of morphisms $E \xrightarrow{f} F \xrightarrow{g} G$, a 2-simplex identifying the composition fg with a morphism $E \xrightarrow{h} G$, and a nullhomotopy of h. The ∞-category $\mathrm{QCoh}(X)$ is equipped with a standard t-structure. We will use that the notion of Tor-amplitude behaves well with respect to this t-structure, i.e., if an object $E \in \mathrm{QCoh}(X)$ is of Tor-amplitude $\leq n$, then $E[m]$ is of Tor-amplitude $\leq n + m$. All details can be found in Lurie's volumes [10, 11].

2. Removing factors

We first introduce the geometric objects we wish to study. These are Artin stacks with a fixed 1-perfect obstruction theory.

Definition 2.1. A *1-perfect obstruction theory* on an Artin stack X locally of finite type over k is given by a morphism

$$\phi : E \longrightarrow L_X$$

in $\mathrm{QCoh}(X)$ such that E is a perfect complex of Tor-amplitude ≤ 1 and $\mathrm{cofib}\,\phi \in \mathrm{QCoh}(X)_{\geq 2}$.

If X is a Deligne–Mumford stack, the morphism $\phi : E \to L_X$ in the above definition is a 1-perfect obstruction theory in the sense of Behrend and Fantechi [1]. We next define morphisms between such objects.

Definition 2.2. A *morphism of Artin stacks with 1-perfect obstruction theories* is a pair

$$(f,\alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y),$$

where $f : X \to Y$ is a morphism of Artin stacks over k, and $\alpha : f^*F \to E$ is a morphism of perfect complexes on X such that

$$\begin{tikzcd} f^*F \arrow[r, "\chi"] \arrow[d, "\alpha"'] & f^*L_Y \arrow[d] \\ E \arrow[r, "\phi"] & L_X \end{tikzcd}$$

commutes in $\mathrm{QCoh}(X)$.

Remark 2.3. Recall that commuting in $\mathrm{QCoh}(X)$ means that we have fixed a homotopy making the diagram commutative. This added information is absolutely essential for all further computations.

We will also need the notion of virtually smooth morphism.

Definition 2.4. Let $(f,\alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y)$ be a morphism of Artin stacks with 1-perfect obstruction theories. Then (f,α) is a *virtually smooth morphism* if $\mathrm{cofib}(\alpha)$ is of Tor-amplitude ≤ 1.

Remark 2.5. Note that a priori $\mathrm{cofib}(\alpha)$ is only of Tor-amplitude ≤ 2.

This, of course, raises the question where to find canonical choices of the homotopy required in Definition 2.2. Derived algebraic geometry provides natural examples of Artin stacks with perfect obstruction theories and morphisms between these including a canonical homotopy.

Example 2.6. Recall that a derived Artin stack X^{d} over k is *quasi-smooth* if its cotangent complex $L_{X^{\mathrm{d}}}$ is of Tor-amplitude ≤ 1 and its underlying Artin stack $X := t_0(X^{\mathrm{d}})$ is locally of finite type over k. By the canonical inclusion $j_X : X \hookrightarrow X^{\mathrm{d}}$ we can obtain the structure of an Artin stack with 1-perfect obstruction theory on X. The perfect obstruction theory is given by the canonical morphism $\phi : j_X^* L_{X^{\mathrm{d}}} \to L_X$. Using the functoriality properties of cotangent complexes, every morphism of quasi-smooth derived Artin stacks gives rise to a morphism of Artin stacks with perfect obstruction theories including the homotopy required in Definition 2.2.

To remove factors from obstruction theories we will make use of smooth Artin stacks equipped with 1-perfect obstruction theories. This is contrary to the philosophy that spaces become smooth after deriving them, or that 1-perfect obstruction theories are only interesting on very singular spaces.

Definition 2.7. Let $(f,\alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y)$ be a morphism of Artin stacks with perfect obstruction theories. We say that (f,α) is a *reduction morphism* if Y is smooth.

Remark 2.8. For applications to virtual classes, X will be assumed to be a Deligne–Mumford stack.

