Math. Res. Lett.
Volume 22, Number 01, 111–140, 2015
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Pairing of zeros and critical points for random meromorphic functions on riemann surfaces

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We prove that zeros and critical points of a random polynomial p_N of degree N in one complex variable appear in pairs. More precisely, suppose p_N is conditioned to have $p_N(\xi) = 0$ for a fixed $\xi \in \mathbb{C}$. For $\epsilon \in \left(0, \frac{1}{2}\right)$ we prove that there is a unique critical point in the annulus $\left\{z \in \mathbb{C} \,\middle|\, N^{-1-\epsilon} < |z - \xi| < N^{-1+\epsilon}\right\}$ and no critical points closer to ξ with probability at least $1 - O(N^{-3/2+3\epsilon})$. We also prove an analogous statement in the more general setting of random meromorphic functions on a closed Riemann surface.

0. Introduction

The purpose of this article is to prove that zeros $\{z_j\}$ and critical points $\{c_j\}$ of random meromorphic functions on a Riemann surface come in pairs (z_j, c_j) with $|z_j - c_j| \approx N^{-1}$, where N is the common number of zeros and poles. To explain the result, consider H_N^{ξ}, the space of polynomials in one complex variable of degree at most N that vanish at a fixed $\xi \in \mathbb{C} = \mathbb{C}P^1 \backslash \{\infty\}$. We equip H_N^{ξ} with a conditional Gaussian measure $\gamma_{h,\xi}^N$ depending on a Hermitian metric h on $\mathcal{O}(1) \twoheadrightarrow \mathbb{C}P^1$ (see Section 0.2 for a precise definition) and study the random variables

$$\mathcal{N}_r := \# \left\{ z \in D_r(\xi) \,\middle|\, \frac{d}{dz} p_N(z) = 0 \right\}, \tag{0.1}$$

where $D_r(\xi)$ is the disk of radius r. Write $e^{-\phi_{z_0}(z)} := \|z_0(z)\|_h^2$, where z_0 is the usual frame of $\mathcal{O}(1)$ over $\mathbb{C}P^1 \backslash \{\infty\}$.

Theorem 1. *Suppose $d\phi_{z_0}(\xi) \neq 0$. For $\epsilon \in (0, \frac{1}{2})$, write $R_{\pm} = N^{-1\pm\epsilon}$. There is $K = K(\epsilon, h, \xi)$ so that for all N*

$$\gamma_{h,\xi}^N \left(\mathcal{N}_{R_+} = 1 \text{ and } \mathcal{N}_{R_-} = 0 \right) \geq 1 - K \cdot N^{-3/2+3\epsilon}.$$

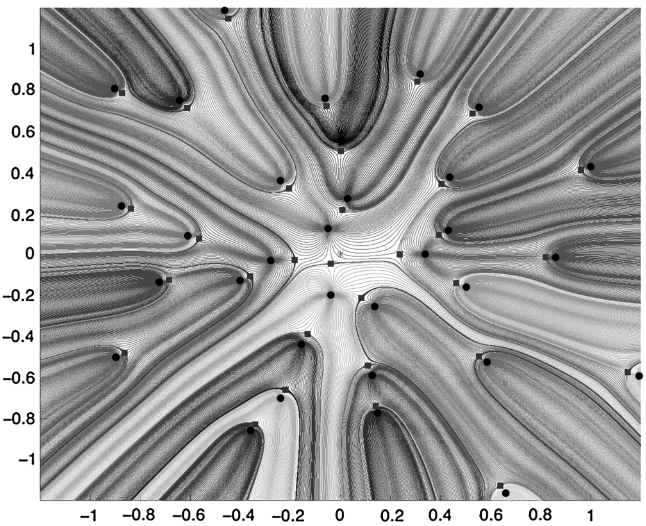

Figure 1: The zeros (black discs) and critical points (blue squares) of a degree 50 $SU(2)$ polynomial p_{50} appear in pairs. The typical distance between each pair is on the order of $\frac{1}{50}$. Each pair lines up with the origin, denoted by a red asterisk. The gradient flow lines for $|p_{50}|^2$ are shown.

A random polynomial distributed according to $\gamma_{h,\xi}^N$ therefore has a unique critical point a distance on the order of N^{-1} from ξ with high probability. The pairing of zeros and critical points are illustrated in Figures 1 and 2. The typical nearest neighbor distance for N i.i.d points on $\mathbb{C}P^1$ is on the order of $N^{-1/2}$. We give a heuristic derivation of the much smaller N^{-1} distance in Theorem 1 in terms of electrostatics on a Riemann surface in Section 2.2. We refer the reader to the article of Dennis and Hannay [6], where a somewhat different electrostatic heuristic is given for why zeros and critical points of certain random polynomials, such as $SU(2)$ polynomials (see (2.1)) and characteristic polynomials of Ginebre matrices, should come in pairs.

In this paper, we focus on understanding the distance from a fixed zero to the nearest critical point for a random polynomial (or more generally meromorphic function on a Riemann surface). The electrostatic heuristics in Section 2.2 also explain why paired zeros and critical points line up with the origin in Figure 1. We do not prove a result taking into account this preferred directions and refer the reader to [15, Theorem 2] for a rigorous statement.

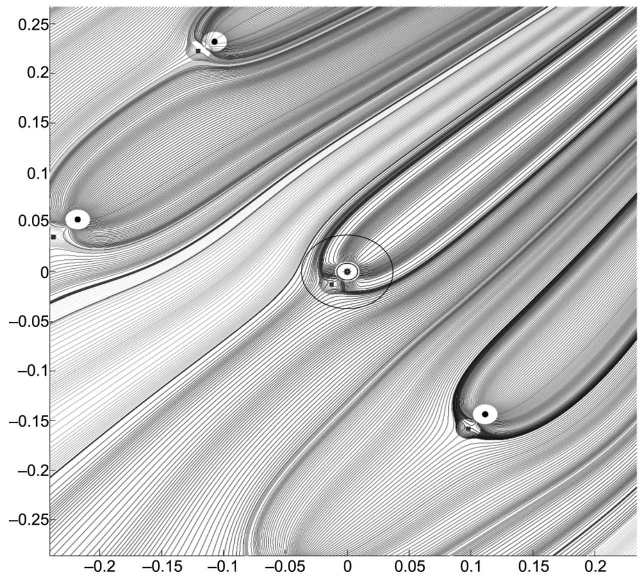

Figure 2: Zeros (black discs) and holomorphic critical points (blue squares) for an $SU(2)$ polynomial p_{50} of degree 50 conditioned to have a zero at $\xi = 1 + 1i$ (denoted by a red asterisk) are drawn in normal coordinates centered at ξ. The annulus with inner radius $N^{-1-1/10}$ and outer radius $N^{-1+1/10}$, which Theorem 1 predicts to have a unique critical point is shown.

0.1. Riemann surfaces

Zeros and critical points of random meromorphic functions on a closed Riemann surface Σ also come in pairs. In this situation, H_N^ξ becomes the space of sections of a very ample line bundle $L^{\otimes N} \twoheadrightarrow \Sigma$ that vanish at a fixed $\xi \in \Sigma$. We generalize $\frac{d}{dz}$ by fixing an arbitrary section $\sigma \in H^0_{\text{hol}}(\Sigma, L)$ and defining the meromorphic connection ∇^σ on L^N by

$$\nabla^\sigma \sigma^N = 0, \tag{0.2}$$

where $\sigma^N = \sigma^{\otimes N}$. The critical points of a section $s_N \in H_N^\xi$ are thus the solutions to

$$\nabla^\sigma s_N(z) = 0. \tag{0.3}$$

Under the identification of polynomials of degree N with sections of $\mathcal{O}(N) \twoheadrightarrow \mathbb{C}P^1$, the usual holomorphic derivative $\frac{d}{dz}$ becomes the connection ∇^{z_0} on

$\mathcal{O}(N)$. For more on meromorphic connections see Section 4. Defining the meromorphic function

$$\gamma_N(z) := \frac{s_N(z)}{\sigma^N(z)}$$

on Σ, we see that

$$\nabla^\sigma s_N(z) \quad \Longleftrightarrow \quad d\log|\gamma_N(z)| = 0$$

if γ_N has simple zeros. The function $\log|\gamma_N|$ is the Coulomb potential for charge distributed according to the divisor of γ_N, and critical points of s_N with respect to ∇^σ are therefore points of equilibrium for the resulting electric field on Σ. This perspective is developed in Section 2.2.

We emphasize that our notion of critical point is purely holomorphic and results in a completely different theory from critical points computed with respect to smooth metric connections studied in [10–14]. We refer the reader to Section 2.4 for a discussion of this point.

