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Cohomology of $GL_4(\mathbb{Z})$ with nontrivial coefficients

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We compute the cohomology groups of $GL_4(\mathbb{Z})$ with coefficients in symmetric powers of the standard representation twisted by the determinant. This problem arises in Goncharov's approach to the study of motivic multiple zeta values of depth 4. We use a result of Harder on Eisenstein cohomology and a computationally effective version for the homological Euler characteristic of arithmetic groups.

1. Introduction

1.1. Main result and applications

The main goal of this paper is to present a computation of cohomology groups

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det),$$

where $S^{n-4}V_4$ is the $(n-4)$th symmetric power of the standard representation V_4 and *det* is the determinant representation.

The above cohomology groups describe certain spaces of motivic multiple zeta values. This relation was revealed by Goncharov who suggested me the problem of computing the cohomology groups of $GL_4(\mathbb{Z})$.

Recall the definition of multiple zeta values

$$\zeta(k_1, \ldots, k_m) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1} \ldots n_m^{k_m}},$$

where $k_1 + \cdots + k_m$ is called weight and m is called depth.

Goncharov has described the cases of depth $= 2$ [Gon98] and of depth $=$ 3 [Gon01]. He relates the space of motivic multiple zeta values of depth $= 2$ and weight $= n$ to the cohomology groups of $GL_2(\mathbb{Z})$ with coefficients in the $(n-2)$-symmetric power of the standard representation V_2. Namely, to

$$H^i(GL_2(\mathbb{Z}), S^{n-2}V_2).$$

He calls this a misterious relation between the multiple zeta values of depth $=$ m and the "modular variety"

$$GL_m(\mathbb{Z})\backslash GL_m(\mathbb{R})/SO_m(\mathbb{R}) \times \mathbb{R}^{\times}_{>0}.$$

In the papers [Gon97], [Gon01], he relates the spaces of motivic multiple zeta values of depth $= 3$ and weight $= n$ to the cohomology of $GL_3(\mathbb{Z})$ with coefficients in the $(n-3)$-symmetric power of the standard representation

V_3, namely,

$$H^i(GL_3(\mathbb{Z}), S^{n-3}V_3).$$

Goncharov has also related the case of multiple zeta values of depth $=$ 4 and weight $= n$ to the computation of the cohomology of $GL_4(\mathbb{Z})$ with coefficients in the $(n-4)$-symmetric power of the standard representation V_4 twisted by the determinant (private communications). That is, in order to compute the spaces of motivic multiple zeta values of depth $= 4$ and weight $= n$ one has to know

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det).$$

The main result of this paper is the following.

Theorem 1.1. *The dimensions of the cohomology groups of $GL_4(\mathbb{Z})$ with coefficients the symmetric powers of the standard representation twisted by the determinant are given by*

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} \mathbb{Q} \oplus H^1_{\text{cusp}}(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det) & \text{for } i = 3, \\ 0 & \text{for } i \neq 3. \end{cases}$$

More explicitly,

$$\dim(H^3(GL_4(\mathbb{Z}), S^{12n-4+k}V_4 \otimes \det)) = \begin{cases} n+1 & \text{for } k = 0, 4, 6, 8, 10, \\ n & \text{for } k = 2, \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

1.2. Computational methods and notation

All representations that we consider are finite-dimensional representations defined over $\mathbb{Q}$ of subgroups of $GL_m(\mathbb{Q})$. We assume that the reader is familiar with group cohomology. A good introduction on this subject and on various Euler characteristics of a group see [Bro82].

We are going to describe briefly various types of cohomologies of arithmetic groups. Namely, boundary cohomology, cohomolgy at the infinity, Eisenstein cohomology, interior cohomology and cuspidal cohomology. All of them are based on a compactification of certain space, called Borel–Serre compactification. The reader who is not familiar with these constructions should not be discouraged. We have tried to present a piece of "Calculus" for cohomology of arithmetic groups. That is, we give the definitions intuitively rather than strictly, and describe the computational tools which we are going to use. The constructions and the proofs of the basic tools could be found in the cited literature. What we do in the main part of this paper is to present the desired computation based on these tools.

We start with the Borel–Serre compactification [BS73]. Let Γ be a subgroup of $GL_m(\mathbb{Q})$ which is commensurable to $GL_m(\mathbb{Z})$. That is, the intersection $\Gamma \cap GL_m(\mathbb{Z})$ is of finite index both in Γ and in $GL_m(\mathbb{Z})$. Let

$$X = GL_m(\mathbb{R})/SO_m(\mathbb{R}) \times \mathbb{R}^{\times}_{>0}.$$

Then X is a contractable topological space on which Γ acts on the left. And let

$$Y_\Gamma = \Gamma \backslash X.$$

Then the Borel–Serre compactification of Y_Γ, denoted by $\overline{Y}_\Gamma$, is a compact space, containing Y_Γ. Moreover, it is of the same homotopy type as Y_Γ. If V is a representation of Γ and $V^\sim$ is the corresponding sheaf then

$$H^i_{\mathrm{top}}(\overline{Y}_\Gamma, V^\sim) = H^i_{\mathrm{group}}(\Gamma, V).$$

The space $\overline{Y}_\Gamma$ can be split into strata, where each stratum corresponds to a parabolic subgroup P of $GL_{m/\mathbb{Q}}$ and the maximal stratum is Y_Γ. Also the closure of a strata corresponding to a parabolic subgroup P consists of all strata corresponding to parabolic subgroups Q so that $Q \subset P$. Let $Y_{\Gamma,P}$ be the stratum corresponding to a parabolic subgroup P. Let

$$P(\mathbb{Z}) = P(\mathbb{Q}) \cap \Gamma.$$

Then the topological cohomology of $\overline{Y}_P$ coincides with the group cohomology of $P(\mathbb{Z})$. More precisely,

$$H^i_{\mathrm{top}}(\overline{Y}_P, j_P^* V^\sim) = H^i_{\mathrm{group}}(P(\mathbb{Z}), V),$$

where V is a representation over the rational numbers and $V^\sim$ the corresponding sheaf on $\overline{Y}_\Gamma$ and $j_P^* V^\sim$ is its restriction on $\overline{Y}_{\Gamma,P}$.

