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On homology of linear groups over $k[t]$

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This note explains how to prove that for any simply-connected reductive group G and any infinite field k, the inclusion $k \hookrightarrow k[t]$ induces an isomorphism on group homology. This generalizes results of Soulé and Knudson.

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1. Introduction

The question of homotopy invariance of group homology is the question under which conditions on a linear algebraic group G and a commutative ring R the natural morphism $G(R) \to G(R[t])$ induces isomorphisms in group homology. This is an unstable version of homotopy invariance for algebraic K-theory as established by Quillen in [Qui73].

The two main results which have been obtained in this direction are due to Soulé and Knudson. In [Sou79], Soulé determined a fundamental domain for the action of $G(k[t])$ on the associated Bruhat–Tits building and deduced homotopy invariance for fields of characteristic $p > 0$ with field coefficients prime to p. In [Knu97], Knudson extended Soulé's approach and deduced homotopy invariance with integral coefficients for SL_n over infinite fields.

In this paper, we generalize Knudson's theorem to arbitrary (connected) reductive groups, cf. Theorem 5.1:

Theorem 1.1. *Let k be an infinite field and let G be a connected reductive smooth linear algebraic group over k. Then the canonical inclusion $k \hookrightarrow k[t]$ induces isomorphisms*

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k.

It follows from the work of Krstić and McCool [KM97] that homotopy invariance does not work for H_1 of rank one groups over integral domains which are not fields. In the case of rank two groups, homotopy invariance fails for H_2 as discussed in [Wen12]. It therefore seems that one cannot hope for an extension of the above result for arbitrary regular rings or even polynomial rings in more than 1 variable.

Structure of the paper: In Section 2, we reduce to simply-connected absolutely almost simple groups. Section 3 recalls the necessary facts on Bruhat–Tits theory and Margaux's generalization of Soulé's theorem. In Section 4, we extend Knudson's computations of the homology of stabilizers to groups other than SL_n; and in Section 5 we establish the general homotopy invariance result.

2. Preliminary reduction

In this section, we provide some preliminary reductions. More precisely, Theorem 1.1 follows for all reductive groups if it can be shown for simply-connected almost simple groups. The arguments are fairly standard reductions, ubiquitous in the theory of algebraic groups.

Recall from [Bor91] the basic notions of the theory of linear algebraic groups. In particular, the *radical* $R(G)$ of a group G is the largest connected solvable normal subgroup of G, and the *unipotent radical* $R_{\mathrm{u}}(G)$ of G is the largest connected unipotent normal subgroup of G. A connected group G is called *reductive* if its unipotent radical is trivial, and it is called *semisimple* if its radical is trivial. A connected algebraic group G is called *simple* if it is non-commutative and has no nontrivial normal algebraic subgroups. It is called *almost simple* if its centre Z is finite and the quotient G/Z is simple. A semisimple group is called *simply-connected*, if there is no nontrivial isogeny $\phi : \tilde{G} \to G$.

The additive group is denoted by $\mathbb{G}_{\mathrm{a}}$, and the multiplicative group by $\mathbb{G}_{\mathrm{m}}$. A *torus* is a linear algebraic group T defined over a field k which over the algebraic closure $\overline{k}$ is isomorphic to $\mathbb{G}_{\mathrm{m}}^n$ for some n.

We now show that the main theorem follows for reductive groups if it can be proved for almost simple simply-connected groups. We first reduce to semisimple groups, the basic idea to keep in mind is the sequence $SL_n \to GL_n \to \mathbb{G}_{\mathrm{m}}$ which reduces homotopy invariance for GL_n to SL_n.

Proposition 2.1. *To prove Theorem 1.1, it suffices to consider the case where G is semisimple over k.*

Proof. For a reductive group G, we have a split extension of linear algebraic groups

$$1 \to (G,G) \to G \to G/(G,G) \to 1,$$

where (G,G) denotes the commutator subgroup in the sense of linear algebraic groups. The quotient $G/(G,G)$ is a torus. We denote $H = (G,G)$ and $T = G/(G,G)$, and obtain a split exact sequence $1 \to H(A) \to G(A) \to T(A) \to 1$ for any k-algebra A. Assuming that A is smooth and essentially of finite type, we have an isomorphism $T(A) \cong T(A[t])$. From the Hochschild–Serre spectral sequence for the above group extensions we conclude that $G(k) \to G(k[t])$ induces an isomorphism on homology if $H(k) \to H(k[t])$ induces an isomorphism on homology. But $H = (G,G)$ is a semisimple algebraic group over k. □

Proposition 2.2. *To prove Theorem 1.1, it suffices to consider the case where G is almost simple simply-connected over k.*

Proof. For a semisimple group G, there is an exact sequence of algebraic groups

$$1 \to \Pi \to \widetilde{G} \to G \to 1,$$

where Π is a finite central group scheme, and $\widetilde{G}$ is a product of simply-connected almost simple groups. If we assume that Theorem 1.1 holds for these simply-connected almost simple groups, then it is also true for their product, by a simple application of the Hochschild–Serre spectral sequence.

