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Symmetry minimizes the principal eigenvalue: an example for the Pucci's sup operator

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We explicitly evaluate the principal eigenvalues of the extremal Pucci's sup–operator for a class of planar domains, and we prove that, for fixed area, the eigenvalue is minimal for the most symmetric set.

1. Introduction

In 1951, Pólya and Szego conjectured:

Of all n-polygons with the same area, the regular n-polygon has the smallest first Dirichlet eigenvalue, referring to the Dirichlet eigenvalue of the Laplacian, see [10]. It is very simple to see that among all rectangles of same area, the one that minimizes the first Laplace Dirichlet eigenvalue is the square. Using Steiner symmetrization, Pólya and Szego proved the conjecture for $n = 3$ and $n = 4$, but it still an open problem for $n > 4$. On the other hand, the well-known Faber–Krahn's inequality affirms that in any dimension, among all domains of same volume, the Euclidean ball has the smallest first Laplace Dirichlet eigenvalue.

The notion of the first Dirichlet eigenvalue for linear elliptic operators has been extended to fully nonlinear ones (see [1, 3, 5, 7]). Indeed, for linear operators, Berestycki, Nirenberg and Varadhan in [2] use the maximum principle to define the principal eigenvalue. So, following their idea it is possible to prove that, if $\mathcal{M}^+_{\lambda,\Lambda}$ denotes the Pucci's supremum operator, with ellipticity constants $0 < \lambda \leq \Lambda$ and if Ω is a bounded Lipschitz domain, then

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there exists $\phi > 0$ in Ω such that

$$\begin{cases} \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu^+(\Omega)\phi = 0 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

for

$$\mu^+(\Omega) = \sup\{\mu \in \mathbb{R} \,:\, \exists\ \phi > 0 \text{ in } \Omega,\ \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \le 0 \quad \text{in } \Omega\}.$$

For

$$\mu^-(\Omega) = \sup\{\mu \in \mathbb{R} \,:\, \exists\ \phi < 0 \text{ in } \Omega,\ \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \ge 0 \quad \text{in } \Omega\}$$

the existence of a negative eigenfunction is similarly proved.

It is hence quite natural to wonder if the Faber–Krahn inequality is valid for these "eigenvalues" associated with $\mathcal{M}^+_{\lambda,\Lambda}$; precisely, given a ball B, is it true that

$$\mu^+(B) \le \mu^+(\Omega), \text{ for any } \Omega \text{ such that } |\Omega| = |B|? \tag{1}$$

Here $|.|$ indicates the volume.

Faber–Krahn inequality is proved in several ways, the most classical one uses Steiner symmetrization together with the Rayleigh quotient that defines the eigenvalue. Clearly these tools are not at all adapted to this non variational fully nonlinear setting. Another possible proof relies on a more geometrical understanding of the problem; as it is well-explained in [6], a domain Ω is critical for the Laplace first eigenvalue functional under fixed volume variation, if and only if the eigenfunction $\phi > 0$ associated with $\mu(\Omega)$ has constant Neumann boundary condition, i.e., if it is a solution of an overdetermined boundary value problem. This is proved using Hadamard's identity (we refer to [6] and references therein). But, by Serrin's classical result, the only bounded domains which admit non trivial solutions satisfying overdetermined boundary conditions are balls. In [4], it is proved that at least for λ and Λ close enough, the only bounded domains for which the overdetermined boundary value problem associated with $\mathcal{M}^+_{\lambda,\Lambda}$ admits a non trivial solution are the balls. This suggests that (1) may be true. Unfortunately, it is not known if, for the eigenvalue functional associated with $\mathcal{M}^+_{\lambda,\Lambda}$, the critical domains under fixed volume have eigenfunctions with constant normal derivative.

Both the Faber–Krahn inequality and the Pólya and Szego conjecture state that symmetry of the domain decreases the Laplace first eigenvalue. If this is true for the Pucci eigenvalue is not known but the scope of this paper is to show that among a family of subsets of $\mathbb{R}^2$ of same area, which are in some sense deformations of rectangles, the one that minimizes $\mu^+(\cdot)$ is the most symmetric one. This minimal domain will be denoted Ω_1^ω for $\omega = \frac{\Lambda}{\lambda}$ and it is, somehow, a deformation of a square. The result is accomplished by explicitly computing the eigenvalue $\mu^+(\Omega_1^\omega)$ and the corresponding eigenfunction. Observe that the square is not the good set to consider, since, as it is proved in Proposition 2.1, the eigenfunction associated with the square is not the product of two functions of one variable.

Remarkably, an analogous explicit computation of $\mu^-(\cdot)$ leads to unbounded sets. In particular, one can construct a symmetric unbounded set D_1^ω such that $\mu^-(D_1^\omega) = \lambda$.

