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Sharp gradient estimate and spectral rigidity for p-Laplacian

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We derive a sharp gradient estimate for positive eigenfunctions of the p-Laplacian on a complete manifold with Ricci curvature bounded below. As an application, we study the rigidity of manifolds achieving the maximum value of the principal eigenvalue of the p-Laplacian.

1. Introduction

In this paper, we consider the p-Laplacian $\mathcal{L}$, $1 < p < \infty$, on a Riemannian manifold M given by

$$\mathcal{L}(v) = \mathrm{div}(|\nabla v|^{p-2}\nabla v).$$

A function v is called p-harmonic if

$$\mathrm{div}(|\nabla v|^{p-2}\nabla v) = 0.$$

For a positive p-harmonic function v, if $u = -(p-1)\ln v$, then it satisfies

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p.$$

As p approaches 1, the equation formally reduces to

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u|.$$

According to Huisken and Ilmanen [4], the level set of a proper solution to this last equation yields a weak solution to the inverse mean curvature flow. Moser [11] has successfully exploited this point of view and established the existence of a weak solution to the inverse mean curvature flow starting from the boundary of any smooth compact domain in the Euclidean space. The crucial point is to justify the limiting process of p approaching 1 as alluded above. This relies on a uniform gradient estimate, independent of p, for function $u = -(p-1)\ln v$ with v being p-harmonic.

Kotschwar and Ni [5] have also succeeded in carrying out the same scheme on general Riemannian manifolds with sectional curvature bounded from below. Again, they proceed by obtaining such a uniform gradient estimate. In fact, the estimate itself only involves the lower bound of the Ricci curvature of the underlying manifold. In view of this, it is natural to wonder if the assumption of the sectional curvature being bounded from below is necessary. In an attempt to address this question, Wang and Zhang [12] have introduced a new approach for such a gradient estimate. Assuming only the Ricci curvature is bounded from below, they were able to obtain a gradient estimate for u. However, their estimate does not seem to be uniform in p.

One of the purposes here is to completely answer the question by providing a sharp gradient estimate for u. In fact, we consider more generally positive eigenfunctions of the p-Laplacian.

Theorem 1.1. Let (M^n, g) be an n-dimensional complete noncompact manifold with $\mathrm{Ric} \geq -(n-1)$. Then for a positive function v satisfying

$$\mathrm{div}(|\nabla v|^{p-2}\nabla v) = -\lambda v^{p-1},$$

we have $|\nabla \ln v| \leq y$, where y is the unique positive root of the equation

$$(p-1)\, y^p - (n-1)\, y^{p-1} + \lambda = 0.$$

In particular, we have the following corollary for positive p-harmonic functions.

Corollary 1.2. Let (M^n, g) be an n-dimensional complete noncompact manifold with $\mathrm{Ric} \geq -(n-1)$. Let v be a positive p-harmonic function and $u = -(p-1)\ln v$. Then $|\nabla u| \leq n-1$.

The estimate in the theorem is sharp as demonstrated by the following example.

Example 1.3. Let $M^n = \mathbb{R} \times N^{n-1}$ with the warped product metric $ds^2 = dt^2 + e^{-2t} ds_N^2$, where N is a complete manifold with nonnegative Ricci curvature. Then it can be directly checked (see [6]) that $\mathrm{Ric}_M \geq -(n-1)$.

Now consider the function $v(t,z) = e^{\alpha t}$ on M with $\frac{n-1}{p} \leq \alpha \leq \frac{n-1}{p-1}$, where $t \in \mathbb{R}$ and $z \in N$. A straightforward calculation shows

$$\mathcal{L}(v) = \mathrm{div}(|\nabla v|^{p-2}\nabla v) = \big((p-1)\,\alpha - (n-1)\big)\,\alpha^{p-1}\, v^{p-1}.$$

Hence,

$$\lambda = \big(n-1-(p-1)\,\alpha\big)\,\alpha^{p-1}$$

and $|\nabla \ln v| = \alpha$. Clearly,

$$(p-1)\,\alpha^{p} - (n-1)\,\alpha^{p-1} + \lambda = 0.$$

We should point out that Theorem 1.1 is known in the case $p = 2$ or $\mathcal{L}$ is the standard Laplacian Δ. This is essentially proved by Yau [13]. We refer to [6] for details.

The second purpose of the paper is to utilize the preceding sharp gradient estimate to study the manifolds whose principal eigenvalue for the p-Laplacian achieves its maximum value. Recall that the principal eigenvalue λ_1 of the p-Laplacian $\mathcal{L}$ is the maximum constant λ such that the equation

$$\mathcal{L}(v) = -\lambda\, v^{p-1}$$

admits a positive solution. Alternatively, λ_1 may be characterized variationally as the best constant of the following Poincaré inequality.

$$\lambda_1 \int_M |\phi|^p \leq \int_M |\nabla \phi|^p.$$

A classical result of Cheng [3] says that $\lambda_1(\Delta) \leq \frac{(n-1)^2}{4}$ on a complete n-dimensional manifold M with $\mathrm{Ric}_M \geq -(n-1)$. The same argument also implies $\lambda_1(\mathcal{L}) \leq \left(\frac{n-1}{p}\right)^p$. This estimate is sharp as demonstrated by Example 1.3 and also the hyperbolic space form $\mathbb{H}^n$.

A natural question is to understand manifolds with $\lambda_1(\mathcal{L})$ achieving its maximum value. When $p = 2$, this question has been studied by Li and the second author in [7, 8]. In particular, they have proved that such a manifold must be connected at infinity unless it is a topological cylinder endowed with an explicit warped product metric. We will take up the general case of $1 < p < \infty$ here and provide a faithful generalization of their result. Our main conclusion can be summarized as follows.