0.2. Definition of Hermitian Guassian ensembles

The ensembles of random sections we study are called Hermitian Gaussian ensembles (cf. [1–3, 8, 9, 15, 17, 19, 20, 22]). Let h be a smooth positive Hermitian metric on an ample holomorphic line bundle $L \twoheadrightarrow \Sigma$ over a closed Riemann surface. We recall the definition of the Hermitian Gaussian ensemble associated to h. A random section of L^N from this ensemble is

$$s_N := \sum_{j=0}^{N} a_j S_j,$$

where $a_j \sim N(0,1)_{\mathbb{C}}$ are i.i.d. standard complex Gaussians and $\{S_j\}_{j=0}^N$ is any orthonormal basis for H_N with respect to the inner product

$$\langle s_1, s_2\rangle_h := \int_\Sigma h^N(s_1(z), s_2(z))\,\omega_h(z), \quad s_1, s_2 \in H_N. \tag{0.4}$$

Here $\omega_h := \frac{i}{2\pi}\partial\overline{\partial}\log h^{-2}$ is the curvature of (L,h). We will write γ_h^N for the law of s_N and abbreviate $s_N \in HGE_N(L,h)$. In this paper, we focus on the following variant of $HGE_N(L,h)$:

Definition 1. For $\xi \in \Sigma$ fixed, we will write $s_N \in HGE_N^\xi(L,h)$ if the law of s_N is the standard Gaussian measure on

$$H_N^\xi = \left\{ s_N \in H^0_{\mathrm{hol}}(\Sigma, L^N) \;\middle|\; s_N(\xi) = 0 \right\}$$

generated by the restriction of the inner product (0.4) to H_N^ξ.

We denote by $\gamma^N_{h,\xi}$ the law of $s_N \in HGE_N^\xi(L,h)$ and will write

$$s_N(z) = \sum_{j=1}^{d_N - 1} a_j S_j^\xi(z),$$

where a_j are i.i.d standard complex Gaussians and $\{S_j^\xi,\ j = 1, \dots, d_N - 1\}$ is any orthonormal basis for H_N^ξ with respect to the inner product (0.4). Every $s_N \in HGE_N^\xi(L,h)$ satisfies $s_N(\xi) = 0$ and may equivalently be defined as the conditional expectation

$$s_N = \mathbb{E}\left[\widetilde{s}_N \mid ev_\xi = 0\right],$$

where $\widetilde{s}_N \in HGE_N(L,h)$ and $ev_\xi : H^0_{\mathrm{hol}}(\Sigma, L^N) \to L^N|_\xi$ is the evaluation map at ξ. See Section 3 of [22] for more details.

1. Main result

Fix $(L,h) \twoheadrightarrow \Sigma$ as above and $\sigma \in H^0_{\mathrm{hol}}(L)$, and define $\phi_\sigma : \Sigma \to (-\infty, \infty]$ by

$$\phi_\sigma(z) := \log \left\|\sigma(z)\right\|_h .$$

Theorem 2. *Fix $\xi \in \Sigma$ such that $d\phi_\sigma(\xi) \notin \{0, \infty\}$. Suppose $s_N \in HGE_N^\xi$ (L,h). For each $\epsilon \in (0, \frac{1}{2})$ define $R_\pm = N^{-1\pm\epsilon}$. Then there exists a constant $K = K(\Sigma, L, h, \epsilon, \xi)$, such that for each $N \geq 1$*

$$\gamma^N_{h,\xi}\left(\mathcal{N}_{R_-} = 0 \quad \text{and} \quad \mathcal{N}_{R_+} = 1\right) \geq 1 - K \cdot N^{-3/2+3\epsilon}, \tag{1.1}$$

where $\mathcal{N}_r$ is defined in (0.1) *and D_r is the geodesic ball of radius r around ξ.*

Remark 1. Let μ be any probability measure on $\prod_{N=1}^\infty H_N^\xi$ with marginals $\gamma^N_{h,\xi}$. Write $A_{N,\epsilon}$ for the event $\left\{\mathcal{N}_{R_+} = 1 \text{ and } \mathcal{N}_{R_-} = 0\right\}$. If $\epsilon < \frac{1}{6}$, then we may apply the Borel–Cantelli lemma to see that the events $A_{N,\epsilon}$ occur for all large enough N μ-almost surely.

Remark 2. That the estimates break down when $\epsilon = \frac{1}{2}$ is natural since the typical distance between zeros of s_N is on the order of $N^{-1/2}$. So if most zeros are paired to a unique nearby critical point, we expect to see quite few critical points a large constant times $N^{-1/2}$ away from ξ.

The novel technical aspect of the proof of Theorem 2 is the precise estimates on Π_N^ξ and its derivatives with respect to ∇^σ are given in Corollaries 1 and 2 in Section 7. We also refer the reader to Sections 1.7 and 3.2 in [15] for more details.

We give a brief explanation for the pairing of zeros and critical points from the perspective of the scaling asymptotics of Bergman kernels. To study the local behavior of $s_N \in HGE_N^\xi(L,h)$ near ξ, it is convenient to write

$$s_N(z) = g_N(z) \cdot e_\xi^N(z),$$

where e_ξ is a preferred frame for L at the point ξ in the sense of Definition 6. The covariance kernel $\Pi_{e_\xi}^N$ of the locally defined Gaussian random functions g_N is the Nth conditional Bergman kernel relative to e_ξ^N (see Section 6). Shiffman and Zelditch show in [21] that, up to rescaling by a power of N, for any $\epsilon > 0$,

$$\Pi_{e_\xi}^N\left(\xi + \frac{u}{N^{1/2}}, \xi + \frac{v}{N^{1/2}}\right) = e^{u\overline{v}} + O(N^{-1/2+\epsilon})$$

in the $C^\infty-$ topology in Kähler normal coordinates at ξ. The function $e^{u\overline{v}}$ is the covariance kernel for the Bargmann–Fock random analytic function

$$g(u) := \sum_{j\geq 0} a_j \cdot \frac{u^j}{\sqrt{j!}} \quad a_j \sim N(0,1)_{\mathbb{C}} \text{ i.i.d}$$

and the monomials $u^j/\sqrt{j!}$ are an orthonormal basis for the Bargmann–Fock space

$$\left\{ f \in L^2(\mathbb{C}, e^{-|z|^2}dz) \,\middle|\, \overline{\partial} f = 0 \right\}.$$

The Gaussian random function f has been studied extensively by Sodin and Tsirelson (cf. e.g., [18, 23–26]). The C^∞ convergence of covariance kernels means the local statistics of g_N in a $N^{-1/2}$-neighborhood of ξ are asymptotically those of g. The frame e_ξ is adapted to the metric h but not to the connection ∇^σ, however. To see this, we write

$$\nabla^\sigma = \nabla^{h^N} + N\partial\phi_\sigma,$$

where ∇^{h^N} is the Chern connection of h^N. If we fix a Kähler normal coordinate z for h near ξ and introduce a new variable $u = N^{1/2}z$, then in these new coordinates

$$\nabla^\sigma = N^{1/2}\partial\phi_\sigma(\xi) + O(1)$$

as $N \to \infty$. Hence, if $d\phi_\sigma(\xi) \neq 0$, differentiation with respect to ∇^σ becomes multiplication by the non-zero constant $\frac{\partial\phi_\sigma}{\partial z}(\xi)$ in the $N \to \infty$ limit. This makes zeros and critical points indistinguishable and explains why they are paired.

2. Discussion

To explain the pairing of zeros and critical points, let us consider a degree N random polynomial drawn from the $SU(2)$ ensemble studied in [15, 21, 22, 24–26]

$$p_N(z) := \sum_{j=0}^{N} a_j \sqrt{\binom{N}{j}}\, z^j. \tag{2.1}$$

Here a_j are i.i.d standard complex Gaussian random variables. The law of p_N is $\gamma^N_{h_{\text{FS}}}$, where h_{FS} is the Fubini–Study metric on $\mathcal{O}(1) \twoheadrightarrow \mathbb{C}P^1$. $\gamma^N_{h_{\text{FS}}}$ is the unique centered Gaussian measure on $\mathcal{P}_N$ for which the expected empirical measure of zeros is uniform on $\mathbb{C}P^1$ (cf. Section 1.2 in [23]). The zeros and critical points of p_{50} are drawn in Figure 1. The colored lines are gradient flow lines for the random morse function $|p_{50}(z)|^2$, whose local minima and saddle points are the zeros and critical points of p_{50}, respectively. There are no local maxima since $|p_{50}(z)|^2$ is subharmonic. Flow lines of the same color terminate in the same zero or critical point.

2.1. Electrostatic explanation for pairing of zeros and critical points

We now explain why most zeros z_j are paired with unique nearby critical points c_j for $SU(2)$ polynomials. We also explain why $\arg z_j \approx \arg c_j$ and $0 < |z_j| - |c_j| \ll 1$. In fact, Theorem 1 shows that $|z_j| - |c_j| \approx N^{-1}$.

Let us distribute N positive and N negative charges on $\mathbb{C}P^1$ according to the divisor of p_N. That is, we place N positive delta charges at infinity and a single negative delta charge at each zero of p_N. Write $E_{p_N}(z) \in T_z^*\mathbb{C}P^1$ for the resulting electric field at z. As explained in Section 2.2, the critical point equation $\frac{d}{dz}p_N(z) = 0$, is equivalent to $E_{p_N}(z) = 0$.

Suppose that $p_N(\xi) = 0$ for some $\xi \neq 0$. The remaining zeros of p_N are, on average, uniformly distributed on $\mathbb{C}P^1$, and the average electric field they produce is therefore zero. For z near ξ, we expect by the central limit theorem that the contribution to $E_{p_N}(z)$ from the remaining zeros is on the order of $N^{1/2}$. To leading order in N, $E_{p_N}(z)$ is thus the deterministic order of N contribution from the N positive delta charges at infinity and the single negative delta charge at ξ.

