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A note on the arithmetic of residual automorphic representations of reductive groups

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We prove an arithmeticity result for a class of cohomological residual automorphic representations of a general reductive group G. More precisely, we show that this class of residual representations is stable under the action of $\mathrm{Aut}(\mathbb{C})$. This complements numerous results on the stability of cohomological cuspidal automorphic representations due by several people. We conclude by showing that the rationality field of such a cohomological residual automorphic representation is a number field.

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Introduction

Let G be a connected reductive group over a number field F. Let $\sigma \in$Aut$(\mathbb{C})$ and let Π be an automorphic representation of $G(\mathbb{A})$. Then, Π splits as $\Pi = \Pi_\infty \otimes \Pi_{\mathrm{f}}$ and we may define a new, "σ-twisted" representation ${}^\sigma\Pi_{\mathrm{f}} := \Pi_{\mathrm{f}} \otimes_\sigma \mathbb{C}$ of $G(\mathbb{A}_{\mathrm{f}})$. A priori, it is by no means clear, if ${}^\sigma\Pi_{\mathrm{f}}$ appears again as the finite part of an automorphic representation of $G(\mathbb{A})$. However, for particular choices of the group G and the automorphic representation Π, there are numerous results in the related literature.

In any case, the key technique, underlying all of these aforementioned results, is to assume that the automorphic representation considered is of *cohomological type*, i.e., that it contributes non-trivially to the relative Lie algebra cohomology of the space of automorphic forms $\mathcal{A}(G)$, twisted by an appropriate finite-dimensional coefficient system.

In order to render the above more precise, let E_μ be a finite-dimensional, irreducible algebraic representation of $G_\infty = R_{F/\mathbb{Q}}(G)(\mathbb{R})$ on a complex vector space. It is given by its highest weight $\mu = (\mu_\iota)_{\iota \in I_\infty}$. Here, $R_{F/\mathbb{Q}}(G)$ denotes restriction of scalars and I_∞ stands for the set of field embeddings $F \hookrightarrow \mathbb{C}$. Similar to the case of an automorphic representation, on the level of G/F, we may form a new representation ${}^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$. On the other hand, we let $E_{^\sigma\mu}$ be the representation of G_∞, which has highest weight $^\sigma\mu := (\mu_{\sigma^{-1}\iota})_{\iota\in I_\infty}$. If G is an inner form of a split algebraic group (which we assume it is from now on), then, as a representation of $G(F)$, $^\sigma E_\mu$ is isomorphic to $E_{^\sigma\mu}$.

As a consequence, if G is inner, we obtain a commutative diagram of $G(\mathbb{A}_\mathrm{f})$-module isomorphisms (with σ-linear columns),

$$\begin{array}{ccc} H^q(S,\mathcal{E}_\mu) & \xleftarrow{\cong} & H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_\mu) \\ \downarrow{\scriptstyle\sigma^*} & & \downarrow{\scriptstyle\sigma^*} \\ H^q(S,{}^\sigma\mathcal{E}_\mu) & \xrightarrow{\cong} & H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_{^\sigma\mu}). \end{array}$$

Using the key result of our recent paper [13], which shows that below a certain, sharp degree of cohomology q_{max}, the space of automorphic cohomology $H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_\mu)$ is isomorphic to the $(\mathfrak{m}_G,K)$-cohomology of the space of *square-integrable* automorphic forms, we can prove the main result of this short paper: We show that — under certain constraints — residuality is an *arithmetic property* of automorphic representations, or — in other words — that the action of $\mathrm{Aut}(\mathbb{C})$ preserves the class of these residual automorphic representations:

Theorem. *Let G be a connected reductive linear algebraic group over a number field F, which is an inner form of a split algebraic group, and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $G(\mathbb{A})$, with non-zero $(\mathfrak{m}_G,K)$-cohomology below degree q_{max}, which does not appear in interior cohomology. Then for all $\sigma\in\mathrm{Aut}(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G(\mathbb{A})$, $(\mathfrak{m}_G,K)$-cohomological with respect to $E_{^\sigma\mu}$, such that $\Xi_\mathrm{f}\cong{}^\sigma\Pi_\mathrm{f}$, and which appears in the complement of interior cohomology.*

We would like to remark that in the case of $G = GL_n/F$, the general linear group over any number field F, our approach leads to a result (cf. Theorem 3.3) which complements Clozel [7, Theorem 3.13], Franke [8, Theorem 20], and Franke–Schwermer [9, Theorem 4.3]. See also Grobner–Raghuram [14, Proposition 7.21, Theorem 7.23].