Now assume that the order of Π is prime to the characteristic of the field k, as in Theorem 1.1. From the universal covering above we have an exact sequence

$$1 \to \Pi(R) \to \tilde{G}(R) \to G(R) \to H^1_{\text{ét}}(R, \Pi) \to H^1_{\text{ét}}(R, \tilde{G})$$

for any k-algebra R. Then we have isomorphisms

$$\Pi(k) \cong \Pi(k[t]), \qquad \text{and} \qquad H^1_{\text{ét}}(k, \Pi) \cong H^1_{\text{ét}}(k[t], \Pi),$$

by our assumption on the characteristic of the base field.

Now the first part of the exact sequence above yields a morphism of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi(k) & \longrightarrow & \tilde{G}(k) & \longrightarrow & \tilde{G}(k)/\Pi(k) & \longrightarrow & 1 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Pi(k[t]) & \longrightarrow & \tilde{G}(k[t]) & \longrightarrow & \tilde{G}(k[t])/\Pi(k[t]) & \longrightarrow & 1.
\end{array}$$

Since Π is in fact abelian, one can consider fibre sequences

$$B\tilde{G}(R) \to B\left(\tilde{G}(R)/\Pi(R)\right) \to K(\Pi(R), 2)$$

for $R = k$ and $R = k[t]$. Then the associated Hochschild–Serre spectral sequence implies that the morphism $\tilde{G}(k)/\Pi(k) \to \tilde{G}(k[t])/\Pi(k[t])$ induces an isomorphism on homology, since we argued before that the morphism $\tilde{G}(k) \to \tilde{G}(k[t])$ induces an isomorphism on homology.

Since

$$H^1_{\text{ét}}(k, \tilde{G}) \to H^1_{\text{ét}}(k[t], \tilde{G})$$

is injective (k is a retract of $k[t]$), the images of $G(k[t])$ and $G(k)$ in $H^1_{\text{ét}}(k, \Pi) \cong H^1_{\text{ét}}(k[t], \Pi)$ are equal — we denote these images by $\pi_0(G(k[t]))$ and $\pi_0(G(k))$, respectively. Therefore we get a morphism of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{G}(k[t])/\Pi(k[t]) & \longrightarrow & G(k[t]) & \longrightarrow & \pi_0(G(k[t])) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
1 & \longrightarrow & \tilde{G}(k)/\Pi(k) & \longrightarrow & G(k) & \longrightarrow & \pi_0(G(k)) & \longrightarrow & 1.
\end{array}$$

The outer vertical arrows are isomorphisms on homology, therefore the comparison theorem for the Hochschild–Serre spectral sequences implies that we obtain an isomorphism on the middle arrow. ☐

Henceforth, we shall only consider linear algebraic groups G, defined over k which are almost simple and simply-connected.

3. Bruhat–Tits buildings and the Soulé–Margaux theorem

In this section, we recall the basics of the theory of buildings which will be needed in the remaining sections. The main references are [BT72, AB08].

Let k be a field. Then we equip the function field $K = k(t)$ with the valuation $\omega_\infty(f/g) = \deg(g) - \deg(f)$, with t^{-1} as uniformizer. We denote by $\mathcal{O}$ the corresponding discrete valuation ring. Alternatively, one can work with $K = k((t^{-1}))$ and the corresponding valuation ring $k[[t^{-1}]]$. The underlying simplicial complex of the building will be the same, only the apartment system will be different.

Let G be a reductive group over k. Then we have two morphisms of groups, the inclusion $G(\mathcal{O}) \hookrightarrow G(K)$ and the reduction $G(\mathcal{O}) \to G(k)$.

3.1. BN-pairs and buildings

We will be concerned with affine buildings associated with reductive groups over discretely valued fields. We recall the definition of buildings based on the notion of BN-pairs. This theory is detailed in [AB08], in particular Section 6.

Definition 3.1. A pair of subgroups B and N of a group G is called a *BN-pair* if B and N generate G, the intersection $T := B \cap N$ is normal in N, and the quotient $W = N/T$ admits a set of generators S such that the following two conditions hold:

(BN1) For $s \in S$ and $w \in W$ we have $sBw \subseteq BswB \cup BwB$.