Given a reduction map, we would like to define a new 1-perfect obstruction theory on X such that the virtual dimension of X increases. The factor we would like to remove from the obstruction theory E is the pull-back to X of the fiber of $\chi : F \to L_Y$. In the following we will show that this is possible if the reduction morphism is virtually smooth. The key to removing a factor is the following lemma, which is true in much greater generality than we actually need. Note that we do not assume (f,α) either to be a reduction morphism or virtually smooth, or any of the perfect obstruction theories to be 1-perfect.

Lemma 2.9. *Let $(f,\alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y)$ be a morphism of Artin stacks with perfect obstruction theories. Let $K = \mathrm{fib}(\chi)$, and define β to be the composition*

$$f^*K \xrightarrow{\gamma} f^*F \xrightarrow{\alpha} E.$$

Then the composition

$$f^*K \xrightarrow{\beta} E \xrightarrow{\phi} L_X$$

is zero.

Proof. By definition, we have a cofiber sequence $K \to F \to L_Y$, and this remains a cofiber sequence after pulling to X. We thus have the following commutative diagram on X:

$$\begin{tikzcd} f^*K \arrow[r] \arrow[d, "\gamma"'] & 0 \arrow[d] \\ f^*F \arrow[r, "\chi"] \arrow[d, "\alpha"'] & f^*L_Y \arrow[d] \\ E \arrow[r, "\phi"] & L_X \end{tikzcd}$$

which gives a homotopy from $\phi \circ \beta$ to zero. □

We can now define our candidate for a reduced obstruction theory. We let $E' := \mathrm{cofib}(\beta)$. By Lemma 2.9, we have a well-defined morphism $\phi' : E' \to L_X$. This is the point where we have used the additional datum of the diagram of obstruction theories commuting up to a fixed homotopy. If we only knew the composition to be zero in the derived category without knowing the precise homotopy this would not be sufficient to obtain a well-defined morphism.

Theorem 2.10. *Let $(f, \alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y)$ be a virtually smooth reduction map. Then*

$$\phi' : E' \to L_X$$

is a 1-perfect obstruction theory on X.

Proof. We first show that E' is perfect. Let K as above denote $\mathrm{fib}(\chi)$, so that we have a cofiber sequence

$$K \longrightarrow F \xrightarrow{\chi} L_Y.$$

Now F is perfect by assumption, and L_Y is perfect since Y is smooth and locally of finite presentation. Since the property of being perfect is stable under cofiber sequences, K is perfect, and thus f^*K is perfect. This shows that E' is the cofiber of a morphism between perfect objects, and thus is perfect.

We now want to show that E' is of Tor-amplitude ≤ 1. Since Y is smooth and L_Y thus is of Tor-amplitude ≤ 0, the above cofiber sequence shows that K is of Tor-amplitude ≤ 1. It follows that f^*K is also of Tor-amplitude ≤ 1.

Let γ denote the morphism $f^*K \to f^*F$. By definition, the diagram

$$\begin{tikzcd} f^*K \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & f^*F \arrow[d, "\alpha"] \\ & E \end{tikzcd}$$

commutes. This gives us a cofiber sequence $\operatorname{cofib}(\gamma) \to \operatorname{cofib}(\beta) \to \operatorname{cofib}(\alpha)$. Since $E' = \operatorname{cofib}(\beta)$ and $f^*L_Y = \operatorname{cofib}(\gamma)$, we have a cofiber sequence

$$f^*L_Y \longrightarrow E' \longrightarrow \operatorname{cofib}(\alpha).$$

Since we assumed (f, α) a virtually smooth morphism, $\operatorname{cofib}(\alpha)$ is of Tor-dimension ≤ 1. Again using that Y is smooth, it follows that E' is of Tor-dimension ≤ 1.

It remains to show that $\operatorname{cofib}(\phi') \in \operatorname{QCoh}(X)_{\geq 2}$, or equivalently that $\operatorname{fib}(\phi') \in \operatorname{QCoh}(X)_{\geq 1}$. Let $K' = \operatorname{fib}(\phi)$. Since the composition $\phi \circ \beta$ factors over zero, we obtain a morphism $\delta : f^*K \to K'$, and we can identify $\operatorname{fib}(\phi')$ with $\operatorname{cofib}(\delta)$. Since f^*K and K' are both in $\operatorname{QCoh}(X)_{\geq 1}$, the claim follows. □