The boundary of the Borel–Serre compactification is

$$\partial \overline{Y}_\Gamma = \overline{Y}_\Gamma - Y_\Gamma = \cup_P \overline{Y}_{\Gamma,P}.$$

The inclusion

$$j : \partial \overline{Y}_\Gamma \subset \overline{Y}_\Gamma$$

induces

$$j^\# : H^i_{\mathrm{top}}(\overline{Y}_\Gamma, V^\sim) \to H^i_{\mathrm{top}}(\partial \overline{Y}_\Gamma, j^* V^\sim).$$

We call the range of the last map $j^{\#}$ cohomology of the boundary. We use the notation

$$H^i_{\partial}(\Gamma, V) := H^i_{\mathrm{top}}(\partial\overline{Y}_{\Gamma}, j^* V^{\sim}).$$

We warn the reader that it is not a standard notation.

The image of the map $j^{\#}$ is called cohomology at the infinity of Γ. We use the notation

$$H^i_{\mathrm{inf}}(\Gamma, V) := \mathrm{Im}(j^{\#}).$$

And the kernel of the map $j^{\#}$ is called interior cohomology of Γ. We use the notation

$$H^i_!(\Gamma, V) := \mathrm{Ker}(j^{\#}).$$

For the representations that we will consider we have that the cohomology at infinity coincides with the Eisenstein cohomology. This is used for describing certain maps between cohomology groups. Also the interior cohomology coincides with e cuspidal cohomology. In the representations that we will consider we are going to use that in order to how that the interior cohomology vanishes.

In our problem we have

$$H^i_{\mathrm{cusp}}(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = 0,$$

where V_m is the standard m-dimensional representation of $GL_m(\mathbb{Q})$. And S^n is the n-th symmetric power. The last equality holds for $n > 4$ because the representation

$$S^{n-4}V_4 \otimes \det$$

is not self-dual. For $n = 4$ it is true because

$$H^i_{\mathrm{cusp}}(SL_4(\mathbb{Z}), \mathbb{Q}) = 0.$$

Thus, we need to compute only the Eisenstein cohomology.

The highest weight representation will be denoted by $L[a_1, \ldots, a_m]$, where the weight $[a_1, \ldots, a_m]$ sends $\mathrm{diag}[H_1, \ldots, H_m]$ to $a_1(H_1) + \cdots + a_m$ (H_m). Sometimes we shall denote the weight simply by λ. At a later stage there will be a number of cohomologies to consider. In order to make the answer more observable, sometimes we abbreviate. For example:

$$H^i(L[a_1,\dots,a_d]) := H^i(GL_d(\mathbb{Z}), L[a_1,\dots,a_d]).$$

For further abbreviation we set

$$\begin{aligned}
&(a_1,a_2|a_3|a_4) := H^1(L[a_1,a_2])\otimes H^0(L[a_3])\otimes H^0(L[a_4]),\\
&(a_1|a_2,a_3|a_4) := H^0(L[a_1])\otimes H^1(L[a_2,a_3])\otimes H^0(L[a_4]),\\
&(a_1|a_2|a_3,a_4) := H^0(L[a_1])\otimes H^0(L[a_2])\otimes H^1(L[a_3,a_4]),\\
&(a_1,a_2|a_3,a_4) := H^1(L[a_1,a_2])\otimes H^1(L[a_3,a_4]),\\
&(a_1|a_2|a_3|a_4) := H^0(L[a_1])\otimes H^0(L[a_2])\otimes H^0(L[a_3])\otimes H^0(L[a_4]).
\end{aligned}$$

We also will use the abbreviation

$$(\overline{a_1,a_2}|a_3|a_4) := H^1_{\mathrm{cusp}}(L[a_1,a_2])\otimes H^0(L[a_3])\otimes H^0(L[a_4]).$$

We consider the parabolic subgroups of GL_4 that contain a fixed Borel subgroup. We shall consider the standard representation of GL_4 with the choice of the Borel subgroup B being the upper triangular matrices. Then the parabolic subgroups can be listed in the following way: P_{ij} is the smallest parabolic subgroup containing a non-zero a_{ji}-entry. And $P_{12,34}$ is the smallest parabolic subgroup containing $a_{21}\neq 0$ and $a_{43}\neq 0$. More precisely, all parabolic subgroups contain B which is upper triangular. Also, P_{12} has a quotient $GL_2\times GL_1\times GL_1$, P_{23} has a quotient $GL_1\times GL_2\times GL_1$, P_{34} has a quotient $GL_1\times GL_1\times GL_2$, P_{13} has a quotient $GL_3\times GL_1$, P_{24} has a quotient $GL_1\times GL_3$, and $P_{12,34}$ has a quotient $GL_2\times GL_2$.

We are going to use the Kostant's theorem [Kos61] in order to obtain information about the parabolic subgroups. To do that we need to examine carefully the action of the Weyl group, W on the root system of gl_n. Also we need the Weyl group, W_P associated with the algebra P. In order to use the Kostant theorem, we need to examine the action of the Weyl group W on the root system of gl_n up to permutation of the root system of P. That is, we need to consider representatives of the quotient $W_P\backslash W$.

Theorem 1.2. *Let V be a representation of highest weight λ. Let N_P be nilpotent radical, and let ρ be half of the sum of the positive roots. Then*

$$H^i(N_P,V)=\oplus_\omega L_{\omega(\lambda+\rho)-\rho},$$

where the sum is taken over the representatives of the quotient $W_P\backslash W$ with minimal length such that their length is exactly i. In the above notation, L_λ means representation of N_P with highest weight λ.

Let $[a, b, c, d]$ denote an element of the root lattice (inside h^*) whose value on the diagonal entry $[H_{11}, H_{22}, H_{33}, H_{44}]$ in h is $aH_{11} + bH_{22} + cH_{33} + dH_{44}$. The Weyl group acts on the weight lattice by permuting the entries of $[a, b, c, d]$. It is well know that the Weyl group is generated by reflections perpendicular to the primitive roots. We can choose positivity so that the primitive roots correspond to the permutation (12), (23) and (34), (having sl_4 in mind; (12) sends $[a, b, c, d]$ to $[b, a, c, d]$.) Then length of an element of the Weyl group is precisely the (minimal) number of successive transpositions, or equivalently, the (minimal) number of reflections w.r.t. the primitive roots. In this setting, the right quotient $W_P\backslash W$ can be interpreted as shuffles in the following way: take for example the parabolic subalgebra P_{23}. Its Levi quotient $M_P = M_{P_{23}}$ is $gl_1 \times gl_2 \times gl_1$. Thus, W_P is generated by (23). Among the representatives, of the quotient $W_P\backslash W$ we can consider the ones that preserve the order of the subset $\{23\}$ inside $\{1234\}$. Thus, we can consider all shuffles of $\{1|23|4\}$. Similarly, if we take the parabolic subalgebra $P_{12,34}$, we need to consider the shuffles of $\{12|34\}$. And for the subalgebras P_{13} we consider the shuffles of the set $\{123|4\}$, which means permutations of $\{1234\}$ such that the order $\{123\}$ is preserved.