2. The principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ in some special domains

In order to fix notations, we recall that the supremum Pucci operator is defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i,$$

where e_i are the eigenvalues of the symmetric matrix X and $\Lambda \geq \lambda > 0$ are fixed constants. The starting point of our analysis is the following observation.

Proposition 2.1. *For $\Lambda > \lambda > 0$, if u is an eigenfunction of $\mathcal{M}^+_{\lambda,\Lambda}$ associated with the positive principal eigenvalue in $Q = \left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right)^2$, then u is not a function of separable variables.*

Proof. By assumption u satisfies

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu\, u & \text{in } Q, \\ u > 0 \quad \text{in } Q, \; u = 0 & \text{on } \partial Q. \end{cases} \tag{2}$$

Assume, by contradiction, that u is a function of separable variables. Then, by symmetry and regularity results, u can be written as

$$u(x, y) = f(x)\, f(y)$$

with $f : \left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right) \to \mathbb{R}$ smooth, positive, even, and, up to a normalization, satisfying $f(0) = 1$. In particular, one has

$$D^2u(0,y) = \begin{pmatrix} f''(0)f(y) & 0 \\ 0 & f''(y) \end{pmatrix}$$

and equation (2) tested at $(0,0)$ yields

$$f''(0) = -\frac{\mu}{2\lambda} < 0.$$

Moreover, if for some $y_0 \in \left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right)$ one has $f''(y_0) = 0$, then from equation (2) written for $(x,y) = (0,y_0)$ we obtain the contradiction

$$-\lambda\, f''(0) = \mu = -2\lambda\, f''(0).$$

Therefore, we have $f'' < 0$ in $\left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right)$ and, again from equation (2), we deduce that f satisfies

$$\begin{cases} f'' = -\frac{\mu}{2\lambda}\, f, \quad f > 0 \quad \text{in } \left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right), \\ f\left(-\frac{\pi}{\sqrt{2}}\right) = f\left(\frac{\pi}{\sqrt{2}}\right) = 0, \; f(0) = 1. \end{cases}$$

Hence, $\mu = \lambda$ and $f(x) = \cos\left(\frac{x}{\sqrt{2}}\right)$. On the other hand, for the function $u(x,y) = \cos\left(\frac{x}{\sqrt{2}}\right)\cos\left(\frac{y}{\sqrt{2}}\right)$ one has, in particular,

$$D^2u(x,x) = \frac{1}{2}\begin{pmatrix} -\cos^2\left(\frac{x}{\sqrt{2}}\right) & \sin^2\left(\frac{x}{\sqrt{2}}\right) \\ \sin^2\left(\frac{x}{\sqrt{2}}\right) & -\cos^2\left(\frac{x}{\sqrt{2}}\right) \end{pmatrix},$$

and, for $\frac{\pi}{2\sqrt{2}} \le |x| < \frac{\pi}{\sqrt{2}}$ we have

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u(x,x)) = \Lambda\cos^2\left(\frac{x}{\sqrt{2}}\right) - \frac{\Lambda-\lambda}{2} \neq \lambda\, u(x,x),$$

unless $\Lambda = \lambda$. $\square$

Let us remark that the function

$$u(x,y) = \cos\left(\frac{x}{\sqrt{2}}\right)\cos\left(\frac{y}{\sqrt{2}}\right) = \frac{1}{2}\left[\cos\left(\frac{x+y}{\sqrt{2}}\right) + \cos\left(\frac{x-y}{\sqrt{2}}\right)\right]$$

is an eigenfunction for the Laplace operator in the squared domain Q relative to the first eigenvalue $\lambda_1(-\Delta, Q) = 1$. On the points where u is concave, it also satisfies the equation

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = -\lambda\,\Delta u = \lambda\, u.$$

Actually this is the case for $(x,y) \in Q_1 = \{(x,y) \in \mathbb{R}^2 \,:\, |x| + |y| < \frac{\pi}{\sqrt{2}}\}$, the rotated squared domain with side π. Moreover, the same holds true for any function of the form

$$u_\gamma(x,y) = \gamma\,\cos\left(\frac{x+y}{\sqrt{2}}\right) + \cos\left(\frac{x-y}{\sqrt{2}}\right),$$

with $\gamma > 0$. In the next result, we suitably extend the function $u_\gamma|_{Q_1}$ in order to obtain an eigenfunction for $\mathcal{M}^+_{\lambda,\Lambda}$ relative to the eigenvalue λ.