The Coulomb force in 1 complex dimension at distance r decays like r^{-1}. Hence, for a configuration of N positive charges at infinity and one negative charge at ξ, a point of equilibrium for the electric field exists at a point z a distance of order N^{-1} away from ξ in the direction of the line from infinity to ξ. This is the electrostatic explanation for the pairing of zeros and critical points shown in Figures 1 and 2.

The pairing of zeros and critical points breaks down near the origin (the south pole) in Figure 1 because the electric field from the N positive charges at infinity vanishes at the south pole. Critical points near $\xi = 0$ are therefore determined by the locations of zeros with small modulus.

2.2. Electrostatics on Riemann surfaces

We describe a theory of electrostatics on a closed Riemann surface Σ that depends only on its complex structure. We will see that solutions to the critical point equation $d\gamma = 0$ for a meromorphic function $\gamma : \Sigma \to \mathbb{C}P^1$ are precisely points of equilibrium for the electric field on Σ from charges distributed according to its divisor

$$D := \mathrm{Div}(\gamma) = \sum_{\gamma(z)=0} m(z)\delta_z - \sum_{\gamma(w)=\infty} m(w)\delta_w.$$

Here $m(\cdot)$ denotes the order of the relevant zero or pole of γ. To begin, observe that $d\gamma = 0$ is equivalent to $d\log|\gamma| = 0$ as long as γ has simple zeros. Let $\Delta = \frac{i}{\pi}\partial\overline{\partial}$ be the Laplacian mapping $\Omega^0(\Sigma)$ to $\Omega^{1,1}(\Sigma)$. By the Poincaré–Lelong formula, $G(z, D) := \log|\gamma(z)|$, solves

$$\Delta G(z, D) = D. \tag{2.2}$$

This is the analog of Poisson's equation, which says that the Laplacian of the Coulomb potential gives the charge density.

Definition 2. The electric co-field at z from charge distribution D is

$$E_\gamma(z) := dG(z, D_\gamma) \in T_z^*\Sigma.$$

Since Σ is compact, the equation $\Delta G = f$ has a solution only if $\int_\Sigma f = 0$. The price we pay for using only the complex structure of Σ to define E_γ is that we may work only with electrically neutral charge distributions. As noted before, the critical point equation $d\gamma(z) = 0$ is generically equivalent to $d \log |\gamma(z)| = 0$ and hence to $E_\gamma(z) = 0$.

2.3. Meaning of ϕ_σ

The quantity $d\phi_\sigma$ plays a key role in our results. To see why, note that

$$\frac{i}{2\pi}\partial\overline{\partial}\log \left\|\sigma^N(z)\right\|_h^{-2} = N\omega_h - NZ_\sigma.$$

As in Section 0.2, ω_h is the curvature form of h and Z_σ is the current of integration over the zero set of σ. The term $N\omega_h$ is essentially $\mathbb{E}\left[Z_{s_N}\right]$ for $s_N \in HGE_N^\xi(L, h)$ (cf. e.g., Theorem 1 in [22], Lemma 3.1 in [19], and Lemma 2 in Section 5 of [15]). Let us write as in Section 2.1 $E_N(\xi)$ for the electric field at ξ from charge distributed according to $Z_{s_N} - \delta_\xi - NZ_\sigma$. Then

$$d\phi_\sigma(\xi) \approx \mathbb{E}\left[E_N(\xi)\right].$$

The contribution of the random zeros of s_N to E_N should heuristically be $N\omega_h$ plus a fluctuation on the order of $N^{1/2}$ by the central limit theorem. The condition that $|d\phi_\sigma(\xi)| \neq 0$ means average electric field $\mathbb{E}\left[E_N(\xi)\right]$ at ξ is of order N and is dominated by three deterministic contributions: the zero of s_N at ξ, the zero current NZ_σ of σ^N, and the *average* distribution of zeros of s_N. Points $z \in \Sigma$ for which $d\phi_\sigma(z) = 0$, e.g., play the same role as the origin for the $SU(2)$ ensemble (cf. the end of Section 2.1).

2.4. Smooth versus holomorphic critical points

The critical points we study solve the equation

$$\nabla^\sigma s_N(z) = 0 \quad \Longleftrightarrow \quad \frac{d}{dz}\left(\frac{s_N(z)}{\sigma^N(z)}\right) = 0. \tag{2.3}$$

Smooth critical points (cf. e.g., [10–12]), in contrast, are solutions of

$$\nabla^h s_N(z) = 0 \quad \Longleftrightarrow \quad \frac{d}{dz}\left\|s_N(z)\right\|_h = 0, \tag{2.4}$$

where ∇^h is the Chern connection of h. The two settings are qualitatively different. For instance, the zeros of $s_N \in HGE_N(L, h)$ repel (cf. e.g., the introductions in [3, 22]). Hence, since zeros and holomorphic critical points tend to appear in pairs, solutions to (2.3) repel as well. This can be seen directly by computing the two point function for holomorphic critical points, although we do not do this in the present paper. In contrast, Baber [1] showed that smooth critical points of s_N actually repel slightly. Further, the number of holomorphic critical points of a generic section is the Chern class of $K_\Sigma \otimes L^N$ plus N times the number of zeros of σ and hence depends only on L, N, and σ. The number of smooth critical points is, on the other hand, a non-degenerate random variable, whose expected value is $5c_1(L)N/3$ to leading order in N (cf. Theorem 1.3, Corollary 5, and Section 6 in [10]).

Smooth critical points were implicitly studied in the work of Nazarov et al. [18], where a so-called "gravitational allocation" was constructed between the counting measure for zeros of a Gaussian analytic function $f(z)$ and Lebesgue measure on $\mathbb{C}$. The allocation is achieved by gradient flow for the potential $\|f(z)\|_h^2$, where $h(z) = \frac{1}{\pi}e^{-|z|^2}$ is the usual Hermitian metric on $\mathbb{C} \times \mathbb{C} \twoheadrightarrow \mathbb{C}$. The saddle points for this potential are critical points of f with respect to the Chern connection of h. The analogous gravitational allocation in Figures 1 and 2 uses $|p_N(z)|^2$ as a potential, omitting the smooth metric factor. Finally, we mention that the expected distribution of critical *values* for smooth critical points was worked out in [13, 14].

3. Outline

The remainder of this paper is organized as follows. First, in Section 4, we give some background on meromorphic connections. Then, in Section 5, we establish notation to be used throughout. In Section 6 we recall relevant facts about Bergman kernels. Namely, in Sections 6.1 and 6.2, we recall their definition and in Sections 6.3 and 6.4, we recall their asymptotic expansions as given by Zelditch and Shiffman [20]. In Section 7, we derive asymptotics for derivatives with respect to ∇^σ of the (conditional) Bergman kernel. These asymptotics will be the key analytic formulas underlying the proof of Theorem 2, which is given in Section 8.

4. Meromorphic connections on $L \twoheadrightarrow \Sigma$

Definition 3. A meromorphic connection on $L \twoheadrightarrow \Sigma$ is a connection ∇ on L that maps holomorphic sections of L to sections of L with values in meromorphic 1-forms on Σ.

We study critical points of random sections of L and its tensor powers with respect to a special class of meromorphic connections. For $\sigma \in H^0_{\text{hol}}(\Sigma, L)$ the equation $\nabla^\sigma \sigma = 0$ defines the meromorphic connection ∇^σ given by

$$\nabla^\sigma s = d\left(\frac{s}{\sigma}\right) \cdot \sigma, \quad s \in H^0_{\text{hol}}(L). \tag{4.1}$$

This formula shows that ∇^σ introduces a pole at each zero of σ. Meromorphic connections L are natural generalizations of the holomorphic derivative $\frac{d}{dz}$. Indeed, if we write z_0 for the usual frame for $\mathcal{O}(1) \twoheadrightarrow \mathbb{C}P^1 \backslash \{[0:1]\}$. The section z_0^N induces the trivialization

$$\alpha_N : \mathcal{O}(N)\big|_{\mathbb{C}P^1 \backslash \{\infty\}} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}$$

which identifies $H^0_{\text{hol}}(\mathbb{C}P^1, \mathcal{O}(N))$ with $\mathcal{P}_N$, the polynomials of degree up to N. We may define then define a meromorphic connection $\nabla^{z_0} := \alpha_N^* d$ on $\mathcal{O}(N)$. We note that the section z_0^N corresponds to the constant polynomial 1 and hence is parallel for ∇^{z_0}, in agreement with our earlier notation. Moreover, z_0 vanishes only at $[0:1]$ and hence ∇^{z_0} has a simple pole at $[0:1]$. See Section 3 in [15] for more details.