(BN2) For $s \in S$, we have $sBs^{-1} \nsubseteq B$.

The group W is called the *Weyl group* of the BN-pair. The tuple (G, B, N, S) is called *Tits system.*

We now describe the BN-pair on $G(K)$ which will be relevant for us. We mostly stick to the notation used in [Sou79]. Choose a maximal torus $T \subseteq G$. This fixes two subgroups $T(k) \subseteq G(k)$ and $T(K) \subseteq G(K)$. Fix a choice of Borel subgroup $\overline{B}$ in $G(k)$ containing $T(k)$.

> For the definition of the BN-pair, we let $B \subseteq G(K)$ be the preimage of $\overline{B}$ under the reduction $G(\mathcal{O}) \to G(k)$. The group N is defined as the normalizer of $T(K)$ in $G(K)$.

This is the usual construction, explained in detail for the case SL_n in [AB08, Section 6.9]. We will not recall the proof that this indeed yields a BN-pair here.

We recall one particular description of the building associated with a BN-pair from [AB08, Section 6.2.6]. Given a Tits system (G, B, N, S), a subgroup $P \subseteq G$ is called *parabolic* if it contains a conjugate of B. The subgroups of G which contain B are called *standard parabolic subgroups.* These are associated with subsets of S.

The building $\Delta(G, B)$ for (G, B, N, S) is the simplicial complex associated with the ordered set of parabolic subgroups of G, ordered by reverse inclusion. The group G acts via conjugation. The fundamental apartment is given by

$$\Sigma = \{wPw^{-1} \mid w \in W, P \geq B\}.$$

The other apartments are of course obtained by using conjugates of the group B above. Alternatively, the building can be described as the simplicial complex associated with the ordered set of cosets of the standard parabolic subgroups, with the group G acting via multiplication.

3.2. Soulé's fundamental domain

We continue to consider the BN-pair defined above. In the standard apartment Σ of $\Delta(G, B)$, there is one vertex fixed by $G(\mathcal{O})$. This vertex is denoted by ϕ. The fundamental chamber containing the vertex ϕ is given by

$$\mathcal{C} = \{P \mid P \geq B\} \subseteq \Sigma.$$

The fundamental sector $\mathcal{Q}$ is the simplicial cone with vertex ϕ which is generated by $\mathcal{C}$.

The following theorem was proved in [Sou79] and subsequently generalized to isotropic simply-connected absolutely almost simple groups, cf. [Mar09].

Theorem 3.2. *The set $\mathcal{Q}$ is a simplicial fundamental domain for the action of $G(k[t])$ on the Bruhat–Tits building $\Delta(G, B)$. In other words, any simplex of $\Delta(G, B)$ is equivalent under the action of $G(k[t])$ to a unique simplex of $\mathcal{Q}$.*

3.3. Stabilizers

We are also interested in the subgroups which stabilize simplices in the fundamental domain $\mathcal{Q}$. We have seen above that the simplices correspond to standard parabolic subgroups. It turns out that the stabilizer of the simplex corresponding to $G \geq P \geq B$ is exactly P, cf. [AB08, Theorem 6.43]. In particular, for the group $G(k[t])$, we find that the stabilizer of a simplex σ_P corresponding to a parabolic subgroup P of $G(K)$ is the following intersection:

$$\mathrm{Stab}(\sigma(P)) = G(k[t]) \cap P.$$

This implies a concrete description of the stabilizers, cf. [Sou79, Paragraph 1.1] resp. [Mar09, Proposition 2.5].

Proposition 3.3. *Let $x \in \mathcal{Q} \setminus \{\phi\}$. We denote by $\mathrm{Stab}(x)$ the stabilizer of x in $G(k[t])$.*

(i) *There is an extension of groups*

$$1 \to \mathrm{Stab}(x) \cap U_x(K) \to \mathrm{Stab}(x) \to L_x(k) \to 1.$$

The group $L_x(k)$ is a reductive subgroup of $G(k)$, in fact it is a Levi subgroup of a maximal parabolic subgroup of $G(k)$ for the spherical BN-pair. The group $\mathrm{Stab}(x) \cap U_x(K)$ is a split unipotent subgroup of $U_x(k[t])$.

