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Hermitian harmonic maps and non-degenerate curvatures

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In this paper, we study the existence of various harmonic maps from Hermitian manifolds to Kähler, Hermitian and Riemannian manifolds, respectively. Using refined Bochner formulas on Hermitian (possibly non-Kähler) manifolds, we derive new rigidity results on Hermitian harmonic maps from compact Hermitian manifolds to Riemannian manifolds, and we also obtain the complex analyticity of pluri-harmonic maps from compact complex manifolds to compact Kähler manifolds (and Riemannian manifolds) with non-degenerate curvatures, which are analogous to several fundamental results in [14, 26, 28].

1. Introduction

In the seminal work of Siu [28], he proved that

Theorem 1.1 ([28, Siu]). *Let $f : (M, h) \to (N, g)$ be a harmonic map between compact Kähler manifolds. If (N, g) has strongly negative curvature and* $\mathrm{rank}_{\mathbb{R}} df \geq 4$*, then f is holomorphic or anti-holomorphic.*

There is a natural question, whether one can obtain similar results when (M, h) is Hermitian but non-Kähler. The main difficulty arises from the torsion of non-Kähler metrics when applying Bochner formulas (or Siu's $\partial\overline{\partial}$-trick) on Hermitian manifolds. On the other hand, it is well known that if the domain manifold (M, h) is non-Kähler, there are various different harmonic maps and they are mutually different (see Section 3 for more details). In particular, holomorphic maps or anti-holomorphic maps are not necessarily harmonic (with respect to the background Riemannian metrics). The first result along this line was proved by Jost and Yau in their fundamental work [14], where they used "Hermitian harmonic map".

Theorem 1.2 ([14, Jost–Yau]). *Let (N,g) be a compact Kähler manifold, and (M,h) a compact Hermitian manifold with $\partial\overline{\partial}\omega_h^{m-2}=0$, where m is the complex dimension of M. Let $f:(M,h)\to(N,g)$ be a Hermitian harmonic map. Then f is holomorphic or anti-holomorphic if (N,g) has strongly negative curvature (in the sense of Siu) and* $\mathrm{rank}_{\mathbb{R}} df \geq 4$.

In the proof of Theorem 1.2, the condition $\partial\overline{\partial}\omega_h^{m-2}=0$ plays the key role. Now a Hermitian manifold (M,h) with $\partial\overline{\partial}\omega_h^{m-2}=0$ is called *astheno-Kähler*.

In this paper, we study various harmonic maps from general Hermitian manifolds and also investigate the complex analyticity of Hermitian harmonic maps and pluri-harmonic maps. Let $f:(M,h)\to(N,g)$ be a smooth map between two compact manifolds. If (M,h) is Hermitian, we can consider the critical points of the partial energies

$$E''(f)=\int_M |\overline{\partial} f|^2\frac{\omega_h^m}{m!},\quad E'(f)=\int_M |\partial f|^2\frac{\omega_h^m}{m!}.$$

If the target manifold (N,g) is a Kähler manifold (resp. Riemannian manifold), the Euler–Lagrange equations of the partial energies $E''(f)$ and $E'(f)$ are $\overline{\partial}_E^*\overline{\partial} f=0$ and $\partial_E^*\partial f=0$, respectively, where E is the pullback vector bundle $f^*(T^{1,0}N)$ (resp. $E=f^*(TN)$). They are called $\overline{\partial}$-harmonic and ∂-harmonic maps, respectively. In general, $\overline{\partial}$-harmonic maps are not necessarily ∂-harmonic and vice versa. In [14], Jost and Yau considered a reduced harmonic map equation

$$-h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}+\Gamma_{jk}^i\frac{\partial f^j}{\partial\overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)=0.$$

Now it is called *Hermitian harmonic map* (or *pseudo-harmonic map*). The Hermitian harmonic map has generalized divergence free structures

$$\left(\overline{\partial}_E-2\sqrt{-1}\partial^*\omega_h\right)^*\left(\overline{\partial} f\right)=0\quad\text{or}\quad\left(\partial_E+2\sqrt{-1}\overline{\partial}^*\omega_h\right)^*(\partial f)=0.$$

The classical harmonic maps, $\overline{\partial}$-harmonic maps, ∂-harmonic maps and Hermitian harmonic maps coincide if the domain manifold (M,h) is Kähler.