Theorem 1.4. Let M^n be a complete manifold of dimension $n \geq 3$ with $\mathrm{Ric}_M \geq -(n-1)$. If $\lambda_1(\mathcal{L}) = \left(\frac{n-1}{p}\right)^p$ for some $p \leq \frac{(n-1)^2}{2(n-2)}$, then either M is connected at infinity or it splits as a warped product $M = \mathbb{R} \times N$ with $ds_M^2 = dt^2 + h^2(t)ds_N^2$, where N is compact, and the function $h(t) = \mathrm{e}^t$ if $n \geq 4$ and $h(t) = \mathrm{e}^t$ or $h(t) = \cosh t$ if $n = 3$.

It is unclear to us at this point whether the restriction on p is necessary. It should also be emphasized that due to the nonlinear nature of the operator $\mathcal{L}$, a different approach from [7, 8] is needed, and the sharp gradient in Theorem 1.1 is crucial here. The following by-product of our argument seems worth mentioning.

Corollary 1.5. Let (M, g) be a complete noncompact manifold with its Ricci curvature bounded below by a constant. If $\lambda_1(\mathcal{L}) > 0$ for some $p > 2$, then $\lambda_1(\Delta) > 0$ as well.

In the last section, we prove a decay estimate in the spirit of Agmon [1] for the p-Laplace equation. Agmon [1] has shown an eigenfunction corresponding to an eigenvalue below the essential spectrum of the Laplacian must decay exponentially provided it does not grow too fast. An optimal version was later given in [10]. Here, we consider the issue for the p-Laplacian. However, our estimate is not as sharp compared to the Laplacian case.

Theorem 1.6. Let M be a complete Riemannian manifold and u a positive function satisfying

$$\mathcal{L}(u) \geq 0$$

on $M \backslash B(R_0)$, where $B(R_0)$ is a geodesic ball in M. Assume the principal eigenvalue λ of $\mathcal{L}$ on $M \backslash B(R_0)$ is positive. If u satisfies the growth condition

$$\int_{B(R)} u^p \exp\left(-\frac{\lambda^{1/p}}{12}\, r\right) = o(R^p),$$

then

$$\int_{B(R+1)\backslash B(R)} u^p \leq C \exp\left(-\frac{\lambda^{1/p}}{12}\, R\right)$$

for some constant $C > 0$.

2. Sharp gradient estimate

In this section, we prove a sharp gradient estimate for positive eigenfunctions of the p-Laplacian. Recall the p-Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L}(v) = \operatorname{div}(|\nabla v|^{p-2} \nabla v).$$

A function v is an eigenfunction of p-Laplacian with corresponding eigenvalue $\lambda \geq 0$ if

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = -\lambda\, |v|^{p-2}\, v.$$

We only consider positive solutions v here and set $u = -(p-1)\ln v$. It can be easily verified that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p + \lambda\,(p-1)^{p-1}.$$

Denote L to be the linear operator given by

$$L(\varphi) = \operatorname{div}\big(f^{\frac{p}{2}-1}A(\nabla\varphi)\big), \tag{2.1}$$

where $f = |\nabla u|^2$ and $A = \mathrm{id} + (p-2)\, f^{-1}\, \nabla u \otimes \nabla u$.

The following lemma is a direct calculation and due to [5].

Lemma 2.1. At points where $f > 0$,

$$\begin{aligned} L(f) = 2f^{\frac{p}{2}-1}(u_{ij}^2 + R_{ij}u_iu_j) + \Big(\frac{p}{2}-1\Big)\, |\nabla f|^2 f^{\frac{p}{2}-2} \\ + p\, f^{\frac{p}{2}-1}\, \langle \nabla u, \nabla f\rangle. \end{aligned}$$

Proof. Note that by the regularity theory v is smooth away from the points where $\nabla v = 0$. By the definition of (2.1),

$$\begin{aligned} L(f) &= f^{\frac{p}{2}-1}\Big(\Delta f + (p-2)\operatorname{div}(f^{-1}\, \langle\nabla u, \nabla f\rangle\, \nabla u\Big) \\ &\quad + \langle \nabla(f^{\frac{p}{2}-1}), \nabla f + (p-2)\, f^{-1}\, \langle \nabla u, \nabla f\rangle\, \nabla u\rangle \\ &= 2\, f^{\frac{p}{2}-1}\, (u_{ij}^2 + R_{ij}\, u_i\, u_j) + \Big(\frac{p}{2}-1\Big)\, f^{\frac{p}{2}-2}\, |\nabla f|^2 \\ &\quad + (p-2)\, \Big(\frac{p}{2}-2\Big)\, f^{\frac{p}{2}-3}\, \langle \nabla u, \nabla f\rangle^2 \\ &\quad + (p-2)\, f^{\frac{p}{2}-2}\, (u_{ij}\, f_i\, u_j + f_{ij}\, u_i\, u_j) \\ &\quad + (p-2)\, f^{\frac{p}{2}-2}\, \Delta u\, \langle \nabla u, \nabla f\rangle + 2\, f^{\frac{p}{2}-1}\, \langle \nabla\Delta u, \nabla u\rangle. \end{aligned} \tag{2.2}$$

Since

$$\operatorname{div}\big(|\nabla u|^{p-2}\, \nabla u\big) = |\nabla u|^p + \lambda(p-1)^{p-1},$$

we have

$$|\nabla u|^{p-2}\Delta u + \langle \nabla|\nabla u|^{p-2}, \nabla u\rangle = |\nabla u|^p + \lambda(p-1)^{p-1}. \tag{2.3}$$