5. Notation

In Section 0.2, we wrote $s_N = \sum_{j=1}^{d_N} a_j S_j \in HGE_N(L, h)$. Consider $\Phi_N : \Sigma \to \mathbb{C}P^{d_N - 1}$ defined by

$$\Phi_N(z) = [S_1(z) : \cdots : S_{d_N}(z)].$$

We will refer to Φ_N as the coherent states embedding generated by h. Since L is ample, the space $H^0_{\text{hol}}(\Sigma, L^N)$ is basepoint free for N large and hence Φ_N is well defined. The map Φ_N is an almost isometry

$$\left\| \omega_h - \frac{1}{N} \Phi_N^* \omega_{\text{FS}} \right\|_{C^k} = O_k(N^{-2}) \quad \forall k \geq 1.$$

Here ω_h is the curvature form of h and ω_{FS} is the Fubini-study metric on $\mathbb{C}P^{d_N}$. This result is Corollary 3 in [28] and was proved independently by Catlin [5]. The result in [28] is stated with N^{-1} on the right-hand side but the same method actually gives N^{-2} (cf. [16, (5.1.23)]). Convergence in the C^2 topology was proved by Tian [27]. We will write

$$s_N = \langle a, \Phi_N \rangle$$

for $a = (a_1, \ldots, a_{d_N})$ a standard complex Gaussian vector on $H^0_{\mathrm{hol}}(\Sigma, L^N)$. We assume fixed throughout a distinguished section $\sigma \in H^0_{\mathrm{hol}}(\Sigma, L)$, which is parallel for the meromorphic connection ∇^σ with respect to which we compute critical points. We will write

$$S_j = f_j \cdot \sigma^N$$

whenever we do local computations. Theorem 2 concerns sections

$$s_N(z) = \sum_{j=1}^{d_N-1} a_j S_j^\xi(z) \in HGE_N^\xi(L, h),$$

where, as before, $\{S_j^\xi\}$ is an orthonormal basis for H_N^ξ (defined in Section 0) with respect to the inner product (0.4). As in the unconditional ensemble, we will write

$$s_N(z) = \left\langle a, \Phi_N^\xi(z) \right\rangle,$$

where $\Phi_N^\xi : \Sigma \to \mathbb{C}P^{d_N-2}$ is given in homogeneous coordinates by $\Phi_N^\xi(z) = [S_1^\xi(z) : \cdots : S_{d_N-1}^\xi(z)]$ and set $S_N^\xi = f_j^\xi \cdot \sigma^N$. Abusing notation, Φ_N^ξ will sometimes denote the locally defined map $\Phi_N^\xi : \Sigma \to \mathbb{C}^{d_N-1}$ given by

$$\Phi_N^\xi(z) = \left(f_1^\xi(z), \ldots, f_{d_N-1}^\xi(z)\right). \tag{5.1}$$

6. Bergman and Szegö kernels

We use this section to give some background on Bergman and Szegö kernels. In Section 6.1, we define the Nth Bergman kernel Π_N of (L, h). The related conditional Bergman kernel Π_N^ξ is the covariance kernel of the Gaussian field $s_N \in HGE_N^\xi(L, h)$. In Section 6.2, we introduce the conditional normalized Bergman kernel P_N^ξ, which plays a key role in the proof of Theorem 2. Our main technical tool is the C^∞ asymptotic expansion for Π_N derived by

Shiffman and Zelditch [20, 21], which we recall in Section 6.4. Off-diagonal Bergman kernel asymptotic expansions are given also in [7, 16]. To explain this asymptotic expansion we recall in Section 6.3 the principal S^1 bundle $X \twoheadrightarrow \Sigma$ associated to $(L, h) \twoheadrightarrow \Sigma$. The family of kernels Π_N are analyzed by lifting to X, where they are naturally interpreted as kernels for the Szëgo projector associated to X. In Section 7, we derive asymptotic expansions for derivatives of Π_N^ξ with respect to the meromorphic connection ∇^σ.

6.1. Definition of Π_N and Π_{σ^N}

We make the following

Definition 4. The covariance kernel for $s_N = \sum_{j=1}^{d_N} a_j S_j \in HGE_N(L, h)$ is called the Nth Bergman kernel for (L, h) :

$$\Pi_N(z, w) := \mathrm{Cov}(p_N(z), p_N(w)) = \sum_{j=0}^{N} S_N^j(z) \otimes \overline{S_N^j(w)} \in H^0_{\mathrm{hol}} (\Sigma \times \overline{\Sigma}, L^N \boxtimes \overline{L^N}). \tag{6.1}$$

The family of Bergman kernels Π_N is well-understood for a positive holomorphic line bundle $(L, h) \twoheadrightarrow M$ over a compact Kähler manifold M (cf. [20, 21]). If we fix a local frame e for L^N and write $S_j = \gamma_j \cdot e$, then we can make the following:

Definition 5. The Nth Bergman kernel for (L, h) relative to the frame e is

$$\Pi_e(z, w) := \sum_{j=1}^{d_N} \gamma_j(z)\overline{\gamma_j(w)}.$$

We observe that $\Pi_e(z, w) \cdot e(z) \otimes \overline{e(w)} = \Pi_N(z, w)$. Writing

$$s_N(z) = \sum_{j=0}^{d_N-1} a_j S_j^\xi(z) \in HGE_N^\xi(L, h)$$

we define the Nth conditional Bergman kernel to be

$$\Pi_N^\xi(z, w) := \sum_{j=1}^{d_N-1} S_j^\xi(z) \otimes \overline{S_j^\xi(w)}.$$

Similarly, writing $S_j^\xi = \gamma_j^\xi \cdot e$ for any frame e of L^N, we define the Nth conditional Bergman kernel relative to the frame e to be

$$\Pi_e^\xi := \sum_{j=1}^{d_N-1} \gamma_j^\xi(z)\overline{\gamma_j^\xi(w)}$$

and note that

$$\Pi_N^\xi(z,w) = \Pi_e^\xi(z,w)\cdot e(z)\otimes \overline{e(w)}. \tag{6.2}$$

6.2. Normalized Bergman kernel

The local statistics of the critical points for $s_N \in HGE_N^\xi(L,h)$ are conveniently expressed in terms of the normalized Bergman kernel for the conditional ensemble $HGE_N^\xi(L,h)$ (cf. [15, 21, 22])

$$P_N^\xi(z,w) := \frac{\left\|\nabla^\sigma \otimes \overline{\nabla^\sigma}\Pi_N^\xi(z,w)\right\|_{h^N}}{\sqrt{\left\|\nabla^\sigma \otimes \overline{\nabla^\sigma}\Pi_N^\xi(z,z)\right\|_{h^N}\left\|\nabla^\sigma \otimes \overline{\nabla^\sigma}\Pi_N^\xi(w,w)\right\|_{h^N}}}. \tag{6.3}$$

The estimates on P_N^ξ in Corollary 2 will be used in the proof of Theorem 2. Probabilitistically, $P_N^\xi(z,w)$ is the correlation between the random variables $\nabla^\sigma s_N(z)$ and $\nabla^\sigma s_N(w)$ for $s_N \in HGE_N^\xi(L,h)$.

6.3. Principal S^1 bundle

Consider a positive line bundle $(L,h) \twoheadrightarrow M$ over a compact Käher manifold and an orthonormal basis $\{S_j\}_{j=0}^{d_N}$ for $H^0_{\text{hol}}(L^N)$ with respect to the inner product (0.4). The Nth Bergman kernel $\Pi_N(z,w) = \sum_{j=0}^{d_N} S_j(z)\otimes\overline{S_j(w)}$ is studied in [20] by lifting sections $s\in H^0_{\text{hol}}(L^{\otimes N})$ to S^1-equivariant functions on the principal S^1 bundle associated to (L,h). More precisely, we write h^* for the dual metric on the dual bundle L^* and define the principle S^1 bundle $X \twoheadrightarrow M$ by

$$X := \{v \in L^* \mid \|v\|_{h^*} = 1\}.$$

Note that X is the boundary the unit co-disc $D = \{v\in L^* \mid \|v\|_{h^*} \le 1\}$. The positivity of h ensures that D is a strictly pseudo-convex manifold. We denote by $\widehat{s}$ the lift of a section s to the function $\widehat{s}(v) := v^{\otimes N}(s)$ on X.

Writing $s = f \cdot e^{\otimes N}$ for a local frame e of L, and using e^* to trivialize X, we may write

$$(6.4) \qquad \widehat{s}(\theta, z) := e^{iN\theta} \left\| e(z) \right\|_h^N \cdot f(z).$$

Observe that

$$(6.5) \qquad |\widehat{s}(\theta, z)| = \left\| s(z) \right\|_{h^N}.$$

The lifted Bergman kernel is then

$$\widehat{\Pi}_N(\alpha, z; \beta, w) := \sum_{j=0}^{N} \widehat{S_j}(\alpha, z)\overline{\widehat{S_j}(\beta, w)}.$$

Since $\sum_N \widehat{\Pi}_N$ is the Szegö kernel for the Hardy space of X, one may apply the parametrix construction of Boutet de Monvel and Sjöstrand from [4] to study $\widehat{\Pi}_N$. We refer the reader to Section 1.2 of [20] for further details. In this paper, we are interested in the special case $M = \Sigma$, a closed Riemann surface. The following two definitions from Section 2.2 of [21] allow us to formulate the C^∞ complete asymptotic expansion for $\widehat{\Pi}_N$ derived there.