In Section 3, we clarify and summarize the definitions of various harmonic maps from Hermitian manifolds to Kähler manifolds, to Hermitian manifolds and to Riemannain manifolds, respectively. Their relations are also discussed.

Using methods developed in [5] and [14], we show in Section 5 that $\overline{\partial}$-harmonic maps and ∂-harmonic maps always exist if the target manifold (N, g) has non-positive Riemannian sectional curvature.

In Section 6, we study Hermitian harmonic maps from Hermitian manifolds to Riemannian manifolds. At first, we obtain the following generalization of a fundamental result of Sampson [26], which is also an analog to Theorem 1.2:

Theorem 1.3. *Let (M, h) be a compact Hermitian manifold with $\partial\overline{\partial}\omega_h^{m-2} = 0$ and (N, g) a Riemannian manifold. Let $f : (M, h) \to (N, g)$ be a Hermitian harmonic map, then* $\text{rank}_{\mathbb{R}} df \leq 2$ *if (N, g) has strongly Hermitian-negative curvature. In particular, if* $\dim_{\mathbb{C}} M > 1$, *there is no Hermitian harmonic immersion of M into Riemannian manifolds of constant negative curvature.*

Here, the strongly Hermitian-negative curvatures on Riemannian manifolds (see Definition 4.2) are analogous to Siu's strongly negative curvatures on Kähler manifolds. For example, Riemannian manifolds with negative constant curvatures have strongly Hermitian-negative curvatures. On the other hand, the condition $\partial\overline{\partial}\omega_h^{m-2} = 0$ can be satisfied on a large class of Hermitian non-Kähler manifolds ([9, 21]), for example, Calabi–Eckmann manifolds $S^{2p+1} \times \mathbb{S}^{2q+1}$ with $p + q + 1 = m$.

In Section 7, we consider the complex analyticity of pluri-harmonic maps from compact *complex manifolds* to compact Kähler manifolds and Riemannian manifolds, respectively. The following results are also analogous to Theorems 1.1 and 1.2.

Theorem 1.4. *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a compact Kähler manifold (N, g). Then it is holomorphic or anti-holomorphic if (N, g) has non-degenerate curvature and* $\text{rank}_{\mathbb{R}} df \geq 4$.

Here, "non-degenerate curvature" (see Definition 4.1) is a generalization of Siu's "strongly positive curvature". For example, both manifolds with strongly positive curvatures and manifolds with strongly negative curvatures have non-degenerate curvatures. Hence, in particular,

Corollary 1.5. *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a compact Kähler manifold (N, g). Then it is holomorphic or anti-holomorphic if (N, g) has strongly negative curvature and* $\text{rank}_{\mathbb{R}} df \geq 4$.

We can see from the proof of Theorem 7.1 that Corollary 1.5 also holds if the target manifold N is the compact quotient of a bounded symmetric domain and f is a submersion. The key ingredients in the proofs are some new observations on refined Bochner formulas on the vector bundle $E = f^*(T^{1,0}N)$ on the Hermitian (possibly non-Kähler) manifold M.

As similar as Theorem 1.4, we obtain

Theorem 1.6. *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a Riemannian manifold (N, g). If the Riemannian curvature R^g of (N, g) is Hermitian non-degenerate at some point p, then* $\mathrm{rank}_{\mathbb{R}} df(p) \leq 2$.