In order to apply Kostant's theorem, we need to examine the length of each element ω in the Weyl group W, and also the resulting weight $\omega(\lambda + \rho) - \rho$, where $\lambda = [a, b, c, d]$ is the weight of V and ρ is half of the sum of the positive roots.

After we obtain the cohomology of the parabolic groups we have to consider a spectral sequence involving these cohomologies in order to obtain the cohomology of the boundary of the Borel–Serre compactification. Then we use trace formulas in order to compute the boundary cohomology of $GL_m(\mathbb{Z})$ for $m = 2, 3, 4$.

2. Homological Euler characteristics of $GL_m(\mathbb{Z})$

We call *homological Euler characteristic of a group* Γ the alternating sum of the dimension of the cohomology of the group. We denote it by $\chi_h(\Gamma, V)$, where V is a finite dimensional representation of Γ. More precisely,

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim H^i(\Gamma, V).$$

In this section, we compute the homological Euler characteristics of $GL_m(\mathbb{Z})$ for $m = 2, 3, 4$ with representations which later will occur in the Kostant's formula applied to $GL_4(\mathbb{Z})$ with coefficients in the representation $(n-4)$-th symmetric power of the standard representation twisted by the determinant which is $L[n-3,1,1,1]$.

The material in this section is in the spirit of the papers [Hora, Hor05]. Most of the formulas and notations are taken from there. Only the computation of $\chi_h(GL_3(\mathbb{Z}), L[n-3,1,0])$ is done here in details, the rest are taken from the above two papers.

We start with $GL_2(\mathbb{Z})$.

Theorem 2.1. *Let S^nV_2 be the nth symmetric power of the standard representation of GL_2. Then*

$$\chi_h(GL_2(\mathbb{Z}), S^{12n+k}V_2) = \begin{cases} -n+1 & k=0, \\ -n & k=2,4,6,8, \\ -n-1 & k=10, \\ 0 & k=\text{odd}, \end{cases}$$

and

$$\chi_h(GL_2(\mathbb{Z}), S^{12n+k}V_2 \otimes \det) = \begin{cases} -n & k=0, \\ -n-1 & k=2,4,6,8, \\ -n-2 & k=10, \\ 0 & k=\text{odd}. \end{cases}$$

For $GL_m(\mathbb{Z})$ $m=3$ and 4, we need to consider the representations

$$L[n-3,1,0] = \mathrm{Ker}(S^{n-3}V_3 \otimes V_3 \to S^{n-2}V_3),$$

$$L[n-2,1,1] = S^{n-3}V_3 \otimes \det,$$

$$L[n-2,2,2] = S^{n-4}V_3,$$

$$L[n-3,1,1,1] = S^{n-4}V_4 \otimes \det.$$

Theorem 2.2. *The homological Euler characteristics of $GL_3(\mathbb{Z})$ and $GL_4(\mathbb{Z})$ with coefficients in the above representation are given by*

(a) $\chi_h(GL_3(\mathbb{Z}), L[n-3,1,0]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4}V_2) - \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2)$,

(b) $\chi_h(GL_3(\mathbb{Z}), L[n-2,1,1]) = \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det)$,

(c) $\chi_h(GL_3(\mathbb{Z}), L[n-2,2,2]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4}V_2)$,

(d) $\chi_h(GL_4(\mathbb{Z}), L[n-3,1,1,1]) = \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det)$.

The technique that we are going to use involves a substantial simplification of the trace formula which works when $\Gamma = GL_m(\mathbb{Z})$ or a group comensurable to $GL_m(\mathbb{Z})$. The simplification of the trace formula for $GL_m(\mathbb{Z})$ was developed in [Hor05, Hora].

Now we present the simplification of the trace formula in the case of $GL_m(\mathbb{Z})$. An arithmetic group Γ has also an orbifold Euler characteristic. We denote it by $\chi(\Gamma)$, without subscript. It is in fact an Euler characteristic of a certain orbifold. There is a more algebraic description. If an arithmetic group, Γ has no torsion then the orbifold Euler characteristic coincides with the homological Euler characteristic with coefficients in the trivial representation.

$$\chi(\Gamma) = \chi_h(\Gamma, \mathbb{Q}).$$

If Γ has torsion choose a torsion-free finite index subgroup Γ_0. Then

$$\chi(\Gamma) = \frac{\chi(\Gamma_0)}{[\Gamma : \Gamma_0]}.$$

Let $C(A)$ denote the centralizer of the element A inside Γ. Then the classical trace formula is

$$\chi_h(\Gamma, V) = \sum_A \chi(C(A))\mathrm{Tr}(A|V),$$

where the sum is taken over all torsion elements considered up to conjugation. And $C(A)$ denotes the centralizer of the element A inside Γ. We remark that in this formula the identity element is also considered as a torsion element.

For the simplification of the trace formula we need the following definition. Let A be an element in $GL_m(\mathbb{Z})$. Consider it as an $m \times m$ matrix. Let f be its characteristic polynomial. Let

$$f = f_1^{a_1} \dots f_l^{a_l}$$

be the factorization of f into irreducible over $\mathbb{Q}$ polynomials. Denote by

$$R(g,h) = \prod_{i,j} (\alpha_i - \beta_j)$$

the resultant of the polynomials

$$g = \prod_i (x - \alpha_i) \text{ and } h = \prod_j (x - \beta_i).$$

Denote by

$$R(A) = \prod_{i<j} R(f_i^{a_i}, f_j^{a_j})$$

Theorem 2.3. *Let V be a finite-dimensional representation of $GL_m(\mathbb{Q})$. Then the homological Euler characteristic of $GL_m(\mathbb{Z})$ with coefficients in V is given by*

$$\chi_h(GL_m(\mathbb{Z}), V) = \sum_A |R(A)| \chi(C(A)) \mathrm{Tr}(A|V),$$

where the sum is taken over torsion matrices A consisting of square blocks $A_{11}, \ldots, A_{ll}$ on the block-diagonal and zero blocks off the diagonal. Also the matrices A_{ii} are non-conjugate to each other. And they are chosen from the set $\{+1, +I_2, -1, -I_2, T_3, T_4, T_6\}$, where

$$T_3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

The blocks on the diagonal are chosen up to a permutation. And the characteristic polynomial f_i of A_{ii} is a power of an irreducible polynomial, and f_i and f_j are relatively prime.

The above theorem gives a computationally effective method for computing the homological Euler characteristic of $GL_m(\mathbb{Z})$ with coefficients in a non-trivial representation. For details about this method see [Hor05, Hora].