Let $\omega \geq 1$ be a parameter to be fixed in the sequel, and, for $\frac{1}{\sqrt{\omega}} \leq \gamma \leq \sqrt{\omega}$ let us introduce the positive even functions defined for $|x| \leq \frac{\pi}{2} + \sqrt{\omega}\arcsin\left(\frac{1}{\gamma\sqrt{\omega}}\right)$ as

$$\phi^\omega_\gamma(x) = \begin{cases} \frac{\pi}{2} + \sqrt{\omega}\arcsin\left(\frac{\gamma}{\sqrt{\omega}}\cos x\right) & \text{if } |x| \leq \frac{\pi}{2}, \\ \arccos\left(\gamma\sqrt{\omega}\sin\left(\frac{|x|-\pi/2}{\sqrt{\omega}}\right)\right) & \text{if } \frac{\pi}{2} < |x| \leq \frac{\pi}{2} + \sqrt{\omega}\arcsin\left(\frac{1}{\gamma\sqrt{\omega}}\right). \end{cases}$$

Note that

$$\phi^\omega_{\gamma^{-1}} = \left(\phi^\omega_\gamma\right)^{-1} \tag{3}$$

so that, in particular, $\phi^\omega_1 = (\phi^\omega_1)^{-1}$.

Next, let us consider the open bounded subsets

$$\Omega^\omega_\gamma := \left\{(x,y) \in \mathbb{R}^2 \,:\, |y| < \phi^\omega_\gamma(x)\right\}.$$

Note that for $\omega = 1$ we have $\gamma = 1$ and Ω^1_1 is nothing but the rotated squared domain with side $\sqrt{2}\pi$. In general, Ω^ω_γ is a Lipschitz domain symmetric both with respect to the x and y axes, and, by (3),

$$\Omega^\omega_{\frac{1}{\gamma}} = \left\{(x,y) \in \mathbb{R}^2 \,:\, (y,x) \in \Omega^\omega_\gamma\right\}.$$

In particular, Ω^ω_1 is symmetric also with respect to the diagonal $y = x$.

Theorem 2.2. *Given $\Lambda \geq \lambda > 0$ let us set $\omega = \frac{\Lambda}{\lambda} \geq 1$. Then, for any $\frac{1}{\sqrt{\omega}} \leq \gamma \leq \sqrt{\omega}$, the positive principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ in the domain Ω^ω_γ is*

$$\mu\left(\Omega^\omega_\gamma\right) = \lambda$$

and the principal eigenfunction is, up to positive constants,

$$u^\omega_\gamma(x,y) = \begin{cases} \gamma\, \cos x + \cos y & \text{if } |x| \leq \frac{\pi}{2}\,,\ |y| \leq \frac{\pi}{2}, \\ \gamma\, \sqrt{\omega} \cos\left(\frac{|x|-\pi/2}{\sqrt{\omega}} + \frac{\pi}{2}\right) + \cos y & \text{if } (x,y) \in \Omega^\omega_\gamma\,,\ |x| \geq \frac{\pi}{2}, \\ \gamma\, \cos x + \sqrt{\omega} \cos\left(\frac{|y|-\pi/2}{\sqrt{\omega}} + \frac{\pi}{2}\right) & \text{if } (x,y) \in \Omega^\omega_\gamma\,,\ |y| \geq \frac{\pi}{2}. \end{cases}$$

Proof. The proof is a straightforward computation. We observe that u^ω_γ is of class C^2 and positive in Ω^ω_γ, and it vanishes on $\partial\Omega^\omega_\gamma$. For $|x| \leq \frac{\pi}{2}\,,\ |y| \leq \frac{\pi}{2}$ one has

$$D^2 u^\omega_\gamma(x,y) = \begin{pmatrix} -\gamma\, \cos x & 0 \\ 0 & -\cos y \end{pmatrix}.$$

Therefore, for $|x| \leq \frac{\pi}{2}$ and $|y| \leq \frac{\pi}{2}$, u^ω_γ is concave and it satisfies

$$-\mathcal{M}^+_{\lambda,\Lambda}\left(D^2 u^\omega_\gamma\right) = -\lambda\, \Delta u^\omega_\gamma = \lambda\, u^\omega_\gamma.$$

For $(x,y) \in \Omega^\omega_\gamma$ and $|x| \geq \frac{\pi}{2}$, one has

$$D^2 u^\omega_\gamma(x,y) = \begin{pmatrix} \frac{\gamma}{\sqrt{\omega}} \sin\left(\frac{|x|-\pi/2}{\sqrt{\omega}}\right) & 0 \\ 0 & -\cos y \end{pmatrix}.$$