As examples, we show

Corollary 1.7. (1) *Any pluri-harmonic map from the Calabi–Eckmann manifold $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ to the real space form $N(c)$ is constant if $p + q \geq 1$.*

(2) *Any pluri-harmonic map from $\mathbb{CP}^n$ to the real space form $N(c)$ is constant if $n \geq 2$.*

2. Connections on vector bundles

2.1. Connections on vector bundles

Let E be a Hermitian complex vector bundle or a Riemannian real vector bundle over a compact Hermitian manifold (M, h) and ∇^E be a metric connection on E. There is a natural decomposition $\nabla^E = \nabla'^E + \nabla''^E$ where

$$\nabla'^E : \Gamma(M, E) \to \Omega^{1,0}(M, E) \quad \text{and} \quad \nabla''^E : \Gamma(M, E) \to \Omega^{0,1}(M, E). \tag{2.1}$$

Moreover, ∇'^E and ∇''^E induce two differential operators. The first one is $\partial_E : \Omega^{p,q}(M, E) \to \Omega^{p+1,q}(M, E)$ defined by

$$\partial_E(\varphi \otimes s) = (\partial\varphi) \otimes s + (-1)^{p+q}\varphi \wedge \nabla'^E s \tag{2.2}$$

for any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$. The operator $\overline{\partial}_E : \Omega^{p,q}(M, E) \to \Omega^{p+1,q}(M, E)$ is defined similarly. For any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$,

$$\left(\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E\right)(\varphi \otimes s) = \varphi \wedge \left(\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E\right)s. \tag{2.3}$$

The operator $\partial_E\overline{\partial}_E + \overline{\partial}_E\partial_E$ is represented by the $(1,1)$-type curvature tensor $R^E \in \Gamma(M, \Lambda^{1,1}T^*M \otimes E^* \otimes E)$. For any $\varphi, \psi \in \Omega^{\bullet,\bullet}(M, E)$, there is a *sesquilinear pairing*

$$\{\varphi, \psi\} = \varphi^\alpha \wedge \overline{\psi^\beta} \langle e_\alpha, e_\beta \rangle \tag{2.4}$$

if $\varphi = \varphi^\alpha e_\alpha$ and $\psi = \psi^\beta e_\beta$ in the local frames $\{e_\alpha\}$ on E. By the metric compatible property of ∇^E,

$$\begin{cases} \partial\{\varphi, \psi\} = \{\partial_E \varphi, \psi\} + (-1)^{p+q}\{\varphi, \overline{\partial}_E \psi\} \\ \overline{\partial}\{\varphi, \psi\} = \{\overline{\partial}_E \varphi, \psi\} + (-1)^{p+q}\{\varphi, \partial_E \psi\} \end{cases} \tag{2.5}$$

if $\varphi \in \Omega^{p,q}(M, E)$. Let ω be the fundamental $(1,1)$-form of the Hermitian metric h, i.e.,

$$\omega = \frac{\sqrt{-1}}{2} h_{i\overline{j}} dz^i \wedge d\overline{z}^j. \tag{2.6}$$

On the Hermitian manifold (M, h, ω), the norm on $\Omega^{p,q}(M, E)$ is defined by

$$(\varphi, \psi) = \int_M \{\varphi, *\psi\} = \int_M \left(\varphi^\alpha \wedge *\overline{\psi^\beta}\right) \langle e_\alpha, e_\beta \rangle \tag{2.7}$$

for $\varphi, \psi \in \Omega^{p,q}(M, E)$. The adjoint operators of $\partial, \overline{\partial}, \partial_E$ and $\overline{\partial}_E$ are denoted by $\partial^*, \overline{\partial}^*, \partial_E^*$ and $\overline{\partial}_E^*$, respectively. We shall use the following computational lemmas frequently in the sequel and the proofs of them can be found in [19].