Additionally, we also prove an algebraicity theorem on the field of rationality,

$$\mathbb{Q}(\Pi_\mathrm{f}) := \{z\in\mathbb{C}|\sigma(z)=z \quad \forall\sigma \text{ satisfying } \Pi_\mathrm{f}\cong{}^\sigma\Pi_\mathrm{f}\}.$$

Corollary. *Let Π be a cohomological residual automorphic representation as in the statement of the above theorem. Then $\mathbb{Q}(\Pi_\mathrm{f})$ is a number field.*

We would like to remark that the above corollary generalizes a result which is well known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15, p. 80], Waldspurger [20, Corollary I.8.3 and first line of p. 153], Clozel [7], and Grobner–Raghuram [14, Theorem 8.1]).

1. Some basics

1.1. Number fields

We let F be an algebraic number field. Its set of places is denoted $V = V_\infty \cup V_{\mathrm{f}}$, where V_∞ stands for the set of archimedean places and V_{f} is the set of non-archimedean places. We shall use the letter I_∞ for the set of field embeddings $\iota : F \hookrightarrow \mathbb{C}$. The ring of adeles of F is denoted $\mathbb{A}$, the subspace of finite adeles is denoted $\mathbb{A}_{\mathrm{f}}$.

1.2. Algebraic groups

In this paper, G is a connected, reductive linear algebraic group over a number field F. We assume to have fixed a minimal parabolic F-subgroup P_0 with Levi decomposition $P_0 = L_0N_0$ and let A_0 be the maximal F-split torus in the center Z_{L_0} of L_0. This choice defines the standard parabolic F-subgroups P with Levi decomposition $P = L_PN_P$, where $L_P \supseteq L_0$ and $N_P \subseteq N_0$. We let A_P be the maximal F-split torus in the center Z_{L_P} of L_P, satisfying $A_P \subseteq A_0$ and denote by $\mathfrak{a}_P$ (resp. $\mathfrak{a}_{P,\mathbb{C}}$) its Lie algebra (resp. its complexification $\mathfrak{a}_{P,\mathbb{C}} = \mathfrak{a}_P \otimes \mathbb{C}$). We write $H_P : L_P(\mathbb{A}) \to \mathfrak{a}_{P,\mathbb{C}}$ for the standard Harish–Chandra height function. The group $L_P(\mathbb{A})^1 := \ker H_P$, admits a direct complement $A_P^{\mathbb{R}} \cong \mathbb{R}_+^{\dim \mathfrak{a}_P}$ in $L_P(\mathbb{A})$ whose Lie algebra is isomorphic to $\mathfrak{a}_P$. With respect to a maximal compact subgroup $K_{\mathbb{A}} \subseteq G(\mathbb{A})$ in good position, cf. [17, I.1.4], we obtain an extension $H_P : G(\mathbb{A}) \to \mathfrak{a}_{P,\mathbb{C}}$ to all of $G(\mathbb{A})$.

1.3. Lie groups and Lie algebras

We put $G_\infty := R_{F/\mathbb{Q}}(G)(\mathbb{R})$, where $R_{F/\mathbb{Q}}$ denotes the restriction of scalars from F to $\mathbb{Q}$. We let $\mathfrak{m}_G := \mathfrak{g}_\infty/\mathfrak{a}_G = Lie(G(\mathbb{A})^1 \cap G_\infty)$ and denote by $\mathfrak{Z}(\mathfrak{g})$ the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}_{\infty,\mathbb{C}}$.

Let $K_\infty \subset G_\infty$ be a maximal compact subgroup (the archimedean factor of the maximal compact subgroup $K_{\mathbb{A}}$ of $G(\mathbb{A})$ in good position) and set once and for all $K := K_\infty^\circ$. (Observe that this choice of a compact subgroup of

G_∞ is in accordance with Franke [8, p. 184].) Please refer to Borel–Wallach [6, I], for the basic facts and notations concerning $(\mathfrak{m}_G, K)$-cohomology.

Furthermore, we let $\mathfrak{h}_\infty$ be a Cartan subalgebra of $\mathfrak{g}_\infty$ that contains $\mathfrak{a}_{0,\infty}$.

1.4. Algebraic representations

In this paper, we always let E_μ be a finite–dimensional irreducible algebraic representation of G_∞ on a complex vector space, given by its highest weight $\mu = (\mu_\iota)_{\iota \in I_\infty} \in \check{\mathfrak{h}}_\infty$. We will assume that $A_G^{\mathbb{R}}$ (and so $\mathfrak{a}_G$) acts trivially on E_μ. There is therefore no difference between the $(\mathfrak{g}_\infty, K)$-module and the $(\mathfrak{m}_G, K)$-module defined by E_μ.