Note that, if $(x,y) \in \Omega^\omega_\gamma$ and $|x| \geq \frac{\pi}{2}$, then $|y| \leq \frac{\pi}{2}$ and $0 \leq \frac{|x|-\pi/2}{\sqrt{\omega}} < \arcsin\left(\frac{1}{\gamma\sqrt{\omega}}\right) \leq \frac{\pi}{2}$; therefore

$$-\mathcal{M}^+_{\lambda,\Lambda}\left(D^2 u^\omega_\gamma\right) = \lambda\, \cos y - \Lambda\, \frac{\gamma}{\sqrt{\omega}} \sin\left(\frac{|x| - \pi/2}{\sqrt{\omega}}\right) = \lambda\, u^\omega_\gamma.$$

Analogously, for $(x,y)\in\Omega^\omega_\gamma$ and $|y|\ge\frac{\pi}{2}$, we have

$$D^2u^\omega_\gamma(x,y)=\begin{pmatrix}-\gamma\,\cos x & 0\\ 0 & \frac{1}{\sqrt{\omega}}\sin\left(\frac{|y|-\pi/2}{\sqrt{\omega}}\right)\end{pmatrix}$$

and, since $|x|\le\frac{\pi}{2}$ and $0\le\frac{|y|-\pi/2}{\sqrt{\omega}}<\arcsin\left(\frac{\gamma}{\sqrt{\omega}}\right)\le\frac{\pi}{2}$, we again conclude

$$-\mathcal{M}^+_{\lambda,\Lambda}\left(D^2u^\omega_\gamma\right)=\lambda\,\gamma\,\cos x-\frac{\Lambda}{\sqrt{\omega}}\sin\left(\frac{|y|-\pi/2}{\sqrt{\omega}}\right)=\lambda\,u^\omega_\gamma.$$

□

Remark 2.3. Let us remark that for $\omega=1$ the only admissible value for γ is $\gamma=1$ and there is only one set Ω^1_1. In this case, up to a rotation, Ω^1_1 is the square $\{|x|<\pi/\sqrt{2},\ |y|<\pi/\sqrt{2}\}$ and $u^1_1(x,y)=\cos\left(\frac{x}{\sqrt{2}}\right)\cos\left(\frac{y}{\sqrt{2}}\right)$ is the first eigenfunction of the Laplace operator, associated with the first eigenvalue $\lambda_1=\mu\left(-\Delta,\Omega^1_1\right)=1$.

For $\omega>1$, we have identified the family of bounded domains Ω^ω_γ, $\frac{1}{\sqrt{\omega}}\le\gamma\le\sqrt{\omega}$, in all of which the positive principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ is λ. Note that Ω^ω_γ is a smooth set except for $\gamma=\sqrt{\omega}$ and the symmetric case $\gamma=1/\sqrt{\omega}$. $\partial\Omega^\omega_{\sqrt{\omega}}$ has singularity points at $\left(0,\pm(1+\sqrt{\omega})\frac{\pi}{2}\right)$, where an angle of amplitude $2\arctan\left(\frac{1}{\sqrt{\omega}}\right)$ occurs (see Figure 1). Moreover, for $(x,y)\in\Omega^\omega_{\sqrt{\omega}}\cap\{|y|>\pi/2\}$, the eigenfunction $u^\omega_{\sqrt{\omega}}$ has the expression

$$u^\omega_{\sqrt{\omega}}(x,y)=2\cos\left(\frac{\frac{|y|-\pi/2}{\sqrt{\omega}}+\frac{\pi}{2}+x}{2}\right)\cos\left(\frac{\frac{|y|-\pi/2}{\sqrt{\omega}}+\frac{\pi}{2}-x}{2}\right)$$

showing that $u^\omega_{\sqrt{\omega}}(x,y)$ vanishes quadratically as $\Omega^\omega_{\sqrt{\omega}}\ni(x,y)\to\left(0,\pm(1+\sqrt{\omega})\frac{\pi}{2}\right)$. This property is consistent with the fact that the homogeneous problem

$$\text{(4)}\qquad\begin{cases}\mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi)=0 & \text{in } \mathcal{C},\\ \Phi=0 & \text{on } \partial\mathcal{C},\end{cases}$$

where $\mathcal{C}$ is the plane cone $\mathcal{C}=\{y>\sqrt{\omega}|x|\}$, has the positive solution, homogeneous of degree 2, $\Phi(x,y)=y^2-\omega\,x^2$ (see [8]). Indeed, by the comparison