Remark 2.11. Since $\phi' : E' \to L_X$ is a perfect obstruction theory on X this automatically poses the question whether this obstruction theory is induced by a natural structure of a derived Artin stack on X. Adding plenty of assumptions such a statement indeed holds. First of all, we have to assume that the perfect obstruction theories $(E \to L_X)$ and $(F \to L_Y)$ are induced by derived stacks X^{d} and Y^{d}, and the compatibility datum α is induced by a morphism $f^{\mathrm{d}} : X^{\mathrm{d}} \to Y^{\mathrm{d}}$. Furthermore, we have to assume that the derived structure on Y^{d} splits as

$$Y^{\mathrm{d}} = Y \times Y^{\mathrm{der}}.$$

The underling stack of Y^{der} is a point. Let $p : Y^{\mathrm{d}} \to Y^{\mathrm{der}}$ be the projection. The homotopy fiber product of the diagram

$$\begin{tikzcd} & X^{\mathrm{d}} \arrow[d, "p \circ f^{\mathrm{d}}"] \\ \operatorname{Spec} k \arrow[r] & Y^{\mathrm{der}} \end{tikzcd}$$

then yields the desired derived Artin stack. Such a splitting exists for the derived Picard stack of a $K3$ surface. It is reasonable to expect such a splitting to exist whenever Y^{d} is a group stack with smooth truncation.

3. Application to deformation theory

In the following, assume that $\phi : E \to L_X$ is a 1-perfect obstruction theory on smooth Deligne–Mumford stack X. Given a reduction map

$$(f, \alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y),$$

we can define a *generalized semi-regularity map*. Given a morphism $p : T \to X$ where $T = \mathrm{Spec}(A)$ is an affine scheme, and a square-zero extension $T \hookrightarrow T'$ classified by a morphism $\eta : L_T \to M[1]$ for some A-module M, $p : T \to X$ extends to a morphism $T' \to X$ if and only if the element in $\mathrm{Ext}^1(p^*E, M)$ defined by the homotopy class of the composition

$$p^*E \xrightarrow{p^*\phi} p^*L_X \longrightarrow L_T \xrightarrow{\eta} M[1]$$

vanishes. We define the generalized semi-regularity map to be the map

$$\mathrm{Ext}^1(p^*E, M) \longrightarrow \mathrm{Ext}^1(p^*f^*F, M)$$

obtained by composition with α. We will now show that realized obstructions lie in the kernel of the generalized semi-regularity map. We first give a definition of realized obstructions following Behrend and Fantechi [1].

Definition 3.1. Let $\phi : E \to L_X$ be a 1-perfect obstruction theory on Deligne–Mumford stack X, and let $p : T \to X$ be a morphism with $T = \mathrm{Spec}(A)$. Let M be an A-module. A non-zero morphism $\alpha : p^*E \to M[1]$ *realizes an obstruction* if there exists a square-zero extension $T \hookrightarrow T'$ classified by $\eta : L_T \to M[1]$ such that

$$\begin{tikzcd} p^*E \arrow[r] \arrow[drr, "\alpha"'] & p^*L_X \arrow[r] & L_T \arrow[d, "\eta"] \\ & & M[1] \end{tikzcd}$$

commutes.

We can now show that obstructions that are actually realized always lie in the kernel of the generalized semi-regularity map.

Proposition 3.2. *Let $(f, \alpha) : (X, \phi : E \to L_X) \longrightarrow (Y, \chi : F \to L_Y)$ be a reduction morphism. Assume that X is a Deligne–Mumford stack, and let p :*

$T \to X$ be a morphism where T is an affine scheme. Then realized obstructions lie in the kernel of the generalized semi-regularity map.

Proof. Since Y is smooth, this allow us to conclude that $\mathrm{Ext}^1(p^* f^* L_Y, M)$ and $\mathrm{Ext}^2(p^* f^* L_Y, M)$ are zero. Using the pull-back of the cofiber sequence

$$K \longrightarrow F \xrightarrow{\chi} L_Y$$

to T we thus have an isomorphism

$$\mathrm{Ext}^1(p^* f^* F, M) \simeq \mathrm{Ext}^1(p^* f^* K, M).$$

Applying Lemma 2.9 the claim follows. □

Remark 3.3. A reduction morphism is virtually smooth if its generalized semi-regularity morphism is surjective.