Note that $f = |\nabla u|^2$. Equation (2.3) becomes

$$f^{\frac{p-2}{2}}\Delta u + \left(\frac{p}{2}-1\right) f^{\frac{p}{2}-2}\langle \nabla f, \nabla u\rangle = f^{\frac{p}{2}} + \lambda(p-1)^{p-1}.$$

Therefore,

$$2\left\langle \nabla\left(f^{\frac{p-2}{2}}\Delta u + \left(\frac{p}{2}-1\right) f^{\frac{p}{2}-2}\langle \nabla f, \nabla u\rangle\right), \nabla u\right\rangle = 2\,\langle \nabla f^{\frac{p}{2}}, \nabla u\rangle.$$

This implies

$$\begin{aligned}(p-2)\left(\frac{p}{2}-2\right) f^{\frac{p}{2}-3}\langle \nabla f, \nabla u\rangle^2 + (p-2) f^{\frac{p}{2}-2}\left(f_{ij}\,u_i\,u_j + u_{ij}\,f_i\,u_j\right) \\ + (p-2) f^{\frac{p}{2}-2}\Delta u\,\langle \nabla f, \nabla u\rangle + 2\,f^{\frac{p}{2}-1}\langle \nabla \Delta u, \nabla u\rangle = p\,f^{\frac{p}{2}-1}\langle \nabla f, \nabla u\rangle.\end{aligned}$$

Combining with (2.2), we obtain Lemma 2.1. □

We are now ready to prove the gradient estimate.

Theorem 2.2. Let (M^n, g) be an n-dimensional complete noncompact manifold with $\mathrm{Ric} \geq -(n-1)$. Then for a positive function v satisfying

$$\mathrm{div}(|\nabla v|^{p-2}\nabla v) = -\lambda v^{p-1},$$

$|\nabla \ln v| \leq y$, where y is the unique positive root of the equation

$$(p-1)\,y^p - (n-1)\,y^{p-1} + \lambda = 0.$$

Proof. Let us first point out that the value $y > 0$ is well defined since $\lambda \leq \frac{(n-1)^p}{p^p}$. As before, we let $u = -(p-1)\ln v$ and $f = |\nabla u|^2$. The result in [12] implies that $f \leq c(n,p)$, a constant depending on p and n.

To obtain the sharp estimate, let x be the unique positive root of the equation

$$x^{\frac{p}{2}} - (n-1)\,x^{\frac{p-1}{2}} + \lambda\,(p-1)^{p-1} = 0.$$

For any $\delta > 0$, consider $\omega = (f - (x+\delta))^+$, that is,

$$\omega = \begin{cases} f - (x+\delta), & f > x+\delta, \\ 0, & \text{otherwise}. \end{cases}$$

Claim: $L(\omega) \geq a\omega - b|\nabla w|$ in the weak sense for some positive constants a and b depending on p, n and δ, that is,

$$\int_M L(\varphi)\,\omega \geq \int_M \varphi\,(a\,\omega - b\,|\nabla\omega|) \tag{2.4}$$

for any nonnegative smooth function φ with compact support on M.

Indeed, if we denote by $\Omega = \{f \geq x + \delta\}$, then

$$\begin{aligned}\int_M L(\varphi)\,\omega &= \int_\Omega L(\varphi)\,\omega \\ &= \int_\Omega \varphi\,L(\omega) + \int_{\partial\Omega} \langle f^{\frac{p}{2}-1} A(\nabla\varphi), \nu\rangle\,\omega - \int_{\partial\Omega} \langle f^{\frac{p}{2}-1}\,A(\nabla\omega), \nu\rangle\,\varphi,\end{aligned}$$

where ν is the outward unit normal vector of $\partial\Omega$. Using $\nu = -\frac{\nabla f}{|\nabla f|} = -\frac{\nabla\omega}{|\nabla\omega|}$ and $\omega = 0$ on $\partial\Omega$, we conclude

$$\begin{aligned}\int_M L(\varphi)\,\omega &= \int_\Omega \varphi\,L(\omega) + \int_{\partial\Omega} f^{\frac{p}{2}-1}\,\frac{\varphi\,\langle A(\nabla\omega), \nabla\omega\rangle}{|\nabla\omega|} \\ &\geq \int_\Omega \varphi\,L(\omega).\end{aligned} \tag{2.5}$$

On the other hand,

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = |\nabla u|^p + \lambda(p-1)^{p-1}. \tag{2.6}$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal frame on M with $|\nabla u|\,e_1 = \nabla u$. Then (2.6) can be written into

$$(p-1)\,u_{11} + \sum_{i=2}^n u_{ii} = f + |\nabla u|^{2-p}\,\lambda\,(p-1)^{p-1}.$$

Therefore,

$$\begin{aligned}\sum_{i,j=1}^n u_{ij}^2 &\geq \frac{(\sum_{i=2}^n u_{ii})^2}{n-1} + \sum_{i=1}^n u_{1i}^2 \\ &\geq \frac{\big(f + |\nabla u|^{2-p}\,\lambda\,(p-1)^{p-1} - (p-1)\,u_{11}\big)^2}{n-1} + \sum_{i=1}^n u_{1i}^2.\end{aligned} \tag{2.7}$$