(ii) *The stabilizer of a simplex σ is the intersection of the stabilizers of the vertices x of σ.*

(iii) *The stabilizers can be described using the valuation of the root system, cf. [Sou79, Section 1.1]. In particular, in the notation of Soulé, we have*

$$\Gamma_x = L_x(k) \cdot U_x(k[t]), \quad L_x(k) = T(k) \cdot \langle x_\alpha(k) \mid \alpha(x) = 0 \rangle,$$

$$U_x(k[t]) = \langle x_\alpha(u), u \in k[t], d \circ (u) \leq \alpha(x), \alpha(x) > 0 \rangle.$$

Without explaining all the notation in detail, this means that an element of $Z_x(k) = L_x(k)$ is a product of an element of the torus and certain root elements, where the roots only depend on the vertex x. An *element in $U_x(k[t])$ is a product of certain root elements, the degree of the polynomials and the roots only depend on the vertex x.*

4. Homology of the stabilizers

In this section, we describe the homology of the stabilizers of simplices in Soulé's domain. In [Knu97], Knudson showed in the case SL_n that the homology of the stabilizers is determined by the homology of a Levi subgroup. We provide below a generalization of this result. The results work in general for rings with many units.

From Proposition 3.3, we know that for a simplex σ in the fundamental domain $\mathcal{Q}$, the stabilizer $\Gamma_\sigma = \mathrm{Stab}(\sigma)$ of σ in $G(k[t])$ sits in an extension

$$1 \to U_\sigma \to \Gamma_\sigma \to L_\sigma \to 1,$$

where U_σ is an abstract group contained in a unipotent subgroup of $G(k[t])$ and L_σ is the group of k-points of a reductive subgroup of G. We will show in this section is that the induced morphism $H_\bullet(\Gamma_\sigma, \mathbb{Z}) \to H_\bullet(L_\sigma, \mathbb{Z})$ is an isomorphism. This is done via the Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(L_\sigma, H_q(U_\sigma)) \Rightarrow H_{p+q}(\Gamma_\sigma)$$

associated with the above group extension. To show the result, it suffices to show that $H_p(L_\sigma, H_q(U_\sigma)) = 0$, for $q > 0$.

The basic idea for showing this latter assertion is to use the action of $k^\times$ on the group U_σ, where $k^\times$ is embedded in L_σ as the k-points of a suitable subtorus. The group $k^\times$ acts via multiplication on the various abelian subquotients constituting the unipotent group U_σ, and an argument as in [Knu01, Theorem 2.2.2] shows that this homology is trivial. This argument is detailed after some introductory remarks in Theorem 4.6.

4.1. A result of Suslin

A ring A is an $S(n)$-ring if there are $a_1, \dots, a_n \in A^\times$ such that the sum of each non-empty subfamily is a unit. If A is an $S(n)$-ring for all n, then A is said to *have many units.*

As explained in [Knu01, Section 2.2.1], the right way to prove that

$$H_p(GL_n(A), H_q(M_{n,m}(A), \mathbb{Z})) = 0$$

for $q > 0$ if A is a $\mathbb{Q}$-algebra is the following: we notice that $M_{n,m}(A)$ is an abelian group, and therefore $H_q(M_{n,m}(A)) = \bigwedge^q M_{n,m}(A)$. There exists a central element $a \in GL_n(A)$ which acts on $M_{n,m}(A)$ by multiplication with a, and therefore by multiplication with a^q on $H_q(M_{n,m}(A))$. This action is trivial, because a is in the centre, and therefore $H_q(M_{n,m}(A))$ is annihilated by $a^q - 1$. But it is a $\mathbb{Q}$-vector space and therefore it is trivial for $q > 0$.

The following result due to Nesterenko and Suslin [NS90] is a generalization of this centre-kills-argument to rings with many units in arbitrary characteristics. For more information on the proof of this result, cf. [Knu01, Section 2.2.1].

Proposition 4.1. *Let A be a ring with many units, and let F be a prime field. Then for all $i \geq 0$ and $j > 0$, we have $H_i(A^\times, H_j(A^s, F)) = 0$, where $A^\times$ acts diagonally on A^s.*

The same conclusion also obtains for actions of $A^\times$ via non-zero powers of units, cf. [Hut90, Lemma 9].