Lemma 2.1. *We have the following formula:*

$$\begin{cases} \overline{\partial}_E^*(\varphi \otimes s) = (\overline{\partial}^* \varphi) \otimes s - h^{i\overline{j}} \left(I_{\overline{j}} \varphi\right) \wedge \nabla_i^E s \\ \partial_E^*(\varphi \otimes s) = (\partial^* \varphi) \otimes s - h^{i\overline{j}} \left(I_i \varphi\right) \wedge \nabla_{\overline{j}}^E s \end{cases} \tag{2.8}$$

for any $\varphi \in \Omega^{p,q}(M)$ and $s \in \Gamma(M, E)$. We use the compact notations

$$I_i = I_{\frac{\partial}{\partial z^i}}, \quad I_{\overline{j}} = I_{\frac{\partial}{\partial \overline{z}^j}}, \quad \nabla_i^E = \nabla^E_{\frac{\partial}{\partial z^i}}, \quad \nabla_{\overline{j}}^E = \nabla^E_{\frac{\partial}{\partial \overline{z}^j}},$$

where I_X the contraction operator by the (local) vector field X.

Lemma 2.2. *Let E be a Riemannian real vector bundle or a Hermitian complex vector bundle over a compact Hermitian manifold (M, h, ω). If ∇ is a metric connection on E and τ is the operator of type $(1,0)$ defined by $\tau = [\Lambda, 2\partial\omega]$ on $\Omega^{\bullet,\bullet}(M, E)$, then we have*

(1) $[\overline{\partial}_E^*, L] = \sqrt{-1}(\partial_E + \tau)$, $[\partial_E^*, L] = -\sqrt{-1}(\overline{\partial}_E + \overline{\tau})$,

(2) $[\Lambda, \partial_E] = \sqrt{-1}(\overline{\partial}_E^* + \overline{\tau}^*)$, $[\Lambda, \overline{\partial}_E] = -\sqrt{-1}(\partial_E^* + \tau^*)$.

Moreover,

$$\Delta_{\overline{\partial}_E} = \Delta_{\partial_E} + \sqrt{-1}[R^E, \Lambda] + (\tau^* \partial_E + \partial_E \tau^*) - \left(\overline{\tau}^* \overline{\partial}_E + \overline{\partial}_E \overline{\tau}^*\right), \tag{2.9}$$

where Λ *is the contraction operator by* 2ω*.*

3. Harmonic map equations

In this section, we shall clarify and summarize the definitions of various harmonic maps between two of the following manifolds: Riemannian manifolds, Hermitian manifolds and Kähler manifolds. There are many excellent references on this interesting topic, and we refer the reader to [6–8] and also references therein.

3.1. Harmonic maps between Riemannian manifolds

Let (M, h) and (N, g) be two compact Riemannian manifolds. Let $f : (M, h) \to (N, g)$ be a smooth map. If $E = f^*(TN)$, then df can be regarded as an E-valued one form. There is an induced connection ∇^E on E by the Levi–Civita connection on TN. In the local coordinates $\{x^\alpha\}_{\alpha=1}^m$, $\{y^i\}_{i=1}^n$ on M and N, respectively, the local frames of E are denoted by $e_i = f^*\left(\frac{\partial}{\partial y^i}\right)$ and

$$\nabla^E e_i = f^*\left(\nabla \frac{\partial}{\partial y^i}\right) = \Gamma_{ij}^k(f) \frac{\partial f^j}{\partial x^\alpha} dx^\alpha \otimes e_k.$$

The connection ∇^E induces a differential operator $d_E : \Omega^p(M, E) \to \Omega^{p+1}(M, E)$ given by $d_E(\varphi \otimes s) = (d\varphi) \otimes s + (-1)^p \varphi \wedge \nabla^E s$ for any $\varphi \in \Omega^p(M)$ and $s \in \Gamma(M, E)$. As a classical result, the Euler–Lagrange equation of the energy

$$E(f) = \int_M |df|^2 dv_M \tag{3.1}$$

is $d_E^* df = 0$, i.e.,

$$h^{\alpha\beta}\left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} - \frac{\partial f^i}{\partial x^\gamma}\Gamma_{\alpha\beta}^\gamma + \Gamma_{jk}^i \frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\right) \otimes e_i = 0. \tag{3.2}$$