Definition 6. Fix $\xi \in \Sigma$ and e a frame for L in a neighborhood U containing ξ. The frame e is called a preferred frame for h at ξ if in a Kähler normal coordinate $z : U \to \mathbb{C}$ centered at ξ, we have

$$\left\| e(z) \right\|_h = 1 - \frac{1}{2} |z|^2 + o(|z|^2), \quad \text{as } z \to 0.$$

Definition 7. Fix $\xi \in \Sigma$, a Kähler normal coordinate $\psi : U \to \mathbb{C}$ centered at ξ, and a preferred frame e for h at ξ. Denoting by π the projection map $\pi : X \twoheadrightarrow M$, a Heisenberg coordinate on X centered at ξ is a coordinate $\rho : S^1 \times \mathbb{C} \to \pi^{-1}(U)$ given by

$$(6.6) \qquad \rho(\theta, \psi(z)) = e^{i\theta} \left\| e(z) \right\|_h e^*(z).$$

A Heisenberg coordinate on X is therefore the choice of a Kähler normal coordinate on Σ centered at ξ and a trivialization of X by a preferred frame at ξ. The role of Heisenberg coordinates is that in these special local coordinates, the Szegö kernels $\widehat{\Pi}_N$ have a universal scaling limit depending only on $\dim_{\mathbb{C}} M$. We refer the interested reader to Section 1.3.2 of [3] for more details.

6.4. Asymptotic expansion for Π_N

We now recall for the particular case of $L \twoheadrightarrow \Sigma$ the on-diagonal, near off-diagonal, and far off-diagonal asymptotics for the Szegö kernels Π_N derived by Shiffman and Zelditch [20, 21]. We note that the on-diagonal asymptotics were obtained also by Catlin [5] and off-diagonal expansions appeared in [7, 16]. The following is a special case of Theorem 2.4 from [21].

Theorem 3. *Fix Heisenberg coordinates on X around $\xi \in \Sigma$. Suppose $b > \sqrt{j+2k}$ and $j, k \geq 0$:*
1. Far off-diagonal. *For $d(z,w) > b\left(\frac{\log N}{N}\right)^{1/2}$ and $j \geq 0$, we have*

$$\nabla^j \widehat{\Pi}_N(\alpha, z; \beta, w) = O(N^{-k}), \tag{6.7}$$

where ∇^j denotes the horizontal lift to X of any j mixed derivatives in $z, \overline{z}, w, \overline{w}$.
2. Near off-diagonal. *Let $\epsilon > 0$. In Heisenberg coordinates (see Definition 7) centered at ξ, we have for $|z| + |w| < b\left(\frac{\log N}{N}\right)^{1/2}$*

$$\widehat{\Pi}_N(\alpha, z; \beta, w) = e^{iN\left[\alpha - \beta + z\cdot\overline{w} - \frac{1}{2}\left(|z|^2 + |w|^2\right)\right]}[1 + R_N(z,w)], \tag{6.8}$$

where

$$\nabla^j R_N(z,w) = O(N^{-1/2+\epsilon}) \tag{6.9}$$

and the implied constant in equation (6.9) is allowed to depend on ϵ. The remainder R_N satisfies in addition, for $j = 0, 1, 2$

$$\left|\nabla^j R_N(z,w)\right| = O(|z-w|^{2-j} N^{-1/2+\epsilon}) \tag{6.10}$$

uniformly for $|z| + |w| < \left(\frac{\log N}{N}\right)^{1/2}$ with the implied constants are independent of N.

7. Bergman kernel derivative estimates

We now apply the asymptotic expansions for the Bergman kernel from Section 6.4 to explicitly write asymptotics for its derivatives with respect to a meromorphic connection. The results we use to prove Theorem 2 are Corollaries 1 and 2, which both following from Lemma 1.

Throughout this section, we fix a positive holomorphic line bundle $(L, h) \twoheadrightarrow \Sigma$ over a closed Riemann surface. We fix $\sigma \in H^0_{\text{hol}}$ and denote by ∇^σ on $L^{\otimes N}$ the corresponding meromorphic connection defined in (4.1). We continue to write $\Pi^\xi_{\sigma^N}$ for the Nth conditional Bergman kernel relative to σ^N (defined in Section 6.1), and in Kähler normal coordinates around ξ, we will write

$$\widehat{u} := u \cdot N^{-1/2}.$$

Our main technical result is Lemma 1.

Lemma 1. *In Kähler normal coordinates around ξ, the following expression is valid uniformly for $|\widehat{z}|, |\widehat{w}| < \sqrt{\log N}$*

$$\frac{d^2}{dz\, d\overline{w}} \Pi^\xi_{\sigma^N}(\widehat{z}, \widehat{w}) = \frac{N^3}{\pi} e^{\frac{1}{2}\left(N[\phi_\sigma(\widehat{z}) + \phi_\sigma(\widehat{w}) + i\widetilde{\gamma_\sigma}(\widehat{z}) - i\widetilde{\gamma}(\widehat{w})] + 2z\overline{w} - |z|^2 - |w|^2\right)} \cdot T_N(z, w) \cdot (1 + R_N(z, w)). \tag{7.1}$$

We've written

$$\begin{aligned} T_N(z,w) = \left(1 - e^{-z\overline{w}}\right) &\left[1 + \frac{1}{4}\left(\frac{\partial \phi_\sigma}{\partial z}(\widehat{z})N^{1/2} + \frac{\partial \gamma_\sigma}{\partial z}(\widehat{z})N^{1/2} - \overline{z} + 2\overline{w}\right)\right. \\ &\times \left.\left(\frac{\partial \phi_\sigma}{\partial \overline{w}}(\widehat{w})N^{1/2} + \frac{\partial \gamma_\sigma}{\partial \overline{w}}(\widehat{w})N^{1/2} - w + 2z\right)\right] \\ &+ e^{-z\overline{w}}\left[1 - z\overline{w} + \frac{z}{2}\left(\frac{\partial \phi_\sigma}{\partial z}(\widehat{z})N^{1/2} + \frac{\partial \gamma_\sigma}{\partial z}(\widehat{z})N^{1/2} - \overline{z} + 2\overline{w}\right)\right. \\ &+ \left.\frac{\overline{w}}{2}\left(\frac{\partial \phi_\sigma}{\partial \overline{w}}(\widehat{w})N^{1/2} + \frac{\partial \gamma_\sigma}{\partial \overline{w}}(\widehat{w})N^{1/2} - w + 2z\right)\right]. \end{aligned}$$

We have set γ_σ to be the "leading harmonic part of ϕ_σ"

$$\gamma_\sigma(z, \overline{z}) := \phi_\sigma(\xi) + \left.\frac{\partial \phi_\sigma}{\partial z}\right|_\xi \cdot z + \left.\frac{\partial \phi_\sigma}{\partial \overline{z}}\right|_\xi \cdot \overline{z} + \frac{1}{2}\left[\left.\frac{\partial^2 \phi_\sigma}{\partial z^2}\right|_\xi \cdot z^2 + \left.\frac{\partial^2 \phi_\sigma}{\partial \overline{z}^2}\right|_\xi \cdot \overline{z}^2\right]$$

and we've written $\widetilde{\gamma_\sigma}$ for its harmonic conjugate. Finally, as in Theorem 3, the remainders $R_N(z, w)$ satisfy the estimates (6.10).

We prove Lemma 1 below. The proof of Theorem 2 relies on two consequences of Lemma 1, given in Corollaries 1 and 2. To state them, we consider $\xi \in \Sigma$ satisfying $d\phi_\sigma(\xi) \neq 0$.

Corollary 1. *Consider $\xi \in \Sigma$ satisfying $d\phi_\sigma(\xi) \neq 0$. With the notation of Lemma 1 and for any $\epsilon > 0$, the following expression is valid uniformly for $|z| < \sqrt{\log N}$*

(7.2)
$$\log\left[\frac{d^2}{dz\,d\overline{w}}\bigg|_{z=w}\Pi^{\xi}_{\sigma^N}(\widehat{z},\widehat{w})\right] = \mathrm{Const} + N\phi_\sigma(\widehat{z}) + \log T_N(z,z) + O(N^{-1/2+\epsilon}),$$

where Const is a constant depending on N and, if we write $z = re^{i\theta}$, we have for r small

(7.3)
$$\frac{d}{dz}\bigg|_{z=re^{i\theta}}\log T_N(z,z) = \frac{r^{-1}\left(e^{-i\theta} + O(N^{-1/2})\right)}{1 + r^{-2}N\cdot\left|\frac{\partial\phi_\sigma}{\partial z}(\xi)\right|^2 + O(r^{-1}N^{-1/2}) + O(r^{-2}N^{-2})}.$$

Proof. Equation (7.2) follows from setting $z = w$ in (7.1). To derive (7.3), we put $z = w$ in the expression for $T_N(z,w)$ given in Lemma 1 to see that

$$T_N(z,z) = \left(1 - e^{-|z|^2}\right)\left(1 + \frac{1}{4}\left|\frac{\partial\phi_\sigma}{\partial z}(\widehat{z})N^{1/2} + \frac{\partial\gamma_\sigma}{\partial z}(\widehat{z})N^{1/2} + z\right|^2\right) + e^{-|z|^2}\mathrm{Re}\left(1 + \overline{z}\left(\frac{\partial\phi_\sigma}{\partial z}(\widehat{z})N^{1/2} + \frac{\partial\gamma_\sigma}{\partial z}(\widehat{z})N^{1/2}\right)\right).$$

Since we are in Kähler normal coordinates, we have

$$\frac{1}{2}\left(\frac{\partial\phi_\sigma}{\partial z}(\widehat{z})N^{1/2} + \frac{\partial\gamma_\sigma}{\partial z}(\widehat{z})N^{1/2} + z\right) = \frac{\partial\phi_\sigma}{\partial z}(\xi)N^{1/2} + O(1).$$

Since $d\phi_\sigma(\xi) \neq 0$, we may set $z = re^{i\theta}$ and Taylor expand to find that

$$T_N(z,z) = r^2\left|\frac{\partial\phi_\sigma}{\partial z}(\xi)\right|^2 N\left(1 + N\left|\frac{\partial\phi_\sigma}{\partial z}(\xi)\right|^2 r^{-2} + O(r^{-1}N^{-1/2}) + O(N^{-1})\right).$$

Similarly, we find that

$$\frac{d}{dz}T_N(z,z) = r^2\left|\frac{\partial\phi_\sigma}{\partial z}(\xi)\right|^2\cdot N^{-1}r^{-1}\left(e^{-i\theta} + O(rN^{-1/2})\right).$$

Combining the expressions for $T_N(z,z)$ and $\frac{d}{dz}T_N(z,z)$ yields (7.3). □

Lemma 1 also yields the following estimates on P_N^ξ, defined in (6.3).