3. Cohomology of $GL_2(\mathbb{Z})$

This section is to show how the computational method works for $GL_2(\mathbb{Z})$. All the results are known, but we need them for the later sections. We are going to compute the cohomology of $GL_2(\mathbb{Z})$, Eisenstein cohomology, and cuspidal cohomology.

First we are going to compute the cohomology of the boundary using Kostant's theorem. Let $L[a,b]$ be the irreducible representation with highest weight $[a,b]$. The group GL_2 has one parabolic subgroup, the Borel subgroup B. It has a nilpotent radical N and a Levi quotient $GL_1 \times GL_1$. The Weyl group has two elements. Also, the half of the "sum" of the positive roots is $\rho = [1/2, -1/2]$ Consider the following table:

$\omega \epsilon W$	length	$\omega(\lambda+\rho)-\rho$
12	0	$[a,b]$
21	1	$[b-1, a+1]$.

From Kostant's theorem we obtain that

$$H^n(N, L[a,b]) = \begin{cases} L[a,b] & n=0, \\ L[b-1,a+1] & n=1. \end{cases}$$

The integral points of the Levi quotient of B are $GL_1(\mathbb{Z}) \times GL_1(\mathbb{Z})$. Using the Hochschild–Serre spectral sequence we compute $H^n(B, L[a,b])$. If both a and b are even then

$$H^0(B, L[a,b]) = H^0(GL_1(\mathbb{Z}), L[a]) \otimes H^0(GL_1(\mathbb{Z}), L[b]) = \mathbb{Q},$$

and the rest of the cohomology groups are trivial. If both a and b are odd then

$$H^1(B, L[a,b]) = H^0(GL_1(\mathbb{Z}), L[b-1]) \otimes H^0(GL_1(\mathbb{Z}), L[a+1]) = \mathbb{Q}.$$

If $a+b$ is odd then $H^n(B, L[a,b]) = 0$ for all n.

There are several cases. If $a+b$ is odd then $-I$ acts non-trivially on $L[a,b]$. So the cohomology of $GL_2(\mathbb{Z})$ vanishes. If $a=b=2k$ then $L[a,b]$ is the trivial representation of $GL_2(\mathbb{Z})$. So

$$H^i(GL_2(\mathbb{Z}), L[2k,2k]) = H^i_{Eis}(GL_2(\mathbb{Z}), L[2k,2k]) = \begin{cases} \mathbb{Q} & i=0, \\ 0 & i=1, \end{cases}$$

and

$$H^i_{\text{cusp}}(GL_2(\mathbb{Z}), L[2k,2k]) = 0.$$

If $a=b=2k+1$, then

$$H^i(GL_2(\mathbb{Z}), L[2k+1,2k+1]) = 0.$$

So the Eisenstein and the cuspidal cohomology also vanish.

The interesting cases are when a and b are both even or when a and b are both odd. For those cases we do not give a complete proof, but rather an interpretation of the cohomologies. In any of these cases, we have

$$H^0(GL_2(\mathbb{Z}), L[a,b]) = 0.$$

Also, if a and b are both odd, we have that the map

$$H^1(GL_2(\mathbb{Z}), L[a,b]) \to H^1(B, L[a,b]) = \mathbb{Q}$$

is surjective. Then

$$H^1_{Eis}(GL_2(\mathbb{Z}), L[a,b]) = \mathbb{Q},$$

and

$$\dim H^1_{\text{cusp}}(GL_2(\mathbb{Z}), L[a,b]) = -1 + \dim H^1(GL_2(\mathbb{Z}), L[a,b]).$$

If the weights a and b are both even, then the Eisenstein cohomology coincides with the whole group cohomology.

Here is one interpretation of the cohomology of $GL_2(\mathbb{Z})$ in cases when both a and b are both even or both odd. We are not going to use the following interpretation, only the above formulas, but it is nice to keep in mind.

Let a and b be both odd. Then

$$\begin{aligned} H^1(SL_2(\mathbb{Z}), L[a,b]) &= H^1_{\text{cusp}}(GL_2(\mathbb{Z}), L_{[a+1,b+1]}) \oplus H^1_{\text{cusp}}(GL_2(\mathbb{Z}), L_{[a,b]}) \\ &\quad \oplus H^1_{Eis}(GL_2(\mathbb{Z}), L_{[a,b]}). \end{aligned}$$

The first direct summand corresponds to holomorphic cuspidal forms of weight $a-b-2$. The second summand correspond to anti-holomorphic cuspidal forms of weight $a-b-2$. And the last summand corresponds to the Eisenstein series of weight $a-b-2$ (when bigger than 2).

Keeping in mind the above decompositions one can compute the dimensions of the cohomlogy groups (or dimensions of cusp forms) using Theorem 2.1. Note that in Theorem 2.1 the homological Euler characteristic is equal to minus the dimension of the first cohomology group, since the higher cohomology groups vanish as well as the zeroth.

4. Cohomology of $GL_3(\mathbb{Z})$

In this section, we compute cohomologies of $GL_3(\mathbb{Z})$ with coefficients in certain representations which are needed for our main problem. They arise as representations of the Levi quotients of two of the maximal parabolic subgroups of GL_4, namely, P_{13} and P_{24}. They lead to computation of cohomology of $GL_3(\mathbb{Z})$ with coefficients in any of the representations $L[0,0,0] = \mathbb{Q}$, $L[w-3,1,0]$, $L[w-2,2,2]$ and $L[w-2,1,1]$.

Theorem 4.1. *The cohomology of $GL_3(\mathbb{Z})$ with coefficients in the above representations are given by*

(a) $H^i(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0, \\ 0 & i \neq 0. \end{cases}$

(b) $H^i(GL_3(\mathbb{Z}), L[n-3,1,0]) = \begin{cases} (\overline{n-3,-1}|2) & i = 2, \\ (-2|\overline{n-2,2}) & i = 3, \\ 0 & i \neq 2,3, \end{cases}$

(c) $H^i(GL_3(\mathbb{Z}), L[n-2,2,2]) = \begin{cases} (0|\overline{n-1,3}) & i = 3, \\ 0 & i \neq 3, \end{cases}$

(d) $H^i(GL_3(\mathbb{Z}), L[n-2,1,1]) = \begin{cases} (0|\overline{n-1,1}) & i = 2, \\ 0 & i \neq 2. \end{cases}$

Proof. Before proving the above theorem, we examine the cohomology of $GL_3(\mathbb{Z})$ with coefficients in $L_{[a,b,c]}$

The algebraic group GL_3 has three parabolic subgroups: B, P_{12} and P_{23}. In order to find their cohomologies, we need the explicit action of the Weyl group; more precisely we need the various $\omega(\lambda+\rho)-\rho$ that enter in Kostant's theorem. Note that half of the sum of the positive roots is $\rho = [1,0,-1]$.