4.2. Example: orthogonal groups

We explain the procedure using the special case of orthogonal groups from [Vog79]. For the groups $O_{n,n}$ over a field k of characteristic $\neq 2$, there are maximal parabolic subgroups P_I which have a non-abelian unipotent radical. They have the following general form, cf. [Vog79, p. 21]:

$$\begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & {}^tA^{-1} \end{pmatrix},$$

where $A \in GL_p(k)$, $B \in O_{n-p,n-p}(k)$ and there are some additional conditions on the $*$-terms ensuring that the whole matrix is in $O_{n,n}(k)$. It is proved on p. 34 of that paper that the unipotent subgroup N of P_I sits in an exact sequence

$$1 \to [N,N] \to N \to N/[N,N] \to 1$$

with the outer terms $[N,N]$ and $N/[N,N]$ abelian groups. It is also proved that the torus

$$\mathrm{diag}(\underbrace{a,\ldots,a}_{p},\underbrace{1,\ldots,1}_{2n-2p},\underbrace{a^{-1},\ldots,a^{-1}}_{p})$$

acts via multiplication with a on $N/[N,N]$ and multiplication with a^2 on $[N,N]$.

We apply the Hochschild–Serre spectral sequence for the extension

$$1 \to [N,N] \to P_I \to P_I/[N,N] \to 1.$$

This has the following form:

$$H_p(P_I/[N,N], H_q([N,N])) \Rightarrow H_{p+q}(P_I).$$

To prove that $H_p(P_I/[N,N], H_q([N,N])) = 0$ for $q > 0$, we use another Hochschild–Serre spectral sequence for the torus action:

$$H_p(P_I/[N,N]/k^\times, H_j(k^\times, H_q([N,N])) \Rightarrow H_{p+j}(P_I/[N,N], H_q([N,N])).$$

Since the action of $k^\times$ on $[N,N]$ is via multiplication by squares, the result of Suslin, cf. Proposition 4.1, implies $H_p(P_I/[N,N], H_q([N,N])) = 0$ for $q > 0$.

The morphism $P_I \to P_I/[N,N]$ thus induces an isomorphism on homology. A similar argument applied to the extension

$$1 \to N/[N,N] \to P_I/[N,N] \to P_I/N \to 1$$

implies that the morphism $P_I/[N,N] \to P_I/N$ also induces an isomorphism on homology.

This amounts to a proof of [Vog79, Proposition 2.2] for infinite fields of characteristic $p \neq 2$. We obtain the following strengthening of Vogtmann's stability result, making explicit a remark in [Knu01, Section 2.4.1].

Corollary 4.2. *Let k be an infinite field of characteristic $\neq 2$. Then the induced morphism*

$$H_i(O_{n,n}(k), \mathbb{Z}) \to H_i(O_{n+1,n+1}(k), \mathbb{Z})$$

is surjective for $n \geq 3i+1$ and an isomorphism for $n \geq 3i+3$.

Remark 4.3. (i) The above stabilization result is the one obtained via Vogtmann's argument in [Vog79] using the improved computation of the homology of stabilizers. Better stabilization results for orthogonal groups are available in the work of Essert, cf. [Ess09]. In Essert's work, dealing with the homological contribution from the unipotent radical as above is not necessary — he uses opposition complexes, where stabilizers are Levi subgroups instead of the full parabolic subgroups.

(ii) The above example for orthogonal groups is an instance of a more general result which can be found, e.g., in [ABS90]. Let G be a reductive group over k, let Φ be the associated root system and assume that $\operatorname{char} k$ is not equal to 2 for Φ doubly laced resp. not equal to 2 or 3 for Φ triply laced. Let P be a parabolic subgroup associated with a subset $I \subseteq \Phi$ of simple roots, and let U be the unipotent radical of P. Then the length of the descending central series of U equals $\sum_{\alpha \in I} m(\alpha)$ where α is the multiplicity of α in the highest root $\widetilde{\alpha}$ of Φ.

In the above example of orthogonal groups, the root system is of type D_n. Numbering the simple roots $\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$ such that α_1, α_{n-1} and α_n correspond to the end-points of the Dynkin diagram, the longest root is

$$\widetilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

The parabolic subgroups discussed in the above example are the ones corresponding to roots $\alpha_2, \ldots, \alpha_{n-2}$.

4.3. Homology of the stabilizers

We now want to compute the homology of the stabilizers. We formulate the proof with the k-points of the stabilizers, where k is an infinite field. The same arguments show the result more generally for an integral domain R with many units having quotient field k. The goal is to compute the homology of the stabilizer $\Gamma_\sigma(k[t])$. The Levi subgroup L_σ is defined as $L_\sigma = Z_G(S_\sigma)$, i.e., as the centralizer of a split torus S_σ in G associated with the simplex σ. We note that it follows from this definition that there is a normal central torus in L_σ. Now consider a subtorus $\mathbb{G}_{\mathrm{m}} \to L_\sigma$. If the corresponding abstract group $k^\times$ acts trivially on the k-points of a unipotent subgroup $U \subseteq G(k[t])$, then $U \subseteq Z_G(\mathbb{G}_{\mathrm{m}})$ and hence U is contained in L_σ. Therefore, for any unipotent subgroup $U \subseteq \Gamma_\sigma$ which is not contained in L_σ, there has to exist a torus $\mathbb{G}_{\mathrm{m}} \subseteq L_\sigma$ such that the corresponding group of k-points $k^\times$ acts nontrivially on the k-points of U.