Corollary 2. *Let $\xi \in \Sigma$ be such that $d\phi_\sigma(\xi) \neq 0$. In a Kähler normal coordinate centered at ξ, write $\widehat{u} = u \cdot N^{-1/2}$. Then, for any $\epsilon > 0$, we have the following C^∞ expansion:*

$$P_N^\xi(\widehat{z},\widehat{w})^2 = \frac{\left|1-e^{z\overline{w}}\right|^2}{\left(1-e^{-|z|^2}\right)\left(1-e^{-|w|^2}\right)} \cdot e^{-|z-w|^2}\left(1+\widetilde{R}_N(z,w)\right). \tag{7.4}$$

The remainder $\widetilde{R}_N(z,w)$ satisfies the estimates (6.10). In particular, we find that for $|z|, |w|$ small, we have for some positive constants C_j, $j = 1, 2$,

$$\frac{d}{dz}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right) = (\overline{z}-\overline{w})\left(C_1 + O(N^{-1/2+\epsilon})\right), \tag{7.5}$$

$$\frac{d}{dw}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right) = (z-w)\left(C_1 + O(N^{-1/2+\epsilon})\right), \tag{7.6}$$

$$\frac{d^2}{dz\,dw}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right) = O(N^{-1/2+\epsilon}), \tag{7.7}$$

$$1-e^{-2\Lambda(z,w)} = 1-P_N^\xi(z,w)^2 = |z-w|^2\left(C_2 + O(|z-w|^2)\right). \tag{7.8}$$

Moreover, the constants C_j are uniformly away from 0 and $+\infty$ independently of N.

Proof. Notice that

$$T(z,w) = N^{1/2}\left(1+Q_N(z,w)\right),$$

where for any $\epsilon > 0$ and $j,k,l,m \geq 0$, we have

$$\frac{\partial Q_N}{\partial z^j \partial \overline{z}^k \partial w^l \partial \overline{w}^m} Q_N(z,w) = O(N^{-1/2+\epsilon})$$

uniformly for $|z| < \log N$. Thus, writing

$$P_N^\xi(\widehat{z},\widehat{w})^2 = \frac{\left|\frac{d^2}{dz\,d\overline{w}}\Pi_{\sigma^N}^\xi(\widehat{z},\widehat{w})\right|^2}{\frac{d^2}{dz\,d\overline{w}}\Big|_{z=w}\Pi_{\sigma^N}^\xi(\widehat{z},\widehat{w}) \cdot \frac{d^2}{dz\,d\overline{w}}\Big|_{w=z}\Pi_{\sigma^N}^\xi(\widehat{z},\widehat{w})} \tag{7.9}$$

and substituting expression (7.1) into (7.9) shows that

$$P_N^\xi(\widehat{z},\widehat{w}) = \frac{\left|1-e^{-z\overline{w}}\right| e^{-\frac{1}{2}|z-w|^2}}{\left(1-e^{-|z|^2}\right)^{1/2}\left(1-e^{-|w|^2}\right)^{1/2}} \cdot \left(1+\widetilde{R}_N(z,w)\right)$$

with $\widetilde{R}_N(z,w)$ and its derivatives bounded by a constant times $N^{-1/2+\epsilon}$.

If we fix w and view $P_N^\xi(z,w)$ as a function of $z,\overline{z}$, we see that P_N^ξ is maximized on the diagonal $z=w$ and achieves a value of 1. Estimates (7.5) and (7.6) now follows. To verify (7.8) we may write

$$P_N^\xi\left(\widehat{z},\widehat{w}\right)^2 = e^{-\Lambda(z,w)},$$

where we may write $\Lambda(z,w) := -\log P_N^\xi\left(\widehat{z},\widehat{w}\right)$ as

$$\frac{1}{2}\left(\log(1-e^{-|z|^2})+\log(1-e^{-|w|^2})-\log\left|1-e^{-z\overline{w}}\right|+|z-w|^2\right)$$

plus $\log(1+\widetilde{R}_N(z,w))$. Therefore,

$$\frac{d^2}{dz\,dw}P_N^\xi\left(\widehat{z},\widehat{w}\right) = \left(-\frac{d^2}{dz\,dw}\Lambda(z,w)+\frac{\partial}{\partial z}\Lambda(z,w)\frac{\partial}{\partial w}\Lambda(z,w)\right)e^{-\Lambda(z,w)}.$$

Differentiating $\Lambda(z,w)$ and using the remainder estimates (6.10) completes the derivation. □

We now turn to the proof of Lemma 1.

Proof of Lemma 1. We fix Kähler normal coordinates around ξ. Our proof is based on Lemma 2. We continue to write ϕ_σ for the Kähler potential of ω_h relative to σ :

$$\phi_\sigma(z) = \log\|\sigma(z)\|_h^{-2}.$$

We will also write γ_σ for the "leading harmonic part" of ϕ_σ (defined in the statement of Lemma 1) and write $\widetilde{\gamma_\sigma}$ for its harmonic conjugate.

Lemma 2. *For each N, we have*

$$\Pi_{\sigma^N}^\xi(z,w)=E_N(z,\alpha,w,\beta)\cdot\left(\widehat{\Pi}_N(z,\alpha,w,\beta)-\frac{\widehat{\Pi}_N(z,\alpha,\xi,0)\overline{\widehat{\Pi}_N(\xi,0,w,\beta)}}{\left|\widehat{\Pi}_N(\xi,\alpha,\xi,\beta)\right|}\right),$$

where

$$E_N(z,\alpha,w,\beta) = e^{\frac{N}{2}(\phi_\sigma(z)+\phi_\sigma(w)-i\widetilde{\gamma_\sigma}(w)+i\widetilde{\gamma_\sigma}(z))+iN(\alpha-\beta)}.$$

Assuming this lemma for the moment, we substitute into (7.11) the C^∞ asymptotic expansion

$$\widehat{\Pi}_N(\widehat{z},\alpha,\widehat{w},\beta) = e^{-z\overline{w}+\frac{1}{2}\left(|z|^2+|w|^2\right)} + O(N^{-1/2+\epsilon})$$

from (6.8), which is valid in Heisenberg coordinates on ξ, to obtain the following expression for $\Pi^{\xi}_{\sigma^N}(\widehat{z},\widehat{w})$:

(7.10)

$$\frac{N}{\pi}e^{\frac{1}{2}\left(N[\phi_\sigma(\widehat{z})+\phi_\sigma(\widehat{w})+i\widetilde{\gamma_N}\sigma(\widehat{z})-i\widetilde{\gamma_\sigma}(\widehat{w})]+2z\overline{w}-|z|^2-|w|^2\right)}\left(1-e^{-z\overline{w}}\right)\left(1+R_N(z,w)\right).$$

Differentiating this expression in z and in $\overline{w}$ shows that

$$\begin{aligned}\frac{d^2}{dz\,d\overline{w}}\Pi^{\xi}_{\sigma^N}(\widehat{z},\widehat{w}) =& \frac{N^3}{\pi}e^{\frac{1}{2}\left(N[\phi_\sigma(\widehat{z})+\phi_\sigma(\widehat{w})+i\widetilde{\gamma_\sigma}(\widehat{z})-i\widetilde{\gamma_\sigma}(\widehat{w})]+2z\overline{w}-|z|^2-|w|^2\right)}\\ &\times T(z,w)\cdot(1+R_N(z,w))\end{aligned}$$

as desired. We now turn to the proof of Lemma 2. □

Proof of Lemma 2. From (6.2), we see immediately that

$$\widehat{\Pi}^{\xi}_N(z,\alpha,w,\beta) = \Pi^{\xi}_{\sigma^N}(z,w)\cdot\widehat{\sigma}_N(z,\alpha)\otimes\widehat{\overline{\sigma^N}}(w,\beta). \tag{7.11}$$

We therefore start with the following:

Claim. *Fix $\xi\in\Sigma$. In Heisenberg coordinates centered at ξ on X, we have*

$$\widehat{\sigma}_N(z,\alpha)\otimes\widehat{\overline{\sigma^N}}(w,\beta) = E_N(z,\alpha,w,\beta)^{-1}. \tag{7.12}$$

Proof. Define the frame

$$e_\xi := e^{\frac{N}{2}(\gamma_\sigma+i\widetilde{\gamma_\sigma})}\cdot\sigma^N$$

for L^N near ξ. We have, in the sense of Definition 6, that e_ξ is a preffered frame for L^N at ξ. Therefore, in Heisenberg coordinates centered at ξ on X, we have

$$\widehat{\sigma^N}(z,\alpha) = \|e_\xi(z)\|_h\,e^{-\frac{N}{2}(\gamma_\sigma+i\widetilde{\gamma_\sigma})}\cdot e^{iN\alpha}.$$