4. Application to moduli of maps

We now apply the formalism developed above in an example, working over $k = \mathbb{C}$. The example we will be concerned with is the moduli space of maps from a fixed curve C to a smooth projective complex surface S satisfying the condition $c_1(\mathbb{R}f_* \mathcal{O}_C) = \beta$, where $\beta \in H^1(S, \Omega^1_S)$. We will denote this space by $\mathrm{Mor}_\beta(C, S)$. It is well known that this space carries a 1-perfect obstruction theory, see Behrend and Fantechi [1]. We will denote this space equipped with its obstruction theory by

$$\left(\mathrm{Mor}_\beta(C, S), \phi : E \to L_{\mathrm{Mor}_\beta(C,S)}\right).$$

To apply the results of [15], it is important to note that the same structure of virtually smooth scheme can also be constructed using Example 2.6. To see this, denote by $i : \mathrm{St}_k \to \mathrm{dSt}_k$ the inclusion functor from stacks over k to derived stacks over k. We can then define the derived moduli space of maps to be the derived scheme parametrizing morphisms in this larger category. We will denote this derived scheme by $\mathbb{R}\mathrm{Mor}_\beta(C, S)$.

In order to remove a factor from the obstruction theory using the above formalism we have to find some Artin stack equipped with a 1-perfect obstruction theory as comparison space. The natural candidate in this example is the Picard stack $\mathrm{Pic}(S) := \underline{\mathrm{Hom}}_{\mathrm{St}_k}(S, B\mathbb{G}_m)$. As above, there again is a derived version of this stack, given by $\mathbb{R}\mathrm{Pic}(S) := \underline{\mathrm{Hom}}_{\mathrm{dSt}_k}(S, B\mathbb{G}_m)$. Denote the canonical inclusion by $j : \mathrm{Pic}(S) \to \mathbb{R}\mathrm{Pic}(S)$. The Artin stack with 1-perfect obstruction theory we will use as target for our potential reduction morphism is

$$\left(\mathrm{Pic}(S), \chi : j^* L_{\mathbb{R}\mathrm{Pic}(S)} \to L_{\mathrm{Pic}(S)}\right).$$

Since the underlying Artin stack $\mathrm{Pic}(S)$ is smooth, this is an excellent candidate for a reduction map.

Finally, we have to give a map of virtually smooth schemes. In [15], a map

$$\mathbb{R}\mathrm{Mor}_\beta(C, S) \xrightarrow{A_S} \mathbb{R}\mathrm{Perf}(S) \xrightarrow{\det} \mathbb{R}\mathrm{Pic}(S)$$

is given. At a point $p : \mathrm{Spec}(k) \to \mathbb{R}\mathrm{Mor}_\beta(C, S)$ corresponding to a morphism $f : C \to S$ the above composition is given by

$$(f : C \to S) \longmapsto \mathbb{R}f_*(\mathcal{O}_C) \longmapsto \det(\mathbb{R}f_*(\mathcal{O}_C)).$$

Using Example 2.6, we obtain a map of Artin stacks with 1-perfect obstruction theories

$$\begin{aligned}(f, \alpha) : &\left(\mathrm{Mor}_\beta(C, S), \phi : E \to L_{\mathrm{Mor}_\beta(C,S)}\right) \\ &\longrightarrow \left(\mathrm{Pic}(S), \chi : j^* L_{\mathbb{R}\mathrm{Pic}(S)} \to L_{\mathrm{Pic}(S)}\right).\end{aligned}$$

Remark 4.1. The generalized semi-regularity map associated with (f, α) is just the semi-regularity map for morphisms of Buchweitz and Flenner [5, Remark 7.24].

We can now define a new 1-perfect obstruction theory on $\mathrm{Mor}_\beta(C, S)$. Let $K := \mathrm{fib}(\chi)$. Note that K is non-trivial if and only if $H^2(S, \mathcal{O}_S)$ is non-trivial. As above, we then have a morphism $\gamma : f^*K \to E$, and can define $E' := \mathrm{cofib}(\gamma)$ as candidate for a reduced obstruction theory.

Corollary 4.2. *Assume that*

$$\begin{aligned}(f, \alpha) : &\left(\mathrm{Mor}_\beta(C, S), \phi : E \to L_{\mathrm{Mor}_\beta(C,S)}\right) \\ &\longrightarrow \left(\mathrm{Pic}(S), \chi : j^* L_{\mathbb{R}\mathrm{Pic}(S)} \to L_{\mathrm{Pic}(S)}\right)\end{aligned}$$

is virtually smooth. Then

$$\phi' : E' \to L_{\mathrm{Mor}_\beta(C,S)}$$

is a 1-perfect obstruction theory on $\mathrm{Mor}_\beta(C, S)$.