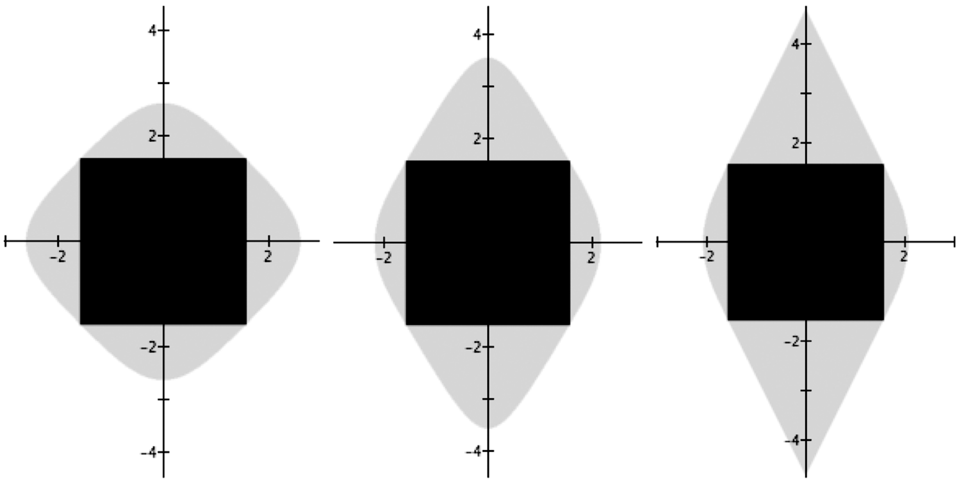

Figure 1: Ω^ω_1, Ω^ω_γ, $\Omega^\omega_{\sqrt{\omega}}$, three domains for which the eigenvalue is λ; in the black square u^ω_γ is concave

principle, it immediately follows that

$$\liminf_{\Omega^\omega_{\sqrt{\omega}}\ni(x,y)\to\left(0,\pm(1+\sqrt{\omega})\frac{\pi}{2}\right)} \frac{u^\omega_{\sqrt{\omega}}(x,y)}{\Phi\left(x,(1+\sqrt{\omega})\frac{\pi}{2}\mp y\right)} > 0.$$

Remark 2.4. The function u^ω_γ can be extended in order to obtain a changing sign eigenfunction for $\mathcal{M}^+_{\lambda,\Lambda}$ in the whole $\mathbb{R}^2$. Precisely, for any $\gamma > 0$, let us define in the square $\left\{|x|,\ |y| \le (1+\sqrt{\omega})\frac{\pi}{2}\right\}$

$$u^\omega_\gamma(x,y) = \begin{cases} \gamma\,\cos x + \cos y & \text{if } |x|\,,\ |y| \le \frac{\pi}{2}, \\ -\gamma\sqrt{\omega}\sin\left(\frac{|x|-\pi/2}{\sqrt{\omega}}\right) + \cos y & \text{if } \frac{\pi}{2} < |x| \le (1+\sqrt{\omega})\frac{\pi}{2},\ |y| \le \frac{\pi}{2}, \\ \gamma\,\cos x - \sqrt{\omega}\sin\left(\frac{|y|-\pi/2}{\sqrt{\omega}}\right) & \text{if } |x| \le \frac{\pi}{2},\ \frac{\pi}{2} < |y| \le (1+\sqrt{\omega})\frac{\pi}{2}, \\ -\sqrt{\omega}\left(\gamma\sin\left(\frac{|x|-\pi/2}{\sqrt{\omega}}\right) + \sin\left(\frac{|y|-\pi/2}{\sqrt{\omega}}\right)\right) & \text{if } \frac{\pi}{2} < |x|\,,\ |y| \le (1+\sqrt{\omega})\frac{\pi}{2} \end{cases}$$

and extend u^ω_γ periodically both with respect to x and y. Then, by arguing as in Theorem 2.2, it is easy to see that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u^\omega_\gamma) + \lambda\,u = 0 \qquad \text{in } \mathbb{R}^2.$$

The set where u_γ^ω is positive has bounded connected components if and only if $\frac{1}{\sqrt{\omega}} \leq \gamma \leq \sqrt{\omega}$, and in this case they are nothing but translations of Ω_γ^ω. Conversely, the connected components of the set $D_\gamma^\omega = \{u_\gamma^\omega < 0\}$ are unbounded for any $\gamma > 0$. For $\frac{1}{\sqrt{\omega}} < \gamma < \sqrt{\omega}$ D_γ^ω is connected and unbounded in both x- and y-directions, whereas either for $\gamma \leq \frac{1}{\sqrt{\omega}}$ or for $\gamma \geq \sqrt{\omega}$ the connected components of D_γ^ω are contained in unbounded respectively horizontal or vertical stripes; see Figure 2. Since u_γ^ω is a negative eigenfunction for $\mathcal{M}^+_{\lambda,\Lambda}$ in each of the connected components of D_γ^ω, we can say that for these sets one has $\mu^- = \lambda$.

We finally remark that this construction does not yield a changing sign eigenfunction for a bounded domain, so that it cannot be applied to calculate eigenvalues different from the principal ones.