$\omega \epsilon W$	length of ω	$\omega(\lambda)$	$\omega(\lambda+\rho)-\rho$
123	0	$[a,b,c]$	$[a,b,c]$
132	1	$[a,c,b]$	$[a,c-1,b+1]$
213	1	$[b,a,c]$	$[b-1,a+1,c]$
231	2	$[b,c,a]$	$[b-1,c-1,a+2]$
312	2	$[c,a,b]$	$[c-2,a+1,b+1]$
321	3	$[c,b,a]$	$[c-2,b,a+2]$

Using the Kostant's theorem we find the cohomology groups of the nilpotent radicals of the parabolic groups.

$$H^q(H, L[a,b,c]) = \begin{cases} L[a,b,c] & q = 0, \\ L[a,c-1,b+1] \oplus L[b-1,a+1,c] & q = 1, \\ L[b-1,c-1,a+2] \oplus L[c-2,a+1,b+1] & q = 2, \\ L[c-2,b,a+2] & q = 3, \end{cases}$$

$$H^q(N_{12}, L[a,b,c]) = \begin{cases} L[a,b,c] & q=0, \\ L[a,c-1,b+1] & q=1, \\ L[b-1,c-1,a+2] & q=2, \end{cases}$$

$$H^q(N_{23}, L[a,b,c]) = \begin{cases} L[a,b,c] & q=0, \\ L[b-1,a+1,c] & q=1, \\ L[c-2,a+1,b+1] & q=2. \end{cases}$$

In order to pass to cohomologies of the parabolic groups, we use the Hochschild–Serre spectral sequence relating the nil radical and the Levi quotient of a parabolic subgroup to the parabolic subgroup itself; namely the short exact sequence $N \to P \to S$. We recall the notation $H^n(L[a_1,\dots,a_k]) = H^n(GL_k(\mathbb{Z}), L[a_1,\dots,a_k])$ and $(a|b|c) = H^0(L[a]) \otimes H^0(L[b]) \otimes H^0(L[c])$.

$$H^i(B, L[a,b,c]) = \begin{cases} (a|b|c) & i=0, \\ (a|c-1|b+1) \oplus (b-1|a+1|c) & i=1, \\ (b-1|c-1|a+2) \oplus (c-2|a+1|b+1) & i=2, \\ (c-2|b|a+2) & i=3, \end{cases}$$

$$E_2^{p,q}(P_{12}, L[a,b,c]) = \begin{cases} H^p(L[a,b]) \otimes H^0(L[c]) & q=0, \\ H^p(L[a,c-1]) \otimes H^0(L[b+1] & q=1, \\ H^p(L[b-1,c-1]) \otimes H^0(L[a+2] & q=2, \end{cases}$$

$$E_2^{p,q}(P_{23}, L[a,b,c]) = \begin{cases} H^0(L[a]) \otimes H^p(L[b,c]) & q=0, \\ H^0(L[b-1] \otimes H^p(L[a+1,c]) & q=1, \\ H^0(L[c-2]) \otimes H^p(L[a+1,b+1]) & q=2. \end{cases}$$

It is true that the above two spectral sequences stabilize at the E_2-level. However, in any particular case the formulas will be much simpler, and one can use them to compute the boundary cohomology.

Let B, P_{12} P_{23} be the parabolic subgroups of $GL_3(\mathbb{Z})$.

$H^i(GL_3(\mathbb{Z}), \mathbb{Q})$

For part (a) we have

$$H^i(B, \mathbb{Q}) = \begin{cases} (0|0|0) & i=0, \\ (-2|0|2) & i=3, \\ 0 & n \neq 0,3, \end{cases}$$

$$H^0(P_{12}, \mathbb{Q}) = H^0(GL_2(\mathbb{Z}), \mathbb{Q}) \otimes H^0(GL_1(\mathbb{Z}), \mathbb{Q}),$$
$$H^0(P_{23}, \mathbb{Q}) = H^0(GL_1(\mathbb{Z}), \mathbb{Q}) \otimes H^0(GL_2(\mathbb{Z}), \mathbb{Q}).$$

From Mayer–Vietoris we obtain that the boundary cohomology of $GL_3(\mathbb{Z})$ is

$$H^i_\partial(GL_3(\mathbb{Z}), \mathbb{Q}) = \begin{cases} (0|0|0) & i = 0, \\ (-2|0|2) & i = 4, \\ 0 & i \neq 0, 4. \end{cases}$$

We have that

$$\chi_h(GL_3(\mathbb{Z}), \mathbb{Q}) = 1.$$

Then the forth cohomology of the boundary component disappears in the Eisenstein cohomology.

Also, the cuspidal cohomology of $GL_3(\mathbb{Z})$ with trivial coefficients is zero. Therefore the Eisenstein cohomology coincides with the whole group cohomology.

$H^i(B, L[n-3, 1, 0])$

We proceed to part(b).

Similarly, we compute the cohomology of B, P_{12} and P_{23} with coefficients in $L[n-3, 1, 1]$ Using Mayer–Vietoris, for the cohomology of the boundary of the Borel–Serre compactification, we obtain

$$H^i_\partial(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \begin{cases} (\overline{n-3, 1}|0) & i = 1, \\ (\overline{n-3, -1}|2) \oplus (0|\overline{n-2, 0}) & i = 2, \\ (-2|\overline{n-2, 2}) & i = 3, \\ 0 & i \neq 1, 2, 3. \end{cases}$$

The representation $L[n-3, 1, 0]$ is not self-dual. So the cohomology of $GL_3(\mathbb{Z})$ with coefficients in $L[n-3, 1, 0]$ coincides with the Eisenstein cohomology, which is a subspace of the cohomology of the boundary. The first cohomology of $GL_3(\mathbb{Z})$ with coefficients in any representation vanishes. For the homological Euler characteristic of $GL_3(\mathbb{Z})$ with coefficients in $L[n-3, 1, 0]$ (Theorem 2.2 part (a)) we have

$$\chi_h(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \chi_h(GL_2(\mathbb{Z}), S^{n-4}V_2) - \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2).$$

We obtain that the dimension of the second cohomology is half of the dimension of the second cohomology of the boundary of the Borel–Serre compactification. That is,

$$\dim H^2_{Eis}(GL_3(\mathbb{Z}), L[n-3, 1, 0]) = \frac{1}{2} \dim H^2_\partial(GL_3(\mathbb{Z}), L[n-3, 1, 0]).$$