Note that the unipotent radicals U_σ of parabolic subgroups of G associated with the simplex σ are actually split, i.e., there is a filtration

$$U_\sigma = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n = \{1\}$$

with each U_n/U_{n+1} being isomorphic to $\mathbb{G}_{\mathrm{a}}$. Since the automorphism group of $\mathbb{G}_{\mathrm{a}}$ is $\mathbb{G}_{\mathrm{m}}$, the multiplicative group $\mathbb{G}_{\mathrm{m}}$ can only act via

$$(a \in k^\times, u \in k) \mapsto a^n u$$

for some n. We have thus established the following:

Lemma 4.4. *Let U_σ be the unipotent radical of the stabilizer Γ_σ, and let*

$$U_\sigma = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n = \{1\}$$

be a filtration such that $U_i/U_{i+1} \cong \mathbb{G}_{\mathrm{a}}$. For each i, there exists a central embedding $k^\times \to L_\sigma$ and a non-zero number n_i such that $a \in k^\times$ acts on U_i/U_{i+1} via multiplication with a^{n_i}.

Note that above we are only talking about algebraic groups, i.e., about the unipotent radical of parabolic subgroups of $G(k(t))$. However, since the $k[t]$-points of the torus are $k^\times$, the action preserves the degree filtration of the $k[t]$-points of unipotent radicals. In particular, the action described above restricts to an action of $k^\times$ on the unipotent part U_σ of the stabilizer subgroup Γ_σ, for any simplex $\sigma \in \mathcal{Q}$.

Example 4.5. (i) The simplest example of this situation is the embedding

$$R^\times \to SL_{n+m}(R) : a \mapsto \operatorname{diag}(\underbrace{a^m, \ldots, a^m}_{n}, \underbrace{a^{-n}, \ldots, a^{-n}}_{m}).$$

The centralizer of the corresponding torus $\mathbb{G}_{\mathrm{m}} \to SL_{n+m}$ is the Levi subgroup of a maximal parabolic subgroup which is the intersection of the following subgroup with SL_{n+m}:

$$\begin{pmatrix} GL_n & 0 \\ 0 & GL_m \end{pmatrix}.$$

The corresponding parabolic subgroup has the form

$$\begin{pmatrix} GL_n & M \\ 0 & GL_m \end{pmatrix} \cap SL_{n+m}$$

and the torus acts on M via multiplication with a^{m+n}, cf. [Hut90].

(ii) Another example of such a situation is the one discussed in the proof of Corollary 4.2, cf. also [Vog79, p. 34]. In these cases, the unipotent radical of a maximal parabolic of a split orthogonal or symplectic group is not abelian, and the torus acts via different powers on the steps of the central series.

□

The above actions now allow us to compute the E_2-term of the Hochschild–Serre spectral sequence. This is done using the composition series of U_I, which induces a sequence

$$\Gamma_I \to \Gamma_I/\mathbb{G}_a = \Gamma_I/U_n \to \Gamma_I/U_2 \to \cdots \to \Gamma_I/U_I = L_I.$$

We will show below that each step induces isomorphisms on homology. The argument is a generalization of the proof of [Knu97, Corollary 3.2].

The following theorem now describes the homology of the stabilizers of the action of $G(k)$ on the Bruhat–Tits building.

Theorem 4.6. *Let R be an integral domain with many units and denote by $k = Q(R)$ its field of fractions. The group $G(R[t])$ acts on the Bruhat–Tits building associated with the group $G(k(t))$, and we consider the stabilizer group Γ_σ of a simplex $\sigma \in \mathcal{Q}$. Then the morphism*

$$H_\bullet(\Gamma_\sigma, \mathbb{Z}) \to H_\bullet(L_\sigma, \mathbb{Z})$$

induced from the projection in Proposition 3.3 is an isomorphism.