Let us now enlarge, by deforming the sets Ω_γ^ω, the class of domains for which we can evaluate the positive principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$. For any $a \in \mathbb{R}$ with $|a| < \pi$ let us consider the non singular matrix

$$C_a = \begin{pmatrix} \sqrt{1 - \left(\frac{a}{\pi}\right)^2} & 0 \\ \frac{a}{\pi} & 1 \end{pmatrix}$$

Figure 2: D_1^ω, $D_{\sqrt{\omega}}^\omega$ and D_γ^ω for $\gamma > \sqrt{\omega}$

and let us denote by $C_a : \mathbb{R}^2 \to \mathbb{R}^2$ also the linear transformation induced by C_a. We observe that C_a maps the square $Q = \{|x| + |y| < \pi\}$ with side $\sqrt{2}\pi$ into the rectangle $R = \left\{|x| + \left|\frac{\sqrt{\pi^2-a^2}y - a\,x}{\pi}\right| < \sqrt{\pi^2 - a^2}\right\}$ with sides $\sqrt{2\pi(\pi - a)}$ and $\sqrt{2\pi(\pi + a)}$, and the square $\{|x|,\ |y| < \pi/2\}$ onto the rhombus $\left\{|x| < \frac{\sqrt{\pi^2-a^2}}{2},\ \left|y - \frac{a}{\sqrt{\pi^2-a^2}}x\right| < \frac{\pi}{2}\right\}$. Let us further set

$$\Omega^{\omega}_{\gamma,a} := C_a\left(\Omega^{\omega}_{\gamma}\right)$$

and

$$u^{\omega}_{\gamma,a}(x, y) := u^{\omega}_{\gamma}\left(C_a^{-1}(x, y)\right), \quad (x, y) \in \Omega^{\omega}_{\gamma,a},$$

where u^{ω}_{γ} is defined in Theorem 2.2; see Figure 3.

In order to prove our next result we will make use of the following property of the positive principal eigenvalue.

Lemma 2.5. *Let $\mu \in \mathbb{R}$ and $\psi \in C(\Omega)$ be positive and satisfying $\mathcal{M}^+_{\lambda,\Lambda}(D^2\psi) + \mu\,\psi \le 0$ in Ω. Then, either $\mu < \mu^+(\Omega)$ or $\mu = \mu^+(\Omega)$ and ψ is an eigenfunction relative to $\mu^+(\Omega)$.*

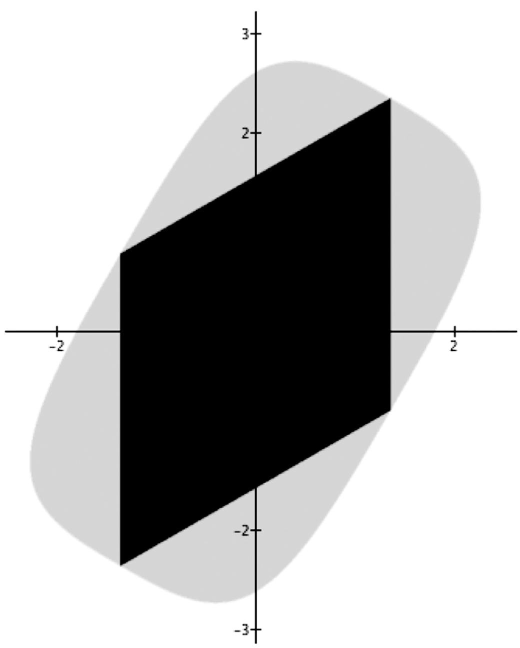

Figure 3: The domain $\Omega^{\omega}_{\gamma,a}$; in the black part $u^{\omega}_{\gamma,a}$ is concave

Proof. See Corollary 2.1 in [2] or Theorem 4.4 in [9]. □

Theorem 2.6. *Given $\Lambda \geq \lambda > 0$ let us set $\omega = \frac{\Lambda}{\lambda} \geq 1$. Then, for any $\frac{1}{\sqrt{\omega}} \leq \gamma \leq \sqrt{\omega}$ and $|a| < \pi$ the function $u^\omega_{\gamma,a}$ satisfies*

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2 u^\omega_{\gamma,a}) \geq \dfrac{\lambda\pi^2}{\pi^2 - a^2} u^\omega_{\gamma,a} \quad \text{in } \Omega^\omega_{\gamma,a}, \\ u^\omega_{\gamma,a} > 0 \text{ in } \Omega^\omega_{\gamma,a}, \ u^\omega_{\gamma,a} = 0 \text{ on } \partial\Omega^\omega_{\gamma,a}. \end{cases} \tag{5}$$

As a consequence, the positive principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ in $\Omega^\omega_{\gamma,a}$ satisfies

$$\mu\left(\Omega^\omega_{\gamma,a}\right) \geq \frac{\lambda\pi^2}{\pi^2 - a^2} \tag{6}$$

and equality holds if and only if either $\omega = 1$ or $a = 0$.