Also,

$$\dim H^3_{Eis}(GL_3(\mathbb{Z}), L[n-3,1,0]) = \dim H^3_{\partial}(GL_3(\mathbb{Z}), L[n-3,1,0]).$$

The second cohomology of the boundary is a direct sum of two spaces with the same dimesions. In order to find out which of the subspaces or which linear combination of the spaces enters in the Eisenstein cohomology, we have to consider the central characters of the two parabolic subgroups. For the parabolic subgroup P_{12} we take the central torus

$$\begin{bmatrix} t & & \\ & t & \\ & & t^{-2} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[n-3,-1,2]$, whose evaluation on the above element is

$$n-3-1-2\times 2 = n-8.$$

For the parabolic subgroup P_{23} we take the central torus

$$\begin{bmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{bmatrix}.$$

The highest weight induces a character on it, namely $[0,n-2,0]$, whose evaluation on the above element is

$$0-(n-2) = -n+2.$$

Their sum is -6. The space which enters in the Eisenstein cohomology has higher weight. Thus, we need to solve

$$n-8 > -n+2.$$

Thus for $n > 5$ we have

$$H^i(GL_3(\mathbb{Z}), L[n-3,1,0]) = \begin{cases} (\overline{n-3,-1}|2) & i=2, \\ (-2|\overline{n-2,2}) & i=3, \\ 0 & i \neq 2,3. \end{cases}$$

The value of n is always even and greater or equal to 4. The other option for n is $n = 4$. Then

$$H^i(GL_3(\mathbb{Z}), L[1,1,0]) = \begin{cases} (0|\overline{4-2,0}) & n = 2, \\ (-2|\overline{4-2,2}) & n = 3, \\ 0 & n \neq 2,3. \end{cases}$$

That is,

$$H^i(GL_3(\mathbb{Z}), L[1,1,0]) = 0.$$

The computation of $H^*(GL_3(\mathbb{Z}), L[n-2,2,2])$ and $H^*(GL_3(\mathbb{Z}), L[n-2,1,1])$, when n is even, is similar. □

5. Cohomologies of the parabolic subgroups of GL_4

This section consists of computation of cohomology of the parabolic subgroups of $GL_4(\mathbb{Z})$ with coefficients in the representation $S^{n-4}V_4 \otimes \det$. We use Kostant's theorem in order to compute these cohomologies. In the process, we reduce the question to computation of the cohomologies of the Levi quotients which have factors $GL_1(\mathbb{Z})$, $GL_2(\mathbb{Z})$ or/and $GL_3(\mathbb{Z})$. For the last three groups we use the computation from the sections on cohomolgy of $GL_2(\mathbb{Z})$ and of $GL_3(\mathbb{Z})$.

Recall the notation of the parabolic subgroups: we choose the Borel subgroup B to be the group of upper triangular matrices. Let N be its unipotent radical of B. Let P_{ij} be the smallest parabolic subgroup containing B and containing a non-zero a_{ji}-entry. Similarly, $P_{12,34}$ is the smallest (parabolic) subgroup containing B and containing non zero a_{21}- and a_{43}-entries. The unipotent radicals of P_{ij} will be denoted by N_{ij}; and the Levi quotient by $S_{ij} = P_{ij}/N_{ij}$.

Proposition 5.1. (Cohomologies of the parabolic subgroups) *Let* $V = S^{n-4}V_4 \otimes \det$. *Then*

$$H^i(B, V) = \begin{cases} (0|n-2|0|2) & i = 2, \\ (0|0|0|n) \oplus (-2|w-2|2|2) & i = 3, \\ (-2|0|2|n) & i = 6, \\ 0 & i \neq 2,3,6. \end{cases}$$

$$H^i(P_{12}, V) = \begin{cases} (n-3,1|0|2) & i = 2, \\ (0|0|0|w) \oplus (n-3,-1|2|2) & i = 3, \\ 0 & i \neq 2,3. \end{cases}$$

$$H^i(P_{23},V) = \begin{cases} (0|n-2,0|2) \oplus (0|0,0|n) & i=3 \\ (-2|n-2,2|2) & i=4, \\ 0 & i \neq 3,4. \end{cases}$$

$$H^n(P_{34},V) = \begin{cases} (0|0|n-1,1) \oplus (-2|n-2|2|2) & i=3, \\ (-2|0|n-1,3) & i=6 \\ 0 & i \neq 3,6. \end{cases}$$

$$H^i(P_{13},V) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3,-1}|2|2) & i=3, \\ (-2|\overline{n-2,2}|2) & i=4, \\ 0 & i \neq 3,4. \end{cases}$$

$$H^i(P_{12,34},V) = \begin{cases} (n-3,-1|2|2) \oplus (0|0|n-1,1) & i=3 \\ 0 & i \neq 3. \end{cases}$$

$$H^i(P_{24},V) = \begin{cases} (0|0|0|n) \oplus (0|n-2,0|2) & i=3, \\ (-2|0|\overline{n-1,3}) & i=6, \\ 0 & i \neq 3,6. \end{cases}$$

Proof. The proof is based on Kostant's theorem, Hochschild–Serre spectral sequence and homological Euler characteristics. For $GL_4(\mathbb{Z})$ the proof is essentially the same as for $GL_3(\mathbb{Z})$, except that the computation is longer. One can find it in details in [Horb] □

6. Boundary cohomology of $GL_4(\mathbb{Z})$

In this section, we compute the cohomology of the boundary of the Borel–Serre compactification associated with $GL_4(\mathbb{Z})$ with coefficients in

$$V = S^{n-4} \otimes \det = L[n-3,1,1,1].$$

The Eisenstein cohomology, which in our case is the whole-group cohomology, injects into the cohomology of the boundary.

We recall briefly several statements about Borel–Serre compactification associated with $GL_m(\mathbb{Z})$. Let

$$X = GL_m(\mathbb{R})/SO_m(\mathbb{R}) \times \mathbb{R}^\times_{>0}.$$

And let

$$Y = GL_m(\mathbb{Z})\backslash X.$$

Then the Borel–Serre compactification of Y, denoted by $\overline{Y}$, is a compact space, containing Y, and of the same homotopy type. The space $\overline{Y}$ is obtained by attaching cell σ_P to X, corresponding to each parabolic subgroup P. Denote by Y_P the projection of σ_P to $\overline{Y}$. Let $\overline{Y}_P$ be the closure of Y_P. Then $\overline{Y}_Q \subset \overline{Y}_P$ when $Q \subset P$. The boundary of $\overline{Y}$ is obtained by gluing together the spaces $\overline{Y}_P$. In the following computation, we shall denote by Y_{ij} the space $Y_{P_{ij}}$. For these spaces we have

$$H^i_{\text{top}}(\overline{Y}_{ij}, i^*F_V) = H^i_{\text{group}}(P_{ij}, V),$$

for a suitable sheaf F_V on $\overline{Y}$, where i is the inclusion of $\overline{Y}_{ij}$ into $\overline{Y}$. For simplification we will not write the restriction functor i^*.