On the other hand, df is also a section of the vector bundle $F := T^*M \otimes f^*(TN)$. Let ∇^F be the induced connection on F by the Levi–Civita connections of M and N. f is said to be *totally geodesic* if $\nabla^F df = 0 \in \Gamma(M, T^*M \otimes T^*M \otimes f^*(TN))$, i.e.,

$$\left(\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} - \frac{\partial f^i}{\partial x^\gamma}\Gamma^\gamma_{\alpha\beta} + \Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\right) dx^\alpha \otimes dx^\beta \otimes e_i = 0. \tag{3.3}$$

Let $f : (M, h) \to (N, g)$ be an immersion, then f is said to be *minimal* if $Tr_h \nabla^F df = 0$. It is obvious that an immersion is minimal if and only if it is harmonic.

3.2. Harmonic maps from Hermitian manifolds to Kähler manifolds

Let (M, h) be a compact Hermitian manifold and (N, g) a compact Kähler manifold. Let $\{z^\alpha\}_{\alpha=1}^m$ and $\{w^i\}_{i=1}^n$ be the local holomorphic coordinates on M and N, respectively, where $m = \dim_{\mathbb{C}} M$ and $n = \dim_{\mathbb{C}} N$. If $f : M \to N$ is a smooth map, the pullback vector bundle $f^*(T^{1,0}N)$ is denoted by E. The local frames of E are denoted by $e_i = f^*(\frac{\partial}{\partial w^i})$, $i = 1, \dots, n$. The metric connection on E induced by the complexified Levi–Civita connection (i.e., Chern connection) of $T^{1,0}M$ is denoted by ∇^E. There are three E-valued 1-forms, namely,

$$\overline{\partial} f = \frac{\partial f^i}{\partial \overline{z}^\beta} d\overline{z}^\beta \otimes e_i, \quad \partial f = \frac{\partial f^i}{\partial z^\alpha} dz^\alpha \otimes e_i, \quad df = \overline{\partial} f + \partial f. \tag{3.4}$$

The $\overline{\partial}$-energy of a smooth map $f : (M, h) \to (N, g)$ is defined by

$$E''(f) = \int_M |\overline{\partial} f|^2 \frac{\omega_h^m}{m!} \tag{3.5}$$

and similarly we can define the ∂-energy $E'(f)$ and the total energy $E(f)$ by

$$E'(f) = \int_M |\partial f|^2 \frac{\omega_h^m}{m!}, \quad E(f) = \int_M |df|^2 \frac{\omega_h^m}{m!}. \tag{3.6}$$

It is obvious that the quantity $E(f)$ coincides with the energy defined by the background Riemannian metrics. The following result is well known and we include a proof here for the sake of completeness.

Lemma 3.1. *The Euler–Lagrange equation of* $\overline{\partial}$*-energy* $E''(f)$ *is* $\overline{\partial}_E^*\overline{\partial} f = 0$*; and the Euler–Lagrange equations of* $E'(f)$ *and* $E(f)$ *are* $\partial_E^*\partial f = 0$ *and* $\overline{\partial}_E^*\overline{\partial} f + \partial_E^*\partial f = 0$*, respectively.*

Proof. Let $F : M \times \mathbb{C} \to N$ be a smooth function such that

$$F(z,0) = f(z), \quad \frac{\partial F}{\partial t}\Big|_{t=0} = v, \quad \frac{\partial F}{\partial \overline{t}}\Big|_{t=0} = \mu.$$