Remark 4.3. Note that the 1-perfect obstruction theories $E' \to L_{\mathrm{Mor}_\beta(C,S)}$ and $E \to L_{\mathrm{Mor}_\beta(C,S)}$ only differ in case $H^2(S, \mathcal{O}_S)$ is non-zero.

Example 4.4. Assume that S is a $K3$ surface. Then the morphism (f, α) is virtually smooth for any class $\beta \neq 0$.

We finally want to state a condition ensuring that (f, α) is virtually smooth. This condition was identified by Kool and Thomas [9] and provided the motivation for this work.

Proposition 4.5. *Assume that*

$$H^1(S, T_S) \xrightarrow{\cup\beta} H^2(S, \mathcal{O}_S)$$

is surjective. Then (f, α) is virtually smooth.

Proof. It suffices to prove the statement on k-points. We thus have to show that at any point $p : \operatorname{Spec} k \to \mathrm{Mor}_\beta(C, S)$ the morphism

$$\pi_1(p^*\alpha) : \pi_1(p^* L_{\mathbb{R}\,\mathrm{Pic}(S)}) \longrightarrow \pi_1(p^* E)$$

is injective. Equivalently, we have to show that the dual of $\pi_1(p^*\alpha)$ is surjective.

Let $g : C \to S$ be the morphism corresponding to p. Recall from Illusie [8, Chapitre V] or explicitly from [6] that for any perfect complex the first Chern class factors as composition of the Atiyah class and the trace map. Let $E = \mathbb{R} g_* \mathcal{O}_C$. Thus, for any class $\alpha \in H^1(S, T_S)$ the operation of cup-product with $\beta = c_1(E)$ factors as

$$\begin{tikzcd} H^1(S, T_S) \arrow[d, "\_\cup \mathrm{at}_E"'] \arrow[dr, "\_\cup\beta"] & \\ \mathrm{Ext}^2_S(E, E) \arrow[r, "\mathrm{tr}"'] & H^2(S, \mathcal{O}_S). \end{tikzcd}$$

Now in [15, Appendix] it is shown that $\_ \cup \mathrm{at}_E$ factors as

$$\begin{tikzcd} & H^1(S, T_S) \arrow[dl] \arrow[d, "\_\cup \mathrm{at}_E"] \\ H^1(C, g^* T_S) \arrow[r, "T_{A_S, g}"'] & \mathrm{Ext}^2_S(E, E). \end{tikzcd}$$

Here $T_{A_S,g}$ is the tangent to

$$A_S : \mathbb{R}\mathrm{Mor}_\beta(C,S) \longrightarrow \mathbb{R}\mathrm{Perf}(S)$$

at the point p. Piecing the two diagrams together, we arrive at a commutative diagram

$$\begin{tikzcd} & H^1(S,T_S) \arrow[dl] \arrow[d, "\_\cup \mathrm{at}_E"'] \arrow[dr, "\_\cup\beta"] & \\ H^1(C,g^*T_S) \arrow[r, "T_{A_S}"'] & \mathrm{Ext}^2_S(E,E) \arrow[r, "\mathrm{tr}"'] & H^2(S,\mathcal{O}_S). \end{tikzcd}$$

Since the bottom row is the dual of $\pi_1(p^*\alpha)$ and by assumption

$$H^1(S,T_S) \xrightarrow{\cup\beta} H^2(S,\mathcal{O}_S)$$

is surjective, the claim follows. □

Remark 4.6. Behrend and Fantechi in [2] suggested removing a factor of $H^0(X,\Omega^2_X)$ from the obstruction theory of the moduli space of stable maps to an irreducible complex symplectic variety of dimension n to perform refined curve counts. The formalism developed here applies as soon as one has an appropriate target for a reduction morphism. Promising candidates are the derived version of the intermediate Jacobian J^p_X with $p = n-1$ constructed recently by Pridham [14] and Iacono and Manetti [7]. More generally, this should work for any variety for which an analog of the surjectivity of cup-product with β holds.

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