Take a field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then we may form the new finite-dimensional irreducible algebraic representation of G_∞, denoted $E_{^\sigma\mu}$, which is given by its highest weight

$$^\sigma\mu := (\mu_{\sigma^{-1}\iota}).$$

Clearly this amounts in a permutation of the components μ_ι of μ. On the level of the algebraic group G/F, there is another representation, denoted $^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$.

1.5. Cohomology of locally symmetric spaces

We let

$$S := G(F)A_G^{\mathbb{R}} \backslash G(\mathbb{A})/K$$

be the projective limit of the "locally symmetric spaces" attached to G. Starting from the algebraic representation E_μ, one obtains a sheaf $\mathcal{E}_\mu$ on S by letting $\mathcal{E}_\mu$ be the sheaf with espace étalé $G(\mathbb{A})/A_G^{\mathbb{R}}K \times_{G(F)} E_\mu$ with the discrete topology on E_μ. In the same way, we obtain a sheaf $^\sigma\mathcal{E}_\mu$ from $^\sigma E_\mu$.

Proposition 1.1. *Let G be a connected reductive linear algebraic group over a number field F and let E_μ be a finite-dimensional algebraic representation of G_∞. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ and all degrees q of cohomology, there is a $G(\mathbb{A}_{\mathrm{f}})$-equivariant, σ-linear isomorphism*

$$H^q(S, \mathcal{E}_\mu) \to H^q(S, {^\sigma\mathcal{E}_\mu}).$$

If G is an inner form of a split algebraic group, then, as a representation of $G(F)$, $^\sigma E_\mu$ is isomorphic to $E_{^\sigma\mu}$. This may fail, if G is an outer form.

Proof. The first assertion is obvious by the definition of Betti cohomology $H^q(S, \bullet)$. For the other assertions see [10, Section 2]. □

2. Automorphic cohomology

2.1. Residual automorphic forms

Our notion of an *automorphic form* $f : G(\mathbb{A}) \to \mathbb{C}$ and of an *automorphic representation* of $G(\mathbb{A})$ is the one of Borel–Jacquet [3, 4.2 and 4.6] . Let $\mathcal{A}(G)$ be the space of all automorphic forms $f : G(\mathbb{A}) \to \mathbb{C}$, which are constant on the real Lie subgroup $A_G^{\mathbb{R}}$. By its very definition, every automorphic form is annihilated by some power of an ideal $\mathcal{J} \triangleleft \mathfrak{Z}(\mathfrak{g})$ of finite codimension. We fix such an ideal $\mathcal{J}$, once and for all. As we will only be interested in cohomological automorphic forms, we take $\mathcal{J}$ to be the ideal which annihilates the contragredient representation $E_\mu^\vee$ of E_μ, cf. Section 1.3, and denote by

$$\mathcal{A}_{\mathcal{J}}(G) \subset \mathcal{A}(G)$$

the space consisting of those automorphic forms which are annihilated by some power of $\mathcal{J}$. Clearly, $\mathcal{A}_{\mathcal{J}}(G)$ carries a commuting $(\mathfrak{g}_\infty, K_\infty)$ and $G(\mathbb{A}_f)$-action and hence defines a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module.

The $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-submodule of all square integrable automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$ is denoted $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)$. An irreducible subquotient of $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)$ will be called a *discrete series automorphic* representation, cf. Borel [2, 9.6]. If $\omega : Z_G(F)\backslash Z_G(\mathbb{A}) \to \mathbb{C}^*$ is a continuous character of the center Z_G of G, we let $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G, \omega)$ be the space of square-integrable automorphic forms with central character ω.

We further recall that $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G, \omega)$ decomposes as a direct sum of automorphic representations Π

$$\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G, \omega) \cong \bigoplus \Pi^{m(\Pi)} \tag{2.1}$$

with finite multiplicities $m(\Pi)$. One has $m(\Pi) = m_{\mathrm{cusp}}(\Pi) + m_{\mathrm{res}}(\Pi)$, where $m_{\mathrm{cusp}}(\Pi)$ is the multiplicity of Π in the space of cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$. If $m_{\mathrm{res}}(\Pi) \neq 0$, we call Π *a residual automorphic representation.* Clearly, it decomposes as $\Pi = \Pi_\infty \otimes \Pi_f$, where Π_∞ is an irreducible $(\mathfrak{m}_G, K)$-module and Π_f an irreducible $G(\mathbb{A}_f)$-module.