Together with Lemma 2.1 and the Ricci curvature lower bound, on Ω we have

$$\begin{aligned}
L(\omega) \geq\, & 2\, f^{\frac{p}{2}-1} \left\{ \frac{\big(f + \lambda\, (p-1)^{p-1}\, |\nabla u|^{2-p} - (p-1)\, u_{11}\big)^2}{n-1} \right. \\
& \left. + \sum_{i=1}^n u_{1i}^2 - (n-1) f \right\} + \left(\frac{p}{2} - 1\right) |\nabla f|^2\, f^{\frac{p}{2}-2} + p\, f^{\frac{p}{2}-1}\, \langle \nabla u, \nabla \omega \rangle \\
\geq\, & 2\, f^{\frac{p}{2}-1} \left\{ \frac{\big(f + \lambda\, (p-1)^{p-1}\, |\nabla u|^{2-p}\big)^2}{n-1} - (n-1) f \right. \\
& \left. + \sum_{i=1}^n u_{1i}^2 - \frac{2(p-1)}{n-1}\, f\, u_{11} - \frac{2\,\lambda\, (p-1)^p}{n-1}\, |\nabla u|^{2-p}\, u_{11} \right\} \\
& + \left(\frac{p}{2} - 1\right)\, |\nabla f|^2\, f^{\frac{p}{2}-2} + p\, f^{\frac{p}{2}-1}\, \langle \nabla u, \nabla \omega \rangle \\
\geq\, & \frac{2\, f^{\frac{p}{2}-1}}{n-1} \left(f + \lambda\, (p-1)^{p-1}\, |\nabla u|^{2-p} + (n-1)\, f^{\frac{1}{2}} \right) \\
& \times \left(f + \lambda\, (p-1)^{p-1}\, |\nabla u|^{2-p} - (n-1)\, f^{\frac{1}{2}} \right) \\
& + \frac{p-1}{2}\, |\nabla f|^2\, f^{\frac{p}{2}-2} + p\, f^{\frac{p}{2}-1}\, \langle \nabla u, \nabla \omega \rangle \\
& - \frac{2(p-1)}{n-1}\, f^{\frac{p}{2}-1}\, \langle \nabla u, \nabla \omega \rangle - \frac{2\lambda\, (p-1)^p}{n-1}\, \frac{\langle \nabla u, \nabla \omega \rangle}{f},
\end{aligned}$$

where we have used the fact that $2\, u_{11} = \langle \nabla u, \frac{\nabla f}{f} \rangle$ and

$$\sum_{i=1}^n u_{1i}^2 \geq \frac{|\nabla f|^2}{4\, f}.$$

Since $0 < x \leq f \leq c(n,p)$ on Ω and $p > 1$, we conclude that, on Ω

$$L(\omega) \geq c_1 \big(f^{\frac{p}{2}} - (n-1) f^{\frac{p-1}{2}} + \lambda (p-1)^{p-1}\big) - c_2 |\nabla \omega|, \tag{2.8}$$

where c_1 and c_2 are positive constants depending on n and p.

We now verify that

$$f^{\frac{p}{2}} - (n-1) f^{\frac{p-1}{2}} + \lambda (p-1)^{p-1} \geq c_3\, \omega \tag{2.9}$$

for some positive constant c_3 depending on n, δ and p.

In fact, if we view both sides as a function of f, (2.9) clearly holds true when $f = x$ for any choice of c_3. Now the left-hand side of (2.9), as a function of f, its derivative is given by

$$\frac{p}{2}\, f^{\frac{p}{2}-1} - \frac{(n-1)(p-1)}{2}\, f^{\frac{p-3}{2}} = \frac{1}{2}\, f^{\frac{p-3}{2}}\big(p\, f^{\frac{1}{2}} - (p-1)(n-1)\big).$$

Note that $\lambda \leq (\frac{n-1}{p})^p$. So $x \geq (p-1)^2\left(\frac{n-1}{p}\right)^2$. Using the fact that $f \geq x+\delta$, we conclude

$$p\, f^{\frac{1}{2}} - (p-1)(n-1) \geq c(p,n,\delta) > 0.$$

Therefore, (2.9) holds true on Ω. In particular, combining with (2.8), we have

$$L(\omega) \geq a\,\omega - b\,|\nabla\omega|$$

on Ω, where a and b are positive constants only depending on p, n and δ. Plugging into (2.5), we arrive at

$$\begin{aligned}\int_M L(\varphi)\,\omega &\geq \int_\Omega \varphi\, L(\omega)\\ &\geq \int_\Omega \varphi\,(a\,\omega - b\,|\nabla\omega|)\\ &= \int_M \varphi\,(a\,\omega - b\,|\nabla\omega|).\end{aligned}$$

In conclusion, $L(\omega) \geq a\,\omega - b\,|\nabla\omega|$ on M in the weak sense as claimed.

We now use the claim to show $\omega \equiv 0$. Observe that for any cut-off function φ on M and $q \geq 1$, the function $\varphi^2\,\omega^q$ may be used as a test function. So we have

$$-\int_\Omega \langle A(\nabla(\varphi^2\,\omega^q)), \nabla\omega\rangle\, f^{\frac{p}{2}-1} \geq \int_M (a\,\varphi^2\,\omega^{q+1} - b\,\varphi^2\,\omega^q\,|\nabla\omega|).$$

This implies

$$\begin{aligned}a\int_M \varphi^2\omega^{q+1} \leq{}& b\int_\Omega \varphi^2\,\omega^q\,|\nabla\omega| + c(n,p,\delta)\int_\Omega \varphi\,|\nabla\varphi|\,|\nabla\omega|\,\omega^q\\ &- \int_\Omega q\,\varphi^2\,\omega^{q-1}\,\langle\nabla\omega, A(\nabla\omega)\rangle\, f^{\frac{p}{2}-1}.\end{aligned}$$