Proof. Consider the composition series of U_σ:

$$U_\sigma = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n = \{1\}.$$

More precisely, the composition series of the unipotent group as an algebraic group induces a similar filtration of the unipotent part of the stabilizer, which is defined inside the unipotent radical by degree bounds as in Proposition 3.3. This induces a sequence of group homomorphisms

$$\Gamma_\sigma \cong \Gamma_\sigma/U_n \to \cdots \to \Gamma_\sigma/U_2 \to \Gamma_\sigma/U_\sigma \cong L_\sigma.$$

Each step in this sequence is a quotient by a subgroup of $\mathbb{G}_a(R[t])$ in Γ_σ/U_i. It therefore suffices to show that each such morphism induces an isomorphism on homology. This is done via the Hochschild–Serre spectral sequence, which then looks like

$$H_p(\Gamma_\sigma/U_i, H_q(U_i/U_{i+1}, \mathbb{Z})) \Rightarrow H_p(\Gamma_\sigma/U_{i+1}, \mathbb{Z}).$$

Thus it suffices to show for any prime field F, we have

$$H_p(\Gamma_\sigma/U_i, H_q(U_i/U_{i+1}, F)) = 0$$

for $q > 0$. But by Lemma 4.4, there is a central embedding $R^\times \to \Gamma_\sigma/U_i$ such that $a \in R^\times$ acts on U_i/U_{i+1} via multiplication by some non-zero power of a.

We have an associated Hochschild–Serre spectral sequence

$$\begin{aligned} E^2_{j,l} = H_j((\Gamma_\sigma/U_i)/R^\times, H_l(R^\times, H_q(U_i/U_{i+1}, F))) \Rightarrow \\ \Rightarrow H_{j+l}(\Gamma_\sigma/U_i, H_q(U_i/U_{i+1}, F)). \end{aligned}$$

From Proposition 4.1, we obtain that $H_l(R^\times, H_q(U_i/U_{i+1}, F)) = 0$ for $q > 0$, which finishes the proof. □

5. Generalization of the theorem of Knudson

We will now prove homotopy invariance in the one-variable case. The following is a generalization of [Knu01, Corollary 4.6.3].

Theorem 5.1. *Let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism*

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k.

Proof. By Proposition 2.2, we can assume that G is simply-connected absolutely almost simple over k. Then we can use the action of the group $G(k[t])$ on the building associated with $G(k(t))$. By Theorem 3.2, the subcomplex $\mathcal{Q}$ is a fundamental domain for this action. There is an associated spectral sequence

$$E^1_{p,q} = \bigoplus_{\dim\sigma = p, \sigma\in\mathcal{Q}} H_q(\Gamma_\sigma, \mathbb{Z}) \Rightarrow H_{p+q}(G(k[t]), \mathbb{Z}).$$

In the above, Γ_σ is the stabilizer of the simplex σ. This is the spectral sequence computing the $G(k[t])$-equivariant homology of the building, cf. [Knu01, p. 162].

From Theorem 4.6, we know the homology of the stabilizers, in particular, that it only depends on the reductive part. In the notation of [Knu97], there is a filtration of the fundamental domain $\mathcal{Q}$ via subsets $E_I^{(k)}$ for any k-element subset I of roots of G. These subsets are simplicial subcones of $\mathcal{Q}$ which consist of all simplices of $\mathcal{Q}$ such that the constant part of the stabilizer is the standard parabolic subgroup of G determined by the subset I. This yields a filtration

$$\mathcal{Q}^{(k)} = \bigcup_I E_I^{(k)}.$$

Now for any two simplices σ, τ in

$$E_I^{(k)} \setminus \bigcup_{J \subset I} E_J^{(k-1)},$$

the stabilizers Γ_σ and Γ_τ have the same reductive part $L_\sigma = L_\tau$, and therefore they also have the same homology, by Theorem 4.6. The coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$ is then locally constant in the sense of [Knu01, Proposition A.2.7], and we obtain an isomorphism

$$H_\bullet(\phi, \mathcal{H}_q) \to H_\bullet(\mathcal{Q}, \mathcal{H}_q).$$

This shows that the argument in the proof of [Knu97, Theorem 3.4] does not depend on SL_n. Therefore, [Knu01, Proposition A.2.7] implies that the E_2-term of the above spectral sequence looks as follows:

$$E^2_{p,q} = \begin{cases} H_q(G(k), \mathbb{Z}) & p = 0, \\ 0 & p > 0. \end{cases}$$

The spectral sequence degenerates and the result is proved. □

Remark 5.2. Theorem 4.6 has been used in [Wen12] to establish homotopy invariance for the homology of Steinberg groups of rank two groups. Apart from this, it seems that the added generality of rings with many units in Theorem 4.6 can not be widely applied. Generalizations of Theorem 5.1 beyond the case $k[t]$ seem to be generally wrong. The failure of homotopy invariance for H_1 of SL_2 follows directly from [KM97]. The failure of homotopy invariance for H_2 of rank two groups has been established in [Wen12]. In these cases, one sees that $\mathcal{Q}$ fails quite badly to be a fundamental domain for the action of $G(R[t])$ if R is not a field — in case SL_2, the subcomplex $SL_2(R[t]) \cdot \mathcal{Q}$ is not connected and in case SL_3, the subcomplex $SL_3(R[t]) \cdot \mathcal{Q}$ is not simply-connected.