Apart from the action of automorphisms $\sigma \in \mathrm{Aut}(\mathbb{C})$ on finite-dimensional algebraic representations E_μ, cf. Section 1.4, there is also an action of σ on the finite part Π_{f} of an automorphic representation. Given $\sigma \in \mathrm{Aut}(\mathbb{C})$ and Π_{f}, we define ${}^{\sigma}\Pi_{\mathrm{f}}$ to be the representation of $G(\mathbb{A}_{\mathrm{f}})$ on $\Pi_{\mathrm{f}} \otimes_{\sigma} \mathbb{C}$ (analogous to ${}^{\sigma}E_{\mu}$). This is in accordance with Waldspurger [20, I.1].

2.2. Parabolic and cuspidal supports

Let $\{P\}$ be the associate class of the parabolic F-subgroup P. It consists by definition of all parabolic F-subgroups $Q = L_Q N_Q$ of G for which L_Q and L_P are conjugate by an element in $G(F)$. We denote by $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ the space of all $f \in \mathcal{A}_{\mathcal{J}}(G)$ which are negligible along every parabolic F-subgroup $Q \notin \{P\}$. There is the following decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ as a $(\mathfrak{m}_G, K, G(\mathbb{A}_{\mathrm{f}}))$-module, cf. [4, Theorem 2.4] or [2, 10.3], first established by Langlands:

$$\mathcal{A}_{\mathcal{J}}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}(G). \tag{2.2}$$

The various summands $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ can be decomposed even further. To this end, recall from [9, 1.2], the notion of an *associate class* φ_P of cuspidal automorphic representations π of the Levi subgroups of the elements in the class $\{P\}$. Given φ_P, represented by a cuspidal representation π, a $(\mathfrak{m}_G, K, G(\mathbb{A}_{\mathrm{f}}))$-submodule

$$\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$$

of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ was defined in [9, 1.3] as follows. It is the span of all possible holomorphic values or residues of all Eisenstein series attached to $\tilde{\pi}$, evaluated at a certain point $\lambda = d\Lambda$ determined by π, together with all their derivatives. (This definition is independent of the choice of the representatives P and π, thanks to the functional equations satisfied by the Eisenstein series considered.) For details, we refer the reader to [9, 1.2–1.4].

The following refined decomposition as $(\mathfrak{m}_G, K, G(\mathbb{A}_{\mathrm{f}}))$-modules of the spaces $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ of automorphic forms was obtained in Franke–Schwermer [9, Theorem 1.4]:

$$\mathcal{A}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G).$$

2.3. Automorphic cohomology

We recall that the $G(\mathbb{A}_f)$-module

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$$

is called the *automorphic cohomology* of G in degree q. From what we recalled in Section 2.2 it inherits a direct sum decomposition

$$\begin{aligned} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu) &\cong \bigoplus_{\{P\}} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_\mu) \\ &\cong \bigoplus_{\{P\}} \bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu). \end{aligned}$$

The summand $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{G\}}(G) \otimes E)$ attached to $\{G\}$ consists precisely of those classes, which are representable by cuspidal automorphic forms in $\mathcal{A}_{\mathcal{J}}(G)$ and will be denoted for short by $H^q_{\mathrm{cusp}}(G, E_\mu)$.

Apart from this, we will need the following result of Franke [8, Theorem 18], which links automorphic cohomology with the sheaf cohomology of S.

Theorem 2.3. *There is an isomorphism of $G(\mathbb{A}_f)$-modules*

$$H^q(S, \mathcal{E}_\mu) \cong H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu).$$

We may therefore compare automorphic cohomology with the cohomology of S. Recall the Borel–Serre compactification of S, cf. [5, 18]. We will denote it by $\overline{S}$ and by $\partial\overline{S}$ its boundary. Let $H^q_!(S, \mathcal{E}_\mu)$ be the space of *interior cohomology*, i.e., the kernel of the natural restriction map,

$$\mathrm{res}^q : H^q(S, \mathcal{E}_\mu) = H^q(\overline{S}, \mathcal{E}_\mu) \to H^q(\partial\overline{S}, \mathcal{E}_\mu).$$

It is well known that $H^q_{\mathrm{cusp}}(G, E_\mu) \subseteq H^q_!(S, \mathcal{E}_\mu)$. This is a direct consequence of the fact that the inclusion of compactly supported differential forms into fast decreasing differential forms defines an isomorphism in sheaf cohomology.