Note that

$$\begin{aligned}\langle\nabla\omega, A(\nabla\omega)\rangle &= |\nabla\omega|^2 + (p-2)\,\frac{u_i\,u_j}{|\nabla u|^2}\,\omega_i\,\omega_j\\ &\geq \min\{p-1, 1\}\,|\nabla\omega|^2.\end{aligned}$$

Thus, for any $\epsilon > 0$,

$$a \int_M \varphi^2 \, \omega^{q+1} \leq \epsilon \int_\Omega \varphi^2 \, \omega^{q+1} + \frac{b}{4\epsilon} \int_\Omega \varphi^2 \, \omega^{q-1} \, |\nabla \omega|^2 + \epsilon \int_\Omega |\nabla \varphi|^2 \, \omega^{q+1}$$
$$+ \frac{c}{4\epsilon} \int_\Omega \varphi^2 \, \omega^{q-1} \, |\nabla \omega|^2 - \tilde{c} \, q \int_\Omega \varphi^2 \, \omega^{q-1} \, |\nabla \omega|^2,$$

where c and $\tilde{c}$ are constants depending on p, n and δ. Choose q such that $b + c = 4 \, \epsilon \, \tilde{c} \, q$. Then

$$a \int_M \varphi^2 \, \omega^{q+1} \leq 2 \, \epsilon \int_M |\nabla \varphi|^2 \, \omega^{q+1}.$$

Now let $\varphi = 1$ on $B(k)$ and 0 outside $B(k+1)$ with $|\nabla \varphi| \leq 2$, where k is a positive integer. Then

$$\frac{c}{\epsilon} \int_{B(k)} \omega^{q+1} \leq \int_{B(k+1)} \omega^{q+1}.$$

Iterating this inequality, we conclude

$$\int_{B(k+1)} \omega^{q+1} \geq \left(\frac{c}{\epsilon} \right)^k \int_{B(1)} \omega^{q+1}.$$

This implies either $\omega \equiv 0$ or for all $R \geq 1$,

$$\int_{B(R)} \omega^{q+1} \geq c_1 \, e^{R \, \ln \frac{c_2}{\epsilon}}$$

for some positive constants c_1 and c_2 independent of ϵ. However, since ω is bounded and the volume of the ball $B(R)$ satisfies $V(R) \leq c \, e^{(n-1)R}$, this leads to a contradiction if ϵ is chosen sufficiently small. Therefore, $\omega \equiv 0$. In other words, $f \leq x$ as $\delta > 0$ is arbitrary. This is obviously equivalent to $|\nabla \ln v| \leq y$. □

As pointed out previously, this theorem is sharp. We now draw a corollary that will be needed later.

Corollary 2.3. Let (M^n, g) be an n-dimensional complete noncompact manifold with $\mathrm{Ric} \geq -(n-1)$. Let v be a positive solution of

$$\mathrm{div}(|\nabla v|^{p-2} \nabla v) = - \left(\frac{n-1}{p} \right)^p v^{p-1}.$$

Then $|\nabla \ln v| \leq \frac{n-1}{p}$.

3. Structure at infinity

In this section, we consider the manifolds with maximum value of $\lambda_1(\mathcal{L})$ and study its structure at infinity. We will first deal with the finite volume ends.

Theorem 3.1. Let (M^n, g) be a complete manifold of dimension $n \geq 2$. Suppose that the Ricci curvature is bounded from below by $\mathrm{Ric}_M \geq -(n-1)$ and $\lambda_1(\mathcal{L}) = (\frac{n-1}{p})^p$. Then either M has no finite volume ends or $M^n = \mathbb{R} \times N^{n-1}$ with $ds^2 = dt^2 + e^{2t} ds_N^2$ and N being a compact manifold of non-negative Ricci curvature.

Proof. Suppose M has a finite volume end E. Let β be the Busemann function associated with a geodesic ray γ contained in E, that is,

$$\beta(x) = \lim_{t\to\infty} \big(t - d(x, \gamma(t))\big).$$

The Laplacian comparison theorem implies

$$\Delta\beta \geq -(n-1).$$

Therefore,

$$\begin{aligned}
\mathcal{L}\big(e^{\frac{(n-1)}{p}\beta}\big) &= \mathrm{div}\Big[\left(\frac{n-1}{p}\right)^{p-2} e^{\frac{(n-1)}{p}(p-2)\beta} \nabla e^{\frac{n-1}{p}\beta}\Big] \\
&= \left(\frac{n-1}{p}\right)^{p-1} e^{\frac{(n-1)}{p}(p-1)\beta} \Delta\beta \\
&\quad + \left(\frac{n-1}{p}\right)^{p-1} \left(\frac{n-1}{p}\right) (p-1) e^{\frac{n-1}{p}(p-1)\beta} |\nabla\beta|^2 \\
&\geq \left(\frac{n-1}{p}\right)^{p-1} e^{\frac{(n-1)}{p}(p-1)\beta} \big[(n-1)\frac{p-1}{p} - (n-1)\big] \\
&= -\left(\frac{n-1}{p}\right)^{p} e^{\frac{(n-1)}{p}(p-1)\beta}.
\end{aligned}$$