Using that $\|e_\xi(z)\|_h = e^{-\frac{N}{2}(\gamma_\sigma(z)-\phi_\sigma(z))}$, we conclude

$$\widehat{\sigma^N}(z,\alpha) = e^{-\frac{N}{2}(\phi_\sigma(z)+i\widetilde{\gamma_\sigma}(z))+iN\alpha}. \tag{7.13}$$

Applying this formula to the lifts of $\sigma^N(z)$ and $\overline{\sigma_N}(w)$ to X and taking their tensor product completes the argument. □

To verify (7.11) it therefore remains to prove that

$$\widehat{\Pi}_N^\xi(z,\alpha,w,\beta) = \widehat{\Pi}_N(z,\alpha,w,\beta) - \frac{\widehat{\Pi}_N(z,\alpha,\xi,0)\overline{\widehat{\Pi}_N(\xi,0,w,\beta)}}{\left|\widehat{\Pi}_N(\xi,\alpha,\xi,\beta)\right|}. \tag{7.14}$$

This is precisely equation (27) from [22]. For the reader's convenience, we reproduce the proof. To do this, we introduce, as in the proofs of Lemma 4 Section 8 of [15] and Proposition 3.9 of [22], the "coherent state" at ξ. To do this,

$$\Psi_N^\xi(z) := \frac{\Pi_{\sigma_N}(z,\xi)}{\Pi_{\sigma_N}(\xi,\xi)^{1/2}} \cdot \sigma_N(z) \in H^0_{\text{hol}}(\Sigma, L^N).$$

We recall from Section 6.1 that Π_{σ_N} is the *unconditional* Nth Bergman kernel relative to σ_N, which we wrote as $\Pi_{\sigma_N}(z,w) = \sum_{j=1}^{d_n} f_j(z)\overline{f_j(w)}$. Hence

$$\Psi_N^\xi(z) = \frac{1}{\left(\sum_{j=1}^{d_N-1} |f_j(\xi)|^2\right)^{1/2}} \sum_{j=1}^{d_N} \overline{f_j(\xi)} f_j(z) \cdot \sigma_N(z).$$

Using the weighted L^2 inner product (0.4), we see that for every $s \in H^0_{\text{hol}}(\Sigma, L^N) = H_N$ satisfying $s(\xi) = 0$

$$\left\langle s, \Psi_N^\xi \right\rangle = 0$$

and that $\left\|\Psi_N^\xi\right\| = 1$. Therefore, Ψ_N^ξ spans the orthocomplement in H_N to H_N^ξ and

$$\Pi_N^\xi(z,w) = \Pi_N(z,w) - \Psi_N^\xi(z) \otimes \overline{\Psi_N^\xi(w)}.$$

Lifting this equation to X, (7.14) reduces to showing that

$$\widehat{\Psi}_N^\xi(z,\alpha) \otimes \widehat{\overline{\Psi_N^\xi}}(w,\beta) = \frac{\widehat{\Pi}_N(z,\alpha,\xi,0)\widehat{\Pi}_N(\xi,0,w,\beta)}{\widehat{\Pi}_N(\xi,\alpha,\xi,\beta)}.$$

To verify this equality, we note that, by formula (7.13) for the lift of σ_N,

$$\widehat{\Psi}_N^\xi(z,\alpha) = \frac{1}{\sqrt{\sum |f_j(\xi)|^2}} \sum \overline{f_j}(\xi) f_j(z) \cdot e^{-\frac{N}{2}(\phi(z)+i\widetilde{\gamma_N}(z))+iN\alpha}.$$

Hence, $\widehat{\Psi}_N^{\xi}(z,\alpha) \otimes \widehat{\overline{\Psi}_N^{\xi}}(w,\beta)$ is

$$\frac{\widehat{\Pi}_N(z,\alpha,\xi,0)E_N(z,\alpha,\xi,0)\widehat{\Pi}_N(\xi,0,w,\beta)E_N(\xi,0,w,\beta)}{\widehat{\Pi}_N(\xi,\alpha,\xi,\beta)E_N(\xi,\alpha,\xi,\beta)}E_N(z,\alpha,w,\beta)^{-1}.$$

Observing that

$$\frac{E_N(z,\alpha,\xi,0)E_N(\xi,0,w,\beta)}{E_N(\xi,\alpha,\xi,\beta)} = E_N(z,\alpha,w,\beta)$$

completes the proof. □

8. Proof of Theorem 2

We first recall the notation. Let $\sigma \in H^0_{\mathrm{hol}}(\Sigma, L)$ be fixed and consider

$$\phi_\sigma(z) = \log \|\sigma(z)\|_h^{-2}.$$

Consider $\xi \in \Sigma \backslash \{\sigma = 0\}$. In a Kähler normal coordinate around ξ, we wrote

$$\mathcal{N}_r := \#\left\{w \in D_{r\cdot N^{-1/2}}(\xi) \mid \nabla^{\sigma_N} s_N(w) = 0\right\},$$

where $D_R(\xi)$ is the disk of radius R in our fixed coordinate system. We fix an $\epsilon \in \left(0, \frac{1}{2}\right)$ and abbreviate $R_\pm := N^{-1/2+\pm\epsilon}$. We start with Lemmas 3 and 4, which are proved in Sections 8.1 and 8.2, respectively.

Lemma 3. *Fix $\epsilon \in \left(0, \frac{1}{2}\right)$, we have*

$$\mathbb{E}\left[\mathcal{N}_{R_+}\right] = 1 + O(N^{-\epsilon}). \tag{8.1}$$

Further,

$$\mathbb{E}\left[\mathcal{N}_{R_-}\right] = O(N^{-\epsilon}). \tag{8.2}$$

The implied constants in $O(N^{-\epsilon})$ depend only on ϵ.

Lemma 4. *For any $\epsilon > 0$, if $r \le N^{-1/2+\epsilon}$, then we have*

$$\mathrm{Var}[\mathcal{N}_r] = O(N^{-3/2+3\epsilon}).$$

The implied constant depends only on ϵ.

Theorem 2 follows from Lemmas 3 and 4 as follows. Since N_r is an integer valued random variance, by Chebyshev's inequality

$$\begin{aligned}P\left(\left(\mathcal{N}_{R_+}=1\cap\mathcal{N}_{R_-}=0\right)^c\right) &\leq P\left(\mathcal{N}_{R_+}\neq 1\right)+P\left(\mathcal{N}_{R_-}\neq 0\right)\\ &\leq P\left(\left|\mathcal{N}_{R_+}-\mathbb{E}\left[\mathcal{N}_{R_+}\right]\right|>1+O(N^{-\epsilon})\right)\\ &\quad+P\left(\left|\mathcal{N}_{R_-}-\mathbb{E}\left[\mathcal{N}_{R_-}\right]\right|>1+O(N^{-\epsilon})\right)\\ &\leq\frac{\mathrm{Var}[\mathcal{N}_{R^+}]+\mathrm{Var}[\mathcal{N}_{R_\epsilon^-}]}{1+O(N^{-\epsilon})}\\ &=O(N^{-3/2+3\epsilon}).\end{aligned}$$

8.1. Proof of Lemma 3

Since $\sigma(\xi)\neq 0$, we may write $s_N=p_N\cdot\sigma^N=\left\langle a,\Phi_N^\xi\right\rangle\cdot\sigma^N$, where $a=(a_0,\ldots,a_{d_N})$ is a standard Gaussian vector on H_N^ξ and $\Phi_N^\xi(z) = \left(f_1^\xi(z),\cdots,f_{d_N}^\xi(z)\right)$ is the Kodaira map defined in (5.1). We will write

$$\frac{d}{dz}\Phi_N^\xi(z)=\left(\frac{d}{dz}f_1^\xi(z),\ldots,\frac{d}{dz}f_{d_N}^\xi(z)\right).$$

The Nth conditional Bergman kernel relative to σ_N (introduced in Section 6.1) is therefore $\Pi_{\sigma_N}^\xi(z,w)=\sum_{j=1}^{d_n}f_j^\xi(z)\overline{f_j^\xi(w)}$.