2.4. A diagram

Assume now that G is an inner form of a split reductive algebraic group. Then, for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, we obtain the following commutative diagram of $G(\mathbb{A}_f)$-module isomorphisms (with σ-linear columns) by Proposition 1.1 and Theorem 2.3:

$$\begin{array}{ccc} H^q(S,\mathcal{E}_\mu) & \xleftarrow{\cong} & H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_\mu) \\ \downarrow{\sigma^*} & & \downarrow{\sigma^*} \\ H^q(S,{}^\sigma\mathcal{E}_\mu) & \xrightarrow{\cong} & H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_{{}^\sigma\mu}). \end{array}$$

Observe the position of the automorphism σ in the various coefficient systems. This is the reason for our assumption that G is an inner form, since otherwise ${}^\sigma E_\mu$ and $E_{{}^\sigma\mu}$ may differ as $G(F)$-modules, cf. Proposition 1.1. Moreover, observe that $H^q_!(S,\mathcal{E}_\mu)$ is stable by σ^*, i.e., $\sigma^*(H^q_!(S,\mathcal{E}_\mu)) = H^q_!(S,{}^\sigma\mathcal{E}_\mu)$. This is due to the rationality of the map res^q, which we used in order to define interior cohomology.

2.5. Certain bounds in cohomology

Given an associate class $\{P\}$ of parabolic F-subgroups, represented by P, we define (with $R = L_R N_R$ denoting a parabolic subgroup of G)

$$q_{\mathsf{max}}(P) := \min_{\substack{R \text{maximal} \\ R \supseteq P}} \left(\sum_{v\in V_\infty} \left\lceil \frac{1}{2} \dim_{\mathbb{R}} N_R(F_v) \right\rceil \right).$$

Moreover, we let

$$q_{\mathsf{max}} := \min_{P \text{standard}} q_{\mathsf{max}}(P).$$

This definition is motivated by our own results in [13]. The two numbers $q_{\mathsf{max}}(P)$ and q_{max} will serve as certain "upper bounds" for the degrees of automorphic cohomology. Indeed, in [13], we proved the following result:

Theorem 2.4. *Let G be a connected, reductive, linear algebraic group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $L^2_{\mathcal{J},\{P\},\varphi_P} := \mathcal{A}_{\mathrm{dis},\mathcal{J}}(G) \cap \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$ be the space of square-integrable automorphic forms in $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$.*

Then, the inclusion $L^2_{\mathcal{J},\{P\},\varphi_P}(G) \hookrightarrow \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ *induces an isomorphism of* $G(\mathbb{A}_f)$*-modules*

$$H^q(\mathfrak{m}_G, K, L^2_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu),$$

in all degrees $q < q_{\mathsf{max}}(P)$*. In particular, below* q_{max}*, automorphic cohomology* $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$ *is identical to* $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathrm{dis},\mathcal{J}}(G) \otimes E_\mu)$*.*

The latter theorem will be the key result for the proof of the main result of this article detailed in the following section.

3. The main results

3.1. Arithmeticity of residual automorphic representations — the general case

Theorem 3.1. *Let* G *be a connected reductive linear algebraic group over a number field* F*, which is an inner form of a split algebraic group, and let* E_μ *be a finite-dimensional algebraic representation of* G_∞*. Let* Π *be a residual automorphic representation of* $G(\mathbb{A})$*, with non-zero* $(\mathfrak{m}_G, K)$*-cohomology below degree* q_{max}*, which does not appear in* $H_!(S, \mathcal{E}_\mu)$*. Then for all* $\sigma \in \mathrm{Aut}(\mathbb{C})$*, there exists a residual automorphic representation* Ξ *of* $G(\mathbb{A})$*,* $(\mathfrak{m}_G, K)$*-cohomological with respect to* $E_{^\sigma\mu}$*, such that* $\Xi_f \cong {}^\sigma\Pi_f$*, and which appears in the complement of* $H_!(S, {}^\sigma\mathcal{E}_\mu)$*.*

Proof. Let Π be as in the statement of the theorem. Let $q_0 < q_{\mathsf{max}}$ be a degree, where the $(\mathfrak{m}_G, K)$-cohomology of Π_∞ (with respect to E_μ) does not vanish. Clearly, $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_f$ as $G(\mathbb{A}_f)$-modules, whence we may fix an injection $\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$. By Theorem 2.4 (resp. [13, Corollary 17]), we hence obtain an *injection*

$$\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu).$$

In other words, we have realized Π_f as a submodule of the space of automorphic cohomology in degree q_0. Chasing the image of Π_f in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$ through our diagram, see Section 2.4, we finally end up with an irreducible subrepresentation $\sigma^*(\Pi_f)$ of $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{^\sigma\mu})$. This is the place, where the assumption that G is an inner form of a split algebraic group comes into play. As mentioned below our diagram in Section 2.4, otherwise, ${}^\sigma E_\mu$ (which defines the sheaf ${}^\sigma\mathcal{E}_\mu$) may not be isomorphic to $E_{^\sigma\mu}$

(which defines the coefficient module in $(\mathfrak{m}_G, K)$-cohomology) as a representation of $G(F)$.