So for $v = e^{\frac{(n-1)}{p}\beta}$, we have

$$\mathcal{L}(v) \geq -\lambda_1\, v^{p-1}.$$

Let φ be a nonnegative compactly supported smooth function on M. Then

$$\lambda_1 \int_M (\varphi\, v)^p \leq \int_M |\nabla(\varphi\, v)|^p.$$

Noting that

$$\int_M \varphi^p \, v \, \mathcal{L}(v) = -\int_M \varphi^p \, |\nabla v|^p - p \int_M \varphi^{p-1} \, \langle \nabla \varphi, \nabla v \rangle \, v \, |\nabla v|^{p-2}$$

and

$$\begin{aligned} |\nabla(\varphi v)|^p &= \left(|\nabla \varphi|^2 \, v^2 + 2\varphi \, v \, < \nabla \varphi, \nabla v > + \varphi^2 |\nabla v|^2 \right)^{\frac{p}{2}} \\ &\le \varphi^p |\nabla v|^p + p\varphi \, v \, < \nabla \varphi, \nabla v > \varphi^{p-2} \, |\nabla v|^{p-2} + c|\nabla \varphi|^2 \, v^p \end{aligned}$$

for some constant c depending on p, we conclude

$$\begin{aligned} (3.1) \quad & \int_M \varphi^p \, v \left(\mathcal{L}(v) + \lambda_1 \, v^{p-1} \right) \\ &= \lambda_1 \int_M (\varphi \, v)^p - \int_M \varphi^p \, |\nabla v|^p - p \int_M \varphi^{p-1} \, \langle \nabla \varphi, \nabla v \rangle \, v \, |\nabla v|^{p-2} \\ &\le \int_M |\nabla(\varphi \, v)|^p - \int_M \varphi^p \, |\nabla v|^p - p \int_M \varphi^{p-1} \, \langle \nabla \varphi, \nabla v \rangle \, v \, |\nabla v|^{p-2} \\ &\le c \int_M |\nabla \varphi|^2 \, v^p. \end{aligned}$$

Now choose

$$\varphi = \begin{cases} 1, & B(R), \\ 0, & M \backslash B(2R) \end{cases}$$

with $|\nabla \varphi| \le \frac{2}{R}$. Then

$$\begin{aligned} (3.2) \qquad \int_M |\nabla \varphi|^2 \, v^p &= \int_M |\nabla \varphi|^2 \, e^{(n-1)\beta} \\ &\le \frac{4}{R^2} \int_{B(2R) \backslash B(R)} e^{(n-1)\beta} \\ &= \frac{4}{R^2} \int_{E \cap (B(2R) \backslash B(R))} e^{(n-1)\beta} \\ &\quad + \frac{4}{R^2} \int_{(M \backslash E) \cap (B(2R) \backslash B(R))} e^{(n-1)\beta}. \end{aligned}$$

Since $\lambda_1(\mathcal{L}) = \left(\frac{n-1}{p} \right)^p$, by Buckley and Koskela [2], we have

$$V(E \backslash B(R)) \le c \, e^{-(n-1)R}.$$

Thus, the first term goes to 0 as R goes to infinity. Note that (see [9])

$$\beta(x) \leq -r(x) + c$$

on $M \backslash E$ and $V(B(R)) \leq c\, e^{(n-1)R}$. The second term of (3.2) converges to zero as well. Combining with (3.1), we conclude

$$\mathcal{L}(v) + \lambda_1\, v^{p-1} \equiv 0.$$

This implies

$$\Delta\beta = -(n-1).$$

The final conclusion that $M^n = \mathbb{R} \times N^{n-1}$ now follows from the argument in [9]. □

We now turn to the infinite volume ends.

Theorem 3.2. Let (M^n, g) be a complete manifold of dimension $n \geq 3$. Suppose that the Ricci curvature is bounded from below by $\mathrm{Ric}_M \geq -(n-1)$ and $\lambda_1(\mathcal{L}) = (\frac{n-1}{p})^p$ for some $p \leq \frac{(n-1)^2}{2(n-2)}$. Then M has only one infinite volume end or $M^n = \mathbb{R} \times N^{n-1}$ for some compact manifold N with $ds_M^2 = dt^2 + \cosh^2(t)\, ds_N^2$.

Proof. We may assume $p \geq 2$ as otherwise by a simple Hölder inequality argument $\lambda_1(\Delta) = \frac{(n-1)^2}{4}$ and the theorem holds true by Li and Wang [7].

We first recall a general fact (see [10]) that the existence of a positive solution u to the equation

$$\Delta u + \langle \nabla h, \nabla u \rangle = -\rho\, u$$

on M implies the validity of the following weighted Poincaré inequality:

$$\int_M \rho\, \phi^2\, e^h \leq \int_M |\nabla \phi|^2\, e^h$$

for any compactly supported smooth function ϕ.

Now let v be a positive eigenfunction of the p-Laplacian such that

$$\operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = -\lambda_1 v^{p-1}.$$

The equation can be re-written into

$$\Delta v + (p-2)\left\langle \nabla v, \frac{\nabla|\nabla v|}{|\nabla v|}\right\rangle = -\lambda_1 \frac{v^{p-2}}{|\nabla v|^{p-2}}\, v.$$

Applying the aforementioned general fact with $h = (p-2)\ln|\nabla v|$ and $\rho = \lambda_1 \frac{v^{p-2}}{|\nabla v|^{p-2}}$, we conclude

$$\int_M \lambda_1 \frac{v^{p-2}}{|\nabla v|^{p-2}}\, \varphi^2\, |\nabla v|^{p-2} \le \int_M |\nabla\varphi|^2\, |\nabla v|^{p-2}$$

for any compactly supported smooth function φ on M. Obviously, the inequality can be simplified into

$$\int_M \lambda_1\, \varphi^2\, v^{p-2} \le \int_M |\nabla\varphi|^2\, |\nabla v|^{p-2}.$$