The cohomology of the boundary can be computed the spectral sequence of the type "Mayer–Vietoris".

$$E_1^{*.q}: \quad \begin{array}{ccccc} H^q(\overline{Y}_{13}, F_V)) & \longrightarrow & H^q(\overline{Y}_{12}, F_V)) & & \\ & \nearrow\!\!\!\!\searrow & & \searrow & \\ H^q(\overline{Y}_{12,34}, F_V) & & H^q(\overline{Y}_{23}, F_V) & \longrightarrow & H^q(\overline{Y}_B, F_V) \\ & \searrow\!\!\!\!\nearrow & & \nearrow & \\ H^q(\overline{Y}_{24}, F_V) & \longrightarrow & H^q(\overline{Y}_{34}, F_V) & & \end{array}$$

The direct sum of the first column will be $E_1^{0,q}$; the direct sum of the second column will be $E_1^{1,q}$; and $E_1^{2,q} = H^q(\overline{Y}_B, F_V)$. We have non-zero terms when $q = 2, 3, 4$ or 6. Similarly, to the Mayer–Vietoris sequence, we want every square at the E_1 level to be anti-commutative. It can be achieved in the following way. First, consider the maps induced by the inclusion of the boundary components. Then the squares will commute. Then change the sign of every other arrow mapping a subspace of $E_1^{0,q}$ to a subspace of $E_1^{1,q}$ as it is done in the definition of the spectral sequence. Then the squares will anti-commute.

Theorem 6.1. *The above spectral sequence stabilizes at E_2 level. It converges to the cohomology of the boundary of the Borel–Serre compactification associated with $GL_4(\mathbb{Z})$, which is*

$$H^i_\partial(GL_4(\mathbb{Z}), V) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3,-1}|2|2) \oplus (\overline{n-3,1}|0|2) & i = 3, \\ 0 & i \neq 3, \end{cases}$$

where

$$(a_1|a_2|\dots|a_k) = \otimes_{i=1}^{k} H^0(GL_1(\mathbb{Z}), L[a_i]),$$

and

$$(\overline{a_1, a_2}|a_3|a_4) = H^1_{\text{cusp}}(GL_2(\mathbb{Z}), L[a_1, a_2]) \otimes (a_3|a_4).$$

Proof. We consider all non-vanishing terms of the spectral sequence at E_1 level. The non-vanishing terms occur at $q = 2, 3, 4$ and 6. For a fixed q we have arrows going in direction of the index p induced by the inclusion of the parabolic subgroups. We compute *kernel/image* for these arrows in order to find the E_2 level of the spectral sequence. As a consequence we find that the spectral sequence degenerates at the E_2 level. Then we compute the cohomology to which it converges, which is the cohomology of the boundary. □

6.1. Computation of $E_2^{*,2}$

For the $E_1^{p,2}$-terms the only non-zero cohomologies come are $H^2(P_{12}, V)$ and $H^2(B, V)$. We have

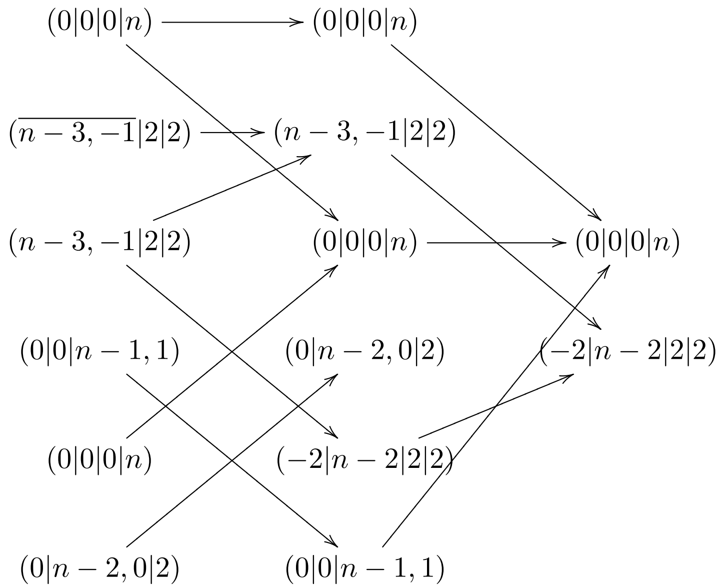
$$(n-3, 1|0|2) \to (0|n-2|0|2).$$

Therefore,

$$E_2^{p,2} = \begin{cases} (\overline{n-3, 1}|0|2) & p = 1, \\ 0 & p \neq 1. \end{cases}$$

6.2. Computation of $E_2^{*,3}$

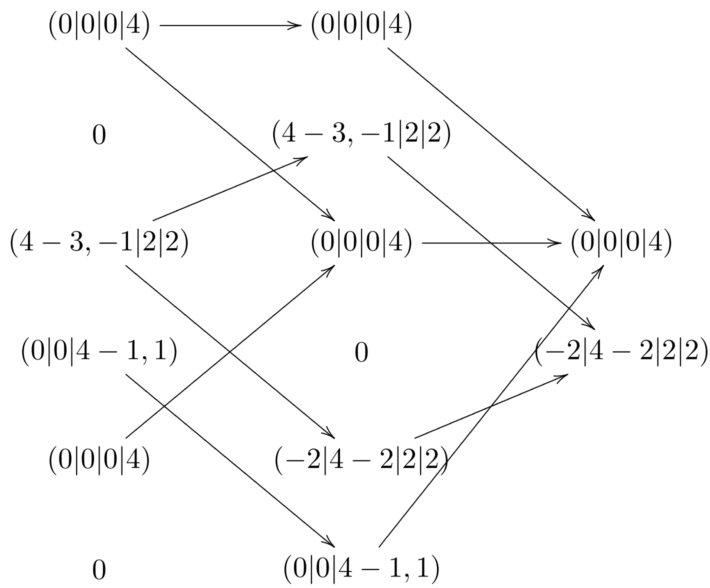
First we consider the case $n > 5$. Now we describe the $E_1^{*,3}$ terms. Consider the columns of the diagram below. Break each column into pairs of vector spaces. Each pair comes one parabolic subgroup. For example $(0|0|0|n)$ and $(\overline{n-3, -1}|2|2)$ come from third cohomology of P_{13}. The two vector spaces below come from the third cohomology of $P_{12,34}$. The maps correspond to the inclusion of the parabolic subgroups.