By the definition of the σ-linear $G(\mathbb{A}_f)$-morphism $H^q(S,\mathcal{E}_\mu)\to H^q(S,{}^\sigma\mathcal{E}_\mu)$, this subrepresentation $\sigma^*(\Pi_f)$ is isomorphic to ${}^\sigma\Pi_f$. By our Theorem 2.4 (resp. our Theorem 18 in [13]),

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)\otimes E_{^\sigma\mu}) \cong H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G)\otimes E_{^\sigma\mu}),$$

so, $\sigma^*(\Pi_f)$, which by the above is isomorphic to ${}^\sigma\Pi_f$, is contained as a subrepresentation of the former cohomology space $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)\otimes E_{^\sigma\mu})$. Now recall that the discrete automorphic spectrum $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)$ decomposes as a direct sum, cf. Section 2.1 (at least, if one fixes a central character). Therefore, there is also a direct sum decomposition of the $G(\mathbb{A}_f)$-module $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)\otimes E_{^\sigma\mu})$. As a consequence, there exists a square-integrable automorphic representation Ξ of $G(\mathbb{A})$, with non-vanishing $(\mathfrak{m}_G, K)$-cohomology with respect to $E_{^\sigma\mu}$, such that $\Xi_f \cong {}^\sigma\Pi_f$.

Since $H^{q_0}_!(S,\mathcal{E}_\mu)$ is preserved by σ^* and contains $H^{q_0}_{\mathrm{cusp}}(G,E_\mu)$, this representation Ξ can be chosen to be residual, i.e., such that $m_{\mathrm{res}}(\Xi)\neq 0$, and even outside of $H_!(S,{}^\sigma\mathcal{E}_\mu)$. This shows the claim. □

Remark 3.2. Let Π be a residual automorphic representation as in the statement of Theorem 3.1. Assume that Π has been obtained using the Langlands–Shahidi method, i.e., by determining the position of poles and zeros of certain automorphic L-functions, which are attached to those cuspidal automorphic representations π of Levi subgroups, which appear in the calculation of the constant term. Our main theorem above hence suggests that if such an L-function — $L(s,\pi,r)$, say — has a pole at the relevant point $s = s_0$ (giving rise to a pole of some Eisenstein series, whose residues span Π), then also $L(s,{}^\sigma\pi,r)$ should have a pole at this point. This may be of some use in the determination of poles of Eisenstein series, respectively, for showing algebraicity results of residual automorphic L-functions.

3.2. Arithmeticity of residual automorphic representations — the case of GL_n/F

In the case of the split general linear group $G = GL_n/F$, we obtain the following refinement[1] of Theorem 3.1. At the same time this result complements the results of Clozel [7, Theorem 3.13] , Franke [8, Theorem 20]

[1]As I learned from L. Clozel, stronger results than the ones of Section 3.2 may be proved using the fact that a residual automorphic representation of $GL_n(\mathbb{A})$ is

and Franke–Schwermer [9, Theorem 4.3]. See also Grobner–Raghuram [14, Proposition 7.21 and Theorem 7.23].

Theorem 3.3. *Let $G = GL_n$ be the split general linear group over F and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $GL_n(\mathbb{A})$, with parabolic support $\{P\}$ and with non-zero $(\mathfrak{m}_G, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{\sigma\mu}$ and such that $\Xi_{\mathrm{f}} \cong {}^\sigma\Pi_{\mathrm{f}}$. If P is maximal parabolic, then the condition $q_0 < q_{\mathsf{max}}(P)$ can be dropped, i.e., the result holds for all cohomological residual automorphic representations Π supported in the maximal parabolic subgroup $P = (GL_{n/2} \times GL_{n/2}) \cdot N$.*

Proof. Let Π be as in the statement of the theorem. Let $q_0 < q_{\mathsf{max}}(P)$ be a degree, where the $(\mathfrak{m}_G, K)$-cohomology of Π_∞ (with respect to E_μ) does not vanish. As, $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_{\mathrm{f}}$ as $G(\mathbb{A}_{\mathrm{f}})$-modules, we may fix an injection $\Pi_{\mathrm{f}} \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$. By Theorem 2.4, we hence obtain an injection

$$\Pi_{\mathrm{f}} \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_\mu).$$