Now let $w = v^{\frac{p-2}{2}}$. Then

$$\lambda_1 \int_M \varphi^2 = \lambda_1 \int_M \left(\frac{\varphi}{w}\right)^2 v^{p-2} \le \int_M |\nabla\left(\frac{\varphi}{w}\right)|^2\, |\nabla v|^{p-2}. \tag{3.3}$$

Expanding the right-hand side of (3.3), we get

$$\begin{aligned}
&\int_M \left|\nabla\left(\frac{\varphi}{w}\right)\right|^2 |\nabla v|^{p-2} \\
&= \int_M \left|\frac{\nabla\varphi}{w} - \frac{\varphi\nabla w}{w^2}\right|^2 |\nabla v|^{p-2} \\
&= \int_M \left\{|\nabla\varphi|^2\, \frac{|\nabla v|^{p-2}}{v^{p-2}} + \varphi^2 \frac{|\nabla w|^2}{w^4}|\nabla v|^{p-2} - 2\frac{\varphi\langle\nabla\varphi,\nabla w\rangle}{w^3}|\nabla v|^{p-2}\right\}.
\end{aligned}$$

Note that

$$\frac{|\nabla w|^2}{w^4}|\nabla v|^{p-2} = \left(\frac{p-2}{2}\right)^2 \frac{|\nabla v|^p}{v^p}.$$

Hence

$$\lambda_1 \int_M \varphi^2 \leq \int_M \Big(|\nabla\varphi|^2 \, \frac{|\nabla v|^{p-2}}{v^{p-2}} + \left(\frac{p-2}{2}\right)^2 \varphi^2 \, \frac{|\nabla v|^p}{v^p} \Big) + \frac{1}{2} \int_M \langle \nabla\varphi^2, \nabla w^{-2} \rangle \, |\nabla v|^{p-2}. \tag{3.4}$$

The last term of (3.4) can be simplified as follows.

$$\begin{aligned} &\frac{1}{2} \int_M \langle \nabla\varphi^2, \nabla w^{-2} \rangle \, |\nabla v|^{p-2} \\ &\quad = -\frac{1}{2} \int_M \varphi^2 \, \Delta(w^{-2}) \, |\nabla v|^{p-2} - \frac{1}{2} \int_M \varphi^2 \, \langle \nabla(w^{-2}), \nabla |\nabla v|^{p-2} \rangle \\ &\quad = -\frac{1}{2} \int_M \varphi^2 \, \Big(|\nabla v|^{p-2} \, \Delta(w^{-2}) + \langle \nabla(w^{-2}), \nabla(|\nabla v|^{p-2}) \rangle \Big). \end{aligned} \tag{3.5}$$

Using

$$\operatorname{div}\big(|\nabla v|^{p-2} \nabla v\big) = -\lambda_1 v^{p-1}$$

or

$$\langle \nabla |\nabla v|^{p-2}, \nabla v \rangle + |\nabla v|^{p-2} \, \Delta v = -\lambda_1 \, v^{p-1},$$

we obtain

$$\begin{aligned} &|\nabla v|^{p-2} \, \Delta(v^{2-p}) + \langle \nabla(v^{2-p}), \nabla(|\nabla v|^{p-2}) \rangle \\ &\quad = |\nabla v|^{p-2} \, (2-p) \, v^{1-p} \, \Delta v + |\nabla v|^{p-2} \, (2-p)(1-p) \, v^{-p} \, |\nabla v|^2 \\ &\qquad + (2-p) \, v^{1-p} \, \langle \nabla v, \nabla |\nabla v|^{p-2} \rangle \\ &\quad = (2-p) \, v^{1-p} \, (-\lambda_1 \, v^{p-1}) + (2-p)(1-p) \, \frac{|\nabla v|^p}{v^p} \\ &\quad = \lambda_1 \, (p-2) + (p-2)(p-1) \, \frac{|\nabla v|^p}{v^p}. \end{aligned} \tag{3.6}$$

Plugging (3.6) into (3.5), we conclude

$$\begin{aligned} &\frac{1}{2} \int_M \langle \nabla\varphi^2, \nabla w^{-2} \rangle \, |\nabla v|^{p-2} \\ &\quad = -\frac{1}{2} \int_M \varphi^2 \, \Big(\lambda_1 \, (p-2) + (p-2)(p-1) \, \frac{|\nabla v|^p}{v^p} \Big). \end{aligned} \tag{3.7}$$

Putting (3.7) into (3.4) and collecting terms, we arrive at

$$\int_M \left(\frac{p}{2} \, \lambda_1 \, \varphi^2 + \frac{p(p-2)}{4} \, \varphi^2 \, \frac{|\nabla v|^p}{v^p} \right) \leq \int_M |\nabla\varphi|^2 \, \frac{|\nabla v|^{p-2}}{v^{p-2}}. \tag{3.8}$$

By Corollary 2.3, $\frac{|\nabla v|}{v} \leq \frac{n-1}{p}$ as $\lambda_1 = (\frac{n-1}{p})^p$. Therefore,

$$\frac{(n-1)^2}{2p} \int_M \varphi^2 \leq \int_M |\nabla \varphi|^2.$$

This shows that $\lambda_1(\Delta) \geq \frac{(n-1)^2}{2p}$. Now by the result in [7], if $\frac{(n-1)^2}{2p} \geq n-2$ or $p \leq \frac{(n-1)^2}{2(n-2)}$, then either M has only one infinite volume end or $M^n = \mathbb{R} \times N^{n-1}$ with N being compact and $ds_M^2 = dt^2 + \cosh^2(t)\, ds_N^2$. $\square$

From (3.8) and Theorem 2.2, one can easily conclude the following.