Concerning the subcomplex $G(R[t]) \cdot \mathcal{Q}$ for R an integral domain, we have the following:

Proposition 5.3. (i) *The complex $E(R[t]) \cdot \mathcal{Q}$ is connected.*

(ii) *The complex $G(R[t]) \cdot \mathcal{Q}$ is connected if $G(R[t]) = E(R[t]) \cdot G(R)$. This in particular holds for isotropic reductive groups G of rank ≥ 2 and R essentially smooth over a field.*

(iii) *The complex $E(R[t]) \cdot \mathcal{Q}$ is simply-connected if $K_2^G(R[t]) \cong K_2^G(R)$, in particular for $G = SL_n$, $n \geq 5$ and $R = k[t_1, \dots, t_m]$.*

Proof. Every elementary matrix for a positive root is contained in some stabilizer, and the stabilizer of ϕ contains the Weyl group. By [Sou79, Theorem 2] the complex $E(\Phi, R[t]) \cdot \mathcal{Q}$ is connected, hence (i). The same argument shows (ii). The additional consequence in (ii) is the work of Suslin [Sus77], Abe [Abe83] and in the non-split case Stavrova [Sta11]. We only sketch (iii): it follows from the assumption on homotopy invariance of K_2 that $E(R[t])$ is an amalgam of the stabilizers along their intersections. Again [Sou79, Theorem 2] shows the claim. The additional assertion for SL_n is a consequence of [Tul82]. □

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References

[Abe83] E. Abe, *Whitehead groups of Chevalley groups over polynomial rings*, Comm. Algebra **11** (1983), 1271–1307.

[AB08] P. Abramenko and K.S. Brown, *Buildings. Theory and applications*, in 'Graduate texts in mathematics', **248**, Springer, 2008.

[ABS90] H. Azad, M. Barry and G. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18**(2) (1990), 551–562.

[Bor91] A. Borel, *Linear algebraic groups*, in 'Graduate texts in mathematics', **126**, Springer, 1991.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[Ess09] J. Essert, *Homological stability for classical groups*, Israel J. Math. **198**(1) (2013), 169–204.

[Hut90] K. Hutchinson, *A new approach to Matsumoto's theorem*, K-Theory **4** (1990), 181–200.

[Knu97] K.P. Knudson, *The homology of special linear groups over polynomial rings*, Ann. Sci. Éc. Norm. Supér. (4) **30** (1997), 385–415.

[Knu01] ———, *Homology of linear groups*, in 'Progress in Mathematics', **193**, Birkhäuser, 2001.

[KM97] S. Krstić and J. McCool, *Free quotients of $SL_2(R[x])$*, Proc. Amer. Math. Soc. **125** (1997), 1585–1588.

[Mar09] B. Margaux, *The structure of the group $G(k[t])$: variations on a theme of Soulé*, Algebra Number Theory **3**(4) (2009), 393–409.

[NS90] Yu. P. Nesterenko and A.A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Math. USSR-Izv. **34** (1990), 121–145.

[Qui73] D.G. Quillen, *Higher algebraic K-theory I, Algebraic K-theory I: higher K-theories*, Lecture Notes in Mathematics **341**, Springer, 1973, 85–147.

[Sou79] Ch. Soulé, *Chevalley groups over polynomial rings*, in 'Homological group theory', London Math. Soc. Lecture Notes **36**, Cambridge University Press, 1979, 359–367.

[Sta11] A. Stavrova, *Homotopy invariance of non-stable K_1-functors*, J. K-theory **13**(2) (2014), 199–248.

[Sus77] A.A. Suslin, *The structure of the special linear group over rings of polynomials*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 235–252.

[Tul82] M.S. Tulenbaev, *The Steinberg group of a polynomial ring*, Mat. Sb. (N.S.) **117**(159) (1982), no. 1, 131–144.

[Vog79] K. Vogtmann, *Homology stability for $O_{n,n}$*, Comm. Algebra **7** (1979), 9–38.

[Wen12] M. Wendt, *On homotopy invariance for homology of rank two groups*, J. Pure Appl. Algebra **216**(10) (2012), 2291–2301.

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