Proof. Let us compute. We have

$$D^2 u^\omega_{\gamma,a}(x,y) = \left(C_a^{-1}\right)^t D^2 u^\omega_\gamma \left(C_a^{-1}(x,y)\right) C_a^{-1}$$

with

$$C_a^{-1} = \begin{pmatrix} \dfrac{\pi}{\sqrt{\pi^2 - a^2}} & 0 \\ -\dfrac{a}{\sqrt{\pi^2 - a^2}} & 1 \end{pmatrix}.$$

Since $D^2 u^\omega_\gamma$ is diagonal, by setting

$$\begin{cases} X = \dfrac{\pi}{\sqrt{\pi^2 - a^2}} x, \\ Y = y - \dfrac{a}{\sqrt{\pi^2 - a^2}} x, \end{cases}$$

we then obtain

$$D^2 u^\omega_{\gamma,a}(x,y) = \begin{pmatrix} \begin{array}{l} \frac{\pi^2}{\pi^2 - a^2}(u^\omega_\gamma)_{xx}(X,Y) \\ + \frac{a^2}{\pi^2 - a^2}(u^\omega_\gamma)_{yy}(X,Y) \end{array} & -\frac{a}{\sqrt{\pi^2 - a^2}}(u^\omega_\gamma)_{yy}(X,Y) \\ & \\ -\frac{a}{\sqrt{\pi^2 - a^2}}(u^\omega_\gamma)_{yy}(X,Y) & (u^\omega_\gamma)_{yy}(X,Y) \end{pmatrix}.$$

Note that, in particular,

$$\operatorname{tr}(D^2 u^\omega_{\gamma,a}(x,y)) = \Delta u^\omega_{\gamma,a}(x,y) = \frac{\pi^2}{\pi^2 - a^2} \Delta u^\omega_\gamma(X,Y)$$

and

$$\det(D^2 u^\omega_{\gamma,a}(x,y)) = \frac{\pi^2}{\pi^2 - a^2} \det(D^2 u^\omega_\gamma(X,Y)).$$

Therefore, for $(x,y) \in C_a\left(\{|X|,\ |Y| \le \frac{\pi}{2}\}\right)$, $u^\omega_{\gamma,a}(x,y)$ is concave like u^ω_γ (X,Y) and it follows that

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 u^\omega_{\gamma,a}) = -\lambda\, \Delta u^\omega_{\gamma,a} = -\frac{\lambda\,\pi^2}{\pi^2 - a^2} \Delta u^\omega_\gamma = \frac{\lambda\,\pi^2}{\pi^2 - a^2} u^\omega_\gamma = \frac{\lambda\,\pi^2}{\pi^2 - a^2} u^\omega_{\gamma,a}.$$

Otherwise, for $(x,y) \in \Omega^\omega_{\gamma,a}$ such that either $|X| > \frac{\pi}{2}$ or $|Y| > \frac{\pi}{2}$, we have $\det(D^2 u^\omega_{\gamma,a}(x,y)) < 0$, and, by computing the eigenvalues of $D^2 u^\omega_{\gamma,a}$ and recalling the expressions of $(u^\omega_{\gamma,a})_{xx}$ and $(u^\omega_{\gamma,a})_{yy}$ from the proof of Theorem 2.2, we get

$$\begin{aligned}
-\mathcal{M}^+_{\lambda,\Lambda}(D^2 u^\omega_{\gamma,a}) &= -\frac{\lambda\,\pi^2}{2(\pi^2 - a^2)} \Bigg[(\omega+1)\left((u^\omega_\gamma)_{xx} + (u^\omega_\gamma)_{yy}\right) \\
&\quad + (\omega - 1)\sqrt{(u^\omega_\gamma)^2_{xx} + (u^\omega_\gamma)^2_{yy} + 2\left(\frac{2a^2}{\pi^2} - 1\right)(u^\omega_\gamma)_{xx}(u^\omega_\gamma)_{yy}}\Bigg] \\
&\ge -\frac{\lambda\,\pi^2}{2(\pi^2 - a^2)} \big[(\omega+1)\left((u^\omega_\gamma)_{xx} + (u^\omega_\gamma)_{yy}\right) \\
&\quad + (\omega-1)\left|(u^\omega_\gamma)_{xx} - (u^\omega_\gamma)_{yy}\right|\big] \\
&= \frac{\lambda\,\pi^2}{(\pi^2 - a^2)}\, u^\omega_{\gamma,a},
\end{aligned}$$

and equality holds in the above if and only if either $\omega = 1$ or $a = 0$. Therefore, $u^\omega_{\gamma,a}$ satisfies (5), and (6) follows immediately from the definition of the positive principal eigenvalue for $\mathcal{M}^+_{\lambda,\Lambda}$. Moreover, by Lemma 2.5, equality holds in (6) if and only if $u^\omega_{\gamma,a}$ is the principal eigenfunction for $\mathcal{M}^+_{\lambda,\Lambda}$ in $\Omega^\omega_{\gamma,a}$. Hence, equality holds in (6) if and only if either $\omega = 1$ or $a = 0$. □