There are many cancellation which occur when passing to E_2 level. In order to follow the cancellation one considers the connected graph of the above diagram. There are 3 connected graphs: one containing the space $(0|0|0|n)$ coming from the 3rd cohomology of the Borel subgroup, and another containing $(-2|n-2|2|2)$ again from the 3rd cohomology of the Borel subgroup, and the 3rd containing $(0|n-2,0|2)$ from the 3rd cohomology of P_{24}. Consider the graph containing $(0|0|0|n)$. The only term that is not cancelled at E_2 level is the vector space $(0|0|0|n)$ which comes from the parabolic group P_{24}. Now consider the second connected graph, containing $(-2|n-2|2|2)$. After cancellation the only vector space left is $(\overline{n-3,-1}|2|2)$ coming from P_{13}. For the 3rd connected graph, there are two vertices corresponding to $(0|n-2,0|2)$. So they cancel and do not contribute to the E_2 level. Thus, for $n > 4$ we have

$$E_2^{p,3} = \begin{cases} (0|0|0|n) \oplus (\overline{n-3,-1}|2|2) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

Now we have to examine the case $n = 4$. The vector spaces are all the same as in the case $n > 4$ except the exchange of $(\overline{n-3,-1}|2|2)$ with $(0|n-2,0|2)$ in the 3rd cohomology of P_{13}. Note also that for $n = 4$, we have $(0|n-2,0|2) = 0$. Then the $E_1^{*,3}$ terms form the following anticommutative diagram:



There are 2 connected graphs in the above diagram. One containing the vector space $(0|0|0|4)$ coming from the Borel subgroup. The other containing the vector space $(-2|4-2|2|2)$ again coming from the Borel subgroup. Consider the graph containing $(0|0|0|4)$. The only terms that is not cancelled at E_2 level is the vector space $(0|0|0|4)$ which comes from the parabolic group P_{24}. Now consider the second connected graph, containing $(-2|4-2|2|2)$. All of its terms of that graph cancel when passing to E_2 level. Thus, for $w = 4$ we have

$$E_2^{p,3} = \begin{cases} (0|0|0|4) & p = 0, \\ 0 & p \neq 0. \end{cases}$$

6.3. Computation of $E_2^{*,4}$

For $q = 4$ the only non-zero terms at the E_1 level come from P_{13} and P_{23}. We have

$$H^4(P_{13}, V)) \to H^4(P_{23}, V)).$$

From the first theorem (Theorem 5.1) in the section "Cohomology of the parabolic subgroups of GL_4" we obtain

$$(-2|n-2, 2|2) \to (-2|n-2, 2|2).$$

Therefore,

$$E_2^{*,4} = 0.$$

6.4. Computation of $E_2^{*,6}$

When $q = 6$, for all even w, the non-zero terms give

$$E_1^{*,6} : H^6(P_{24}, V) \to H^6(P_{34}, V) \to H^6(B, V),$$

which are isomorphic to

$$(-2|0|\overline{w-1,3}) \to (-2|0|w-1,3) \to (-2|0|2|w)$$

from Theorem 5.1. The above sequence is exact. Therefore,

$$E_2^{*,6} = 0.$$

The spectral sequence degenerates at E_2 level. Therefore, we can find what is the cohomology of the boundary of the Borel–Serre compactification associated with $GL_4(\mathbb{Z})$ with coefficients in the sheaf

$$F_V$$

associated with

$$V = S^{n-4}V_4 \otimes \det .$$

Let us recall the notation that we are going to use. By $H^i_\partial(GL_4(\mathbb{Z}), V)$ we mean the cohomology of the boundary of the Borel–Serre compactification associated with $GL_4(\mathbb{Z})$ with coefficients the sheaf F_V For even n greater than 4 we have

$$H^i_\partial(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3,-1}|2|2) \oplus (\overline{n-3,1}|0|2) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

Note that the first two summands for the 3rd cohomology of the boundary come from 3rd cohomology of the maximal parabolic subgroups. And the last summand comes from the 2nd cohomology of a non-maximal parabolic subgroup. Since it comes from second cohomology of a parabolic subgroup, but it contributes in the 3rd cohomology of the boundary, it is called a ghost class.

7. Cohomology of $GL_4(\mathbb{Z})$

In this section, we present the last steps of the Proof of Theorem 1.1. We are going to show that the ghost class does not enter in the Eisenstein cohomology of $GL_4(\mathbb{Z})$ which coincides with the whole cohomology of $GL_4(\mathbb{Z})$.

Since the cohomology of the boundary is concentrated in degree 3, it is enough to compute homological Euler characteristic of $GL_4(\mathbb{Z})$ with coefficients in $S^{n-4}V_4 \otimes \det$. Recall the homological Homological Euler characteristic of an arithmetic group Γ with coefficients in a finite-dimensional representation is

$$\chi_h(\Gamma, V) = \sum_i (-1)^i \dim H^i(\Gamma, V).$$

Note that $S^{n-4}V_4 \otimes \det = L[n-3, 1, 1, 1]$ and $S^{n-2}V_2 \otimes \det = L[n-1, 1] = L[n-3, -1]$ Form [Hor05] and [Hora] we know that

$$\chi_h(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \chi_h(GL_2(\mathbb{Z}), S^{n-2}V_2 \otimes \det).$$

Therefore, for even n greater than 4, we have

$$H^i(GL_4(\mathbb{Z}), S^{n-4}V_4 \otimes \det) = \begin{cases} (0|0|0|n) \oplus (\overline{n-3, -1}|2|2) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

In the case $n = 4$, we use the same argument:

$$H^i_\partial(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

Also, the homological Euler characteristic gives

$$\chi_h(GL_4(\mathbb{Z}), \det) = -1.$$

Therefore, for $n = 4$ the cohomology of the boundary coincides with the Eisenstein cohomology. And we have

$$H^i_{Eis}(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

On the other hand,

$$H^i_{\text{cusp}}(SL_4(\mathbb{Z}), \mathbb{Q}) = 0.$$

Therefore,

$$H^i_{\text{cusp}}(GL_4(\mathbb{Z}), \det) = 0.$$

And we conclude that

$$H^i(GL_4(\mathbb{Z}), \det) = \begin{cases} (0|0|0|4) & i = 3, \\ 0 & i \neq 3. \end{cases}$$

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