Lemma 5. *We have*

$$\mathbb{E}\left[\mathcal{N}_r\right]=\frac{r}{2\pi}\int_0^{2\pi}\frac{d}{dz}\log\left(\left.\frac{d^2}{dz\,d\overline{w}}\right|_{z=w=re^{i\theta}}\Pi_{\sigma_N}^\xi(\widehat{z},\widehat{w})\right)\cdot e^{i\theta}d\theta. \tag{8.3}$$

Proof. Note that $\nabla^{\sigma_N}s_N=dp_N\cdot\sigma_N$. Equation (8.3) is then obtained by integrating by parts in

$$\mathbb{E}\left[\mathcal{N}_r\right]=\int_{D_r(\xi)}\frac{i}{2\pi}\frac{\partial^2}{\partial z\partial\overline{z}}\log\left.\frac{d^2}{dz\,d\overline{w}}\right|_{z=w}\Pi_{\sigma_N}^\xi(z,w)dz\wedge d\overline{z}$$

which is proved in [21, Proposition 2.1]. □

Combining formula (8.3) with (7.2), we find

$$\begin{aligned}\mathbb{E}\left[\mathcal{N}_r\right]=\frac{r}{2\pi}\int_0^{2\pi}&\left(N^{1/2}\left.\frac{\partial}{\partial z}\right|_{z=re^{i\theta}}\phi_\sigma(\widehat{z})+\left.\frac{d}{dz}\right|_{z=re^{i\theta}}\log T(z,z)\right.\\ &\left.+O(N^{-1/2+\epsilon})\right)e^{i\theta}d\theta.\end{aligned} \tag{8.4}$$

Since we are in Kähler normal coordinates, we have

$$N^{1/2}\frac{\partial\phi_\sigma}{\partial z}(\widehat{z}) = N^{1/2}\frac{\partial\phi_\sigma}{\partial z}(\xi) + z\frac{\partial^2}{\partial z^2}\phi_\sigma(\xi) + \overline{z} + O(|z|\,N^{-1/2}).$$

Hence,

$$\frac{r}{2\pi}\int_0^{2\pi} N^{-1/2}\frac{\partial}{\partial z}\bigg|_{z=re^{i\theta}}\phi_\sigma(\widehat{z})e^{i\theta}d\theta = O(r^2),$$

and consequently

$$\mathbb{E}\left[\mathcal{N}_r\right] = \frac{r}{2\pi}\int_0^{2\pi}\frac{d}{dz}\bigg|_{z=re^{i\theta}}\log T(z,z)e^{i\theta}d\theta + O(rN^{-1/2+\epsilon}).$$

Finally, using the estimate (7.3), we have

$$\frac{d}{dz}\bigg|_{z=R_-\cdot e^{i\theta}}\log T(z,z) = \frac{R_-^{-1}\left(e^{-i\theta}+O(N^{-1/2})\right)}{1+N^{2\epsilon}\left(1+O(N^{-\epsilon})\right)}.$$

Substituting this expression into the integral (8.4), we conclude (8.2). Similarly, the estimate (7.3) yields

$$\frac{d}{dz}\bigg|_{z=R_+\cdot e^{i\theta}}\log T(z,z) = \frac{R_+^{-1}\left(e^{-i\theta}+O(N^{-1/2})\right)}{1+O(N^{-\epsilon})}$$

from which we deduce (8.1). □

8.2. Proof of Lemma 4

Note that we may write

$$P_N^\xi(z,w) = \frac{\left|\frac{d^2}{dz\,d\overline{w}}\Pi_{\sigma_N}^\xi(z,w)\right|}{\sqrt{\frac{d^2}{dz\,dw}\big|_{w=z}\Pi_{\sigma_N}^\xi(z,w)\cdot\frac{d^2}{dz\,dw}\big|_{z=w}\Pi_{\sigma_N}^\xi(z,w)}},$$

where $\Pi_{\sigma_N}^\xi(z,w) = \sum_{j=1}^{d_n} f_j^\xi(z)\overline{f_j^\xi(w)}$ is the Bergman kernel relative to σ_N (cf. Section 6.1).

Lemma 6. *Let us write $\widehat{u} = u \cdot N^{-1/2}$. We have the following formula for* $\mathrm{Var}[\mathcal{N}_r]$:

$$\frac{r^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d}{dz}\frac{d}{dw}\bigg|_{z=re^{i\theta_1},\,w=re^{i\theta_2}} G(P_N^{\xi}(\widehat{z},\widehat{w}))e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2, \tag{8.5}$$

where

$$G(t) = \frac{\gamma^2}{4} - \frac{1}{4}\int_0^{t^2}\frac{\log(1-s)}{s}ds.$$

Proof. Using that

$$\mathcal{N}_r = \int_{D_{r\cdot N^{-1/2}}} Z_{p_N} = \int_{D_{r\cdot N^{-1/2}}} \frac{i}{2\pi}\partial_z\overline{\partial}_z \log\left|\frac{d}{dz}p_N(z)\right|^2$$

we have

$$\mathbb{E}\left[\mathcal{N}_r^2\right] = \frac{r^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d^2}{dz\,dw}\bigg|_{z=w=re^{i\theta}} \mathbb{E}\left[\log|p_N(\widehat{z})|\log|p_N(\widehat{w})|\right]e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2. \tag{8.6}$$

For any vector $v \in \mathbb{C}^{d_N-1}\backslash\{0\}$, we will write $\widetilde{v} = \frac{v}{\|v\|}$. We have

$$\log|p_N(\widehat{z})| = \log\left|\left\langle a, \Phi_N^{\xi}(\widehat{z})\right\rangle\right| = \log\left|\left\langle a, \widetilde{\Phi_N^{\xi}(\widehat{z})}\right\rangle\right| + \log\left\|\Phi_N^{\xi}(\widehat{z})\right\|$$

and similarly for $\log|p_N(\widehat{w})|$. We therefore find that

$$\mathbb{E}\left[\mathcal{N}_r^2\right] = \frac{r^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d^2}{dz\,dw}\bigg|_{z=re^{i\theta_1},\,w=re^{i\theta_2}} (E_1+E_2+E_3+E_4)\,e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2,$$

where

$$E_1(z,w) := \log\left\|\Phi_N^{\xi}(\widehat{z})\right\|\log\left\|\Phi_N^{\xi}(\widehat{w})\right\|,$$

$$E_2(z,w) := \mathbb{E}\left[\log\left|\left\langle a, \widetilde{\Phi_N^{\xi}(\widehat{z})}\right\rangle\right|\right]\cdot\log\left\|\Phi_N^{\xi}(\widehat{w})\right\|,$$

$$E_3(z,w) := \mathbb{E}\left[\log\left|\left\langle a, \widetilde{\Phi_N^{\xi}(\widehat{w})}\right\rangle\right|\right]\cdot\log\left\|\Phi_N^{\xi}(\widehat{z})\right\|,$$

$$E_4(z,w) := \mathbb{E}\left[\log\left|\left\langle a, \widetilde{\Phi_N^{\xi}(\widehat{z})}\right\rangle\right|\log\left|\left\langle a, \widetilde{\Phi_N^{\xi}(\widehat{w})}\right\rangle\right|\right].$$

Since the Gaussian measure a is unitarily invariant, we see that $E_2(z,w)$, $E_3(z,w)$ are independent of z, w, respectively and hence are annihilated by $\frac{d^2}{dz\,dw}$. Moreover

$$\frac{r^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d^2}{dz\,dw}\bigg|_{z=re^{i\theta_1},\,w=re^{i\theta_2}} E_1(z,w)e^{i(\theta_1+\theta_2)} = \mathbb{E}\left[N_r\right]^2. \tag{8.7}$$

In order to interpret $E_4(z,w)$, we now recall the following result.

Lemma 7 (Lemma 3.3 from [21]). *Let a be a standard Gaussian random vector in $\mathbb{C}^{N+1}$ and let $u, v \in C^{N+1}$ denote unit vectors. Then*

$$\mathbb{E}\left[\log|\langle a,u\rangle|\log|\langle a,v\rangle|\right] = G(|\langle u,v\rangle|),$$

where $\langle\cdot,\cdot\rangle$ is the usual Hermitian inner product on $\mathbb{C}^{N+1}$.

Observe that $\left|\left\langle \widetilde{\Phi_N^\xi}(\widehat{z}), \widetilde{\Phi_N^\xi}(\widehat{w})\right\rangle\right| = P_N^\xi(\widehat{z},\widehat{w})$. Putting this together with (8.7), we find that

$$\mathrm{Var}[\mathcal{N}_r] = \frac{r^2}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d^2}{dz\,dw}\bigg|_{z=re^{i\theta_1},\,w=re^{i\theta_2}} G(P_N^\xi(\widehat{z},\widehat{w}))e^{i(\theta_1+\theta_2)}d\theta_1 d\theta_2$$

as claimed. □

To complete the proof, note that

$$G'(t) = -\frac{\log(1-t^2)}{2t} \qquad G''(t) = \frac{1}{1-t^2} + \frac{\log(1-t^2)}{2t^2}. \tag{8.8}$$

Hence, we can write $\mathrm{Var}[\mathcal{N}_r]$ as

$$\frac{r^2}{2\pi}\int_0^{2\pi}\int_0^{2\pi} e^{i(\theta_1+\theta_2)}I(z,w)\bigg|_{z=re^{i\theta_1},\,w=re^{i\theta_2}} d\theta_1 d\theta_2, \tag{8.9}$$

where

$$I(z,w) := G''(P_N^\xi(\widehat{z},\widehat{w}))\frac{\partial}{\partial z}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right)\cdot\frac{\partial}{\partial w}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right) + G'(P_N^\xi(\widehat{z},\widehat{w}))\frac{d^2}{dz\,dw}\left(P_N^\xi(\widehat{z},\widehat{w})^2\right).$$

Substituting (7.5)–(7.8) of Corollary 2 into (8.9) and noting that $\log\left|re^{i\theta_1} - re^{i\theta_2}\right|$ has finite integral completes the proof. □

Acknowledgements

I am grateful to S. Zelditch for many helpful conversations about his work previous work on statistics of zeros and critical points and for his comments on an earlier draft of this paper. I would also like to thank two anonymous referees for the their helpful comments and R. Peled for sharing with me M. Krishnapur's code that I modified to generate the figures in this paper.

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Received May 27, 2013