As a consequence of Theorems 2.2 and 2.6, we can deduce that, among all sets $\Omega^\omega_{\gamma,a}$ and their rescaled $\delta\,\Omega^\omega_{\gamma,a}$ with $\delta > 0$, for equal area the minimum of the principal eigenvalue for $\mathcal{M}^+_{\lambda,\Lambda}$ is achieved on the most symmetric domain, that is some rescaled of Ω^ω_1. We will denote by $|\Omega|$ the area (two-dimensional Lebesgue measure) of any set $\Omega \subset \mathbb{R}^2$, and by $\mu(\Omega)$ the positive principal eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$ in the domain Ω.

Corollary 2.7. *Given* $\Lambda \geq \lambda > 0$*, let us set* $\omega = \frac{\Lambda}{\lambda} \geq 1$*. Then*

$$\mu\left(\frac{\Omega_1^\omega}{\sqrt{|\Omega_1^\omega|}}\right) = \min\left\{\mu\left(\frac{\Omega_{\gamma,a}^\omega}{\sqrt{\left|\Omega_{\gamma,a}^\omega\right|}}\right) \ : \ \frac{1}{\sqrt{\omega}} \leq \gamma \leq \sqrt{\omega},\ |a| < \pi\right\}.$$

Proof. By the homogeneity of the principal eigenvalue and by Theorem 2.6, we have

$$\mu\left(\frac{\Omega_{\gamma,a}^\omega}{\sqrt{\left|\Omega_{\gamma,a}^\omega\right|}}\right) = \left|\Omega_{\gamma,a}^\omega\right| \mu\left(\Omega_{\gamma,a}^\omega\right) \geq \frac{\lambda\,\pi^2}{\pi^2 - a^2}\left|\Omega_{\gamma,a}^\omega\right|.$$

Moreover, one has

$$\left|\Omega_{\gamma,a}^\omega\right| = \left|C_a\left(\Omega_\gamma^\omega\right)\right| = \left|\det\left(C_a\right)\right| \left|\Omega_\gamma^\omega\right| = \frac{\sqrt{\pi^2 - a^2}}{\pi}\left|\Omega_\gamma^\omega\right|,$$

so that

$$\mu\left(\frac{\Omega_{\gamma,a}^\omega}{\sqrt{\left|\Omega_{\gamma,a}^\omega\right|}}\right) \geq \frac{\lambda\,\pi}{\sqrt{\pi^2 - a^2}}\left|\Omega_\gamma^\omega\right| \geq \lambda\left|\Omega_\gamma^\omega\right|.$$

On the other hand, by the definition of Ω_γ^ω, we get

$$\left|\Omega_\gamma^\omega\right| = \pi^2 + 4\sqrt{\omega}\int_0^{\pi/2}\left[\arcsin\left(\frac{\gamma}{\sqrt{\omega}}\cos x\right) + \arcsin\left(\frac{1}{\gamma\sqrt{\omega}}\cos x\right)\right]dx\,;$$

hence,

$$\begin{aligned}\frac{d}{d\gamma}|\Omega_\gamma^\omega| &= \frac{4\sqrt{\omega}}{\gamma}\int_0^{\pi/2}\left[\frac{\cos x}{\sqrt{\frac{\omega}{\gamma^2} - \cos^2 x}} - \frac{\cos x}{\sqrt{\omega\gamma^2 - \cos^2 x}}\right]dx \\ &\begin{cases}\geq 0 & \text{for } \gamma \geq 1, \\ \leq 0 & \text{for } \gamma \leq 1,\end{cases}\end{aligned}$$

which shows that $\left|\Omega_\gamma^\omega\right|$ is minimal for $\gamma = 1$. In conclusion, using also Theorem 2.2, we deduce

$$\mu\left(\frac{\Omega_{\gamma,a}^\omega}{\sqrt{\left|\Omega_{\gamma,a}^\omega\right|}}\right) \geq \lambda\left|\Omega_\gamma^\omega\right| \geq \lambda\left|\Omega_1^\omega\right| = \mu\left(\frac{\Omega_1^\omega}{\sqrt{\Omega_1^\omega}}\right).$$

□

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