Corollary 3.3. Let (M, g) be a complete noncompact manifold with its Ricci curvature bounded below by a constant. If $\lambda_1(\mathcal{L}) > 0$ for some $p > 2$, then $\lambda_1(\Delta) > 0$ as well.

4. Decay estimate

In this section, we prove a decay estimate for subsolutions to the p-Laplace equation in the spirit of Agmon [1].

Theorem 4.1. Let M be a complete Riemannian manifold and u a positive function satisfying

$$\mathcal{L}(u) \geq -\mu\, u^{p-1}$$

on $M \backslash B(R_0)$, where $B(R_0)$ is a geodesic ball in M and μ a constant strictly smaller than the principal eigenvalue λ of $\mathcal{L}$ on $M \backslash B(R_0)$. If u satisfies the growth condition

$$\int_{B(R)} u^p \exp\left(-\frac{\lambda - \mu}{12\lambda^{\frac{p-1}{p}}}\, r\right) = o(R^p),$$

then

$$\int_{B(R+1)\backslash B(R)} u^p \leq C \exp\left(-\frac{\lambda - \mu}{12\lambda^{\frac{p-1}{p}}}\, R\right)$$

for some constant $C > 0$.

Proof. For a compactly supported smooth function φ on $M \setminus B(R_0)$, the variational characterization for λ implies

$$\begin{aligned}\lambda \int_M (\varphi\, u)^p &\leq \int_M |\nabla(\varphi u)|^p \\ &\leq \left(\frac{3p}{\epsilon}\right)^{p-1} \int_M |\nabla\varphi|^p\, u^p + (1+\epsilon) \int_M \varphi^p\, |\nabla u|^p \end{aligned} \tag{4.1}$$

for any $0 < \epsilon < 1$. On the other hand, we have

$$\begin{aligned}-\mu \int_M \varphi^p\, u^p &\leq \int_M \varphi^p\, u\, \mathrm{div}(|\nabla u|^{p-2}\nabla u) \\ &= -\int_M \langle \nabla(\varphi^p u), \nabla u\rangle\, |\nabla u|^{p-2} \\ &= -\int_M \Big(\varphi^p\, |\nabla u|^p + p\, \varphi^{p-1}\, \langle \nabla\varphi, \nabla u\rangle\, u\, |\nabla u|^{p-2}\Big).\end{aligned}$$

Thus,

$$\begin{aligned}\int_M \varphi^p |\nabla u|^p &\leq p \int_M \varphi^{p-1}\, |\nabla\varphi|\, |\nabla u|^{p-1}\, u + \mu \int_M \varphi^p\, u^p \\ &\leq \frac{p^{p-1}}{\epsilon^{p-1}} \int_M |\nabla\varphi|^p\, u^p + \epsilon \int_M \varphi^p\, |\nabla u|^p + \mu \int_M \varphi^p\, u^p.\end{aligned}$$

So we get

$$(1-\epsilon) \int_M \varphi^p\, |\nabla u|^p \leq \frac{p^{p-1}}{\epsilon^{p-1}} \int_M |\nabla\varphi|^p\, u^p + \mu \int_M \varphi^p\, u^p. \tag{4.2}$$

Putting (4.2) into (4.1) and optimizing over ϵ, we conclude

$$(\lambda - \mu)^p \int_M (\varphi\, u)^p \leq 6\, (12\lambda\, p)^{p-1} \int_M |\nabla\varphi|^p\, u^p. \tag{4.3}$$

Let us now choose $\varphi = \phi\, e^h$, where

$$\phi = \begin{cases} r(x) - R_0, & B(R_0+1)\backslash B(R_0), \\ 1, & B(R), \\ \dfrac{r(x) - R}{R}, & B(2R)\backslash B(R), \\ 0, & M\backslash B(2R) \end{cases}$$

and

$$h = \begin{cases} \delta\, r(x), & r \leq \dfrac{K}{2\,\delta}, \\ K - \delta\, r(x), & r \geq \dfrac{K}{2\,\delta} \end{cases}$$

for some fixed K, where $\delta = \frac{\lambda-\mu}{12p\,\lambda^{\frac{p-1}{p}}}$. Substituting into (4.3), we obtain

$$\begin{aligned}(\lambda-\mu)^p & \int_M (\phi e^h u)^p \\ & \leq 6\,(12\lambda\, p)^{p-1} \int_M |\nabla(\phi\, e^h)|^p\, u^p \\ & \leq 6\,(12\lambda\, p)^{p-1} \int_M \left((3p)^{p-1}\, |\nabla\phi|^p\, e^{p\,h}\, u^p + 2\,\phi^p\, u^p\, \delta^p\, e^{p\,h}\right).\end{aligned}$$

Noting that $12\,(12\lambda\, p)^{p-1}\,\delta^p = \frac{(\lambda-\mu)^p}{p} < (\lambda-\mu)^p$, we have

$$\begin{aligned}(\lambda-\mu)^p & \left(1-\frac{1}{p}\right) \int_{B\left(\frac{K}{2\delta}\right)\setminus B(R_0)} \phi^p\, e^{\delta\, p\, r}\, u^p \\ & \leq c(p) \int_{B(R_0+1)\setminus B(R_0)} e^{\delta pr}\, u^p + \frac{c(p,K)}{R^p} \int_{B(2R)\setminus B(R)} e^{-\delta pr}\, u^p.\end{aligned}$$

Using the growth assumption on u, one sees the last term goes to 0 as $R \to \infty$. Therefore, by first letting $R \to \infty$ and then $K \to \infty$, we arrive at

$$\int_{M\setminus B(R_0)} e^{\delta pr}\, u^p < \infty.$$

The theorem is proved. $\square$

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