We now use a result of Franke: In [8, Theorem 20], he proved that the decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ along the parabolic supports $\{P\}$, cf. (2.2), is respected by the σ-linear isomorphism σ^*. As a consequence of our diagram, cf. Section 2.4, $\sigma^*(\Pi_{\mathrm{f}}) \cong {}^\sigma\Pi_{\mathrm{f}}$ appears not only in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu})$, but also in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_{\sigma\mu})$. By our Theorem 2.4,

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_{\sigma\mu}) \cong H^{q_0}\left(\mathfrak{m}_G, K, \bigoplus_{\varphi_P} L^2_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_{\sigma\mu}\right),$$

whence $\sigma^*(\Pi_{\mathrm{f}}) \cong {}^\sigma\Pi_{\mathrm{f}}$ appears as an irreducible submodule of the latter cohomology space. Again we recall that the discrete automorphic spectrum $\mathcal{A}_{\mathrm{dis},\mathcal{J}}(G)$, which contains $\bigoplus_{\varphi_P} L^2_{\mathcal{J},\{P\},\varphi_P}(G)$, decomposes as a direct sum, cf. 2.1 (at least, if one fixes a central character). Hence, we find a square-integrable automorphic representation Ξ of $G(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{\sigma\mu}$ and such that $\Xi_{\mathrm{f}} \cong$

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a Mœglin–Waldspurger quotient, cf. [16]. However, I decided to leave this section, in order to exemplify the technique of Theorem 3.1.

${}^{\sigma}\Pi_{\mathrm{f}}$. This representation must be residual by multiplicity one for square-integrable automorphic representations of $GL_n(\mathbb{A})$. Indeed, Π being assumed to be residual implies that $P \neq G$ by multiplicity one in the discrete automorphic spectrum. Therefore, the parabolic support of Ξ also defers from G. In particular, Ξ is residual itself (and we could avoid the assumption that Π_{f} does not appear in interior cohomology $H_!(S, \mathcal{E}_\mu)$). This shows the first claim.

We will now prove the last assertion, claiming that the condition $q_0 < q_{\mathsf{max}}(P)$ can be dropped in the case of maximal parabolic subgroups. For a moment, let Π be any cohomological residual automorphic representation of $GL_n(\mathbb{A})$ supported in a pair $(\{P\}, \varphi_P)$. In [12, , Theorem 4.1], we showed that its cohomology $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ always injects into $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$ in the lowest degree $q = q_{\mathsf{min}}$, where $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ does not vanish identically. In particular, by our above considerations, resp. our diagram in Section 2.4, $\sigma^*(\Pi_{\mathrm{f}}) \cong {}^{\sigma}\Pi_{\mathrm{f}}$ appears as an irreducible submodule of $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},{}^{\sigma}\varphi_P}(G) \otimes E_{{}^{\sigma}\mu})$ for an appropriate cuspidal support denoted ${}^{\sigma}\varphi_P$ here. We use [12] again. In Section 6B therein, we showed that the lowest degree of cohomology q_{min} is always smaller than a certain bound $q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$ of degrees of cohomology. (A precise definition of this degree may be found in [13, Section 6.1] and [12, Section 3C].) It is not important to know how this degree is defined in details; rather one should recall the main result of our paper [13, Theorem 15], which says that for all $q < q_{\mathsf{res}}$,

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E),$$

where $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G)$ is the deepest step in Franke's filtration of the space $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Again, it is not important to know precisely how this filtration is defined (all details may be found in [13, Section 3]); however, by our concrete calculations in [12, Section 6] (cf. Proposition 6.4), it is shown that

$$q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P) = q_{\mathsf{res}}(\{P\}, {}^{\sigma}\varphi_P),$$

whence summarizing what we said above, $\sigma^*(\Pi_{\mathrm{f}}) \cong {}^{\sigma}\Pi_{\mathrm{f}}$ appears as an irreducible submodule of the cohomology of the deepest filtration step $H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^{\sigma}\varphi_P}(G) \otimes E_{{}^{\sigma}\mu})$. We would like to point out that all of this still holds for all parabolic subgroups.