Now we set $K = F^*\left(T^{1,0}N\right)$. The connection on K induced by the Chern connection on $T^{1,0}N$ is denoted by ∇^K. Its $(1,0)$ and $(0,1)$ components are denoted by ∂_K and $\overline{\partial}_K$, respectively. The induced bases $F^*(\frac{\partial}{\partial w^i})$ of K are denoted by $\widehat{e}_i$, $i = 1, \ldots, n$. Since the connection ∇^K is compatible with the Hermitian metric on K, we obtain

$$\frac{\partial}{\partial t} E''(f_t) = \int_M \left\langle \left(\partial_K \overline{\partial} f_t\right)\left(\frac{\partial}{\partial t}\right), \overline{\partial} f_t \right\rangle \frac{\omega_h^m}{m!} + \int_M \left\langle \overline{\partial} f_t, \left(\overline{\partial}_K \overline{\partial} f_t\right)\left(\frac{\partial}{\partial \overline{t}}\right)\right\rangle \frac{\omega_h^m}{m!}. \tag{3.7}$$

On the other hand,

$$\partial_K \overline{\partial} f_t = \partial_K\left(\overline{\partial} f_t^i \otimes \widehat{e}_i\right) = \partial_t\left(\overline{\partial} f_t^i\right) \otimes \widehat{e}_i - \overline{\partial} f_t^i \wedge \nabla'^K \widehat{e}_i, \tag{3.8}$$

where ∂_t is ∂-operator on the manifold $M \times \mathbb{C}$. By definition,

$$\nabla^K \widehat{e}_i = F^*\left(\nabla \frac{\partial}{\partial w^i}\right) = F^*\left(\Gamma_{ji}^k dz^j \otimes \frac{\partial}{\partial w^k}\right) = \Gamma_{ji}^k dF^j \otimes \widehat{e}_k.$$

Therefore, $\left(\nabla'^K \widehat{e}_i\right)\left(\frac{\partial}{\partial t}\right) = \Gamma_{ji}^k \frac{\partial F^j}{\partial t} \otimes \widehat{e}_k$ and

$$\left(\partial_K \overline{\partial} f_t\right)\left(\frac{\partial}{\partial t}\right) = \left(\overline{\partial}\left(\frac{\partial F^i}{\partial t}\right) + \overline{\partial} F^k \frac{\partial F^j}{\partial t} \Gamma_{jk}^i\right) \otimes \widehat{e}_i. \tag{3.9}$$

When $t = 0$,

$$\left(\partial_K \overline{\partial} f_t\right)\left(\frac{\partial}{\partial t}\right)\Big|_{t=0} = \left(\overline{\partial} v^i + \overline{\partial} f^k \cdot v^j \cdot \Gamma_{jk}^i\right) \otimes e_i = \overline{\partial}_E v, \tag{3.10}$$

since (N, g) is Kähler, i.e., $\Gamma_{jk}^i = \Gamma_{kj}^i$. Similarly, we get

$$\left(\overline{\partial}_K \overline{\partial} f_t\right)\left(\frac{\partial}{\partial \overline{t}}\right)\Big|_{t=0} = \overline{\partial}_E \mu. \tag{3.11}$$

Finally, we obtain

$$\frac{\partial}{\partial t}E''(f_t)\big|_{t=0} = \int_M \langle \overline{\partial}_E v, \overline{\partial} f\rangle \frac{\omega_h^m}{m!} + \int_M \langle \overline{\partial} f, \overline{\partial}_E \mu\rangle \frac{\omega_h^m}{m!}. \tag{3.12}$$

Hence the Euler–Lagrange equation of $E''(f)$ is $\overline{\partial}_E^*\overline{\partial} f = 0$. Similarly, we can get the Euler–Lagrange equations of $E'(f)$ and $E(f)$. □

For any smooth function Φ on the compact Hermitian manifold (M,h), we know

$$\begin{cases} \Delta_{\overline{\partial}}\Phi = \overline{\partial}^*\overline{\partial}\Phi = -h^{\alpha\overline{\