Let now P be maximal. Then it is shown in Grbac [11, Theorem 3.1] that the deepest step $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^{\sigma}\varphi_P}(G)$ in Franke's filtration is identical to

$L^2_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G)$, i.e., every automorphic form in $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G)$ is square integrable. This is wrong if P is not maximal. As a consequence, ${}^\sigma\Pi_f$ must be the finite part of a square-integrable automorphic representation Ξ. By multiplicity one, Ξ must be residual. □

Example 3.4. Let $G = GL_4$ over $F = \mathbb{Q}$. Then, $\mathfrak{m}_G = \mathfrak{sl}_4(\mathbb{R})$ and $K = SO(4)$. Let P be the standard maximal parabolic subgroup with Levi factor $L \cong GL_2 \times GL_2$. We take furthermore $E_\mu = \mathbb{C}$, the trivial representation and let $D(2)$ be the discrete series representation of $GL_2(\mathbb{R})$ of lowest nonnegative $O(2)$-type 2. Then, there exists a cuspidal automorphic representation π of $GL_2(\mathbb{A})$, with $\pi_\infty \cong D(2)$ and $\Pi = |.|^{1/2}\pi \times |.|^{-1/2}\pi$ (Langlands quotient) is a residual automorphic representation of $GL_4(\mathbb{A})$ with nontrivial $(\mathfrak{sl}_4(\mathbb{R}), SO(4))$-cohomology with respect to $\mathbb{C}$ in degrees $q = 3, 6$. By Theorem 3.3, and a simple uniqueness argument at the archimedean place, ${}^\sigma\Pi = \Pi_\infty \otimes {}^\sigma\Pi_f$ is a residual automorphic representation of $GL_4(\mathbb{A})$, for all $\sigma \in$Aut$(\mathbb{C})$. Indeed, the archimedean component must be Π_∞ again, since this is the only irreducible unitary representation of $GL_4(\mathbb{R})$, which has nontrivial cohomology in degree $q = 3, 6$. By construction it is the Langlands quotient $|.|^{1/2}D(2) \times |.|^{-1/2}D(2)$. See, e.g., [14, Section 5.4].

Remark 3.5. Let G' be an inner form of GL_n/F and let E_μ be an irreducible algebraic representation of G'_∞, which is of regular highest weight. Then Theorem 7.23 in [14] together with multiplicity one of the discrete automorphic spectrum of $G'(\mathbb{A})$, cf. [1], may be used to show the first part of Theorem 3.3 for residual automorphic representation of $G'(\mathbb{A})$, too. More precisely, if Π is a residual automorphic representation of $G'(\mathbb{A})$, with parabolic support $\{P\}$ and with non-zero $(\mathfrak{m}_{G'}, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$, then for all $\sigma \in$ Aut$(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G'(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_{G'}, K)$-cohomological with respect to $E_{{}^\sigma\mu}$ and such that $\Xi_f \cong {}^\sigma\Pi_f$.

3.3. Rationality fields

We conclude this article proving a result on the rationality field of a cohomological residual automorphic representation Π. Recall that this field is defined by

$$\mathbb{Q}(\Pi_f) := \{z \in \mathbb{C} | \sigma(z) = z \quad \forall \sigma \text{ satisfying } \Pi_f \cong {}^\sigma\Pi_f\}.$$

Corollary 3.6. *Let G be a connected reductive linear algebraic group over a number field F, which is an inner form of a split algebraic group, and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $G(\mathbb{A})$, with non-zero $(\mathfrak{m}_G, K)$-cohomology below degree q_{max}, which does not appear in $H_!(S, \mathcal{E}_\mu)$. Then $\mathbb{Q}(\Pi_{\mathrm{f}})$ is a number field. If $G = GL_n/F$, then we may lighten the assumptions on Π and may take Π to be a residual automorphic representation of $GL_n(\mathbb{A})$, with parabolic support $\{P\}$ and with non-zero $(\mathfrak{m}_G, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$. If P is maximal, then the condition $q_0 < q_{\mathsf{max}}(P)$ may be dropped without replacement.*

Proof. Let Π be any of the representations in the statement of the corollary. Then, for all automorphisms σ, there is a residual automorphic representation Ξ of $G(\mathbb{A})$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{^\sigma\mu}$, such that $\Xi_f \cong {^\sigma\Pi_f}$ by Theorem 3.1 or Theorem 3.3, respectively. Now fix an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$, such that $\Pi_f^{K_f}$ is non-zero. Then also $({^\sigma\Pi})_{\mathrm{f}}^{K_{\mathrm{f}}}$ is non-zero. Since $H^q(S, \mathcal{E}_\mu)^{K_{\mathrm{f}}}$ is finite-dimensional, the result follows from the finiteness of the set $\{E_{^\sigma\mu} : \sigma \in \mathrm{Aut}(\mathbb{C})\}$. □

Remark 3.7. The above corollary generalizes a result which is well known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15, p. 80], Waldspurger [20, Corollary I.8.3 and first line of p. 153], Clozel [7], and Grobner–Raghuram [14, Theorem 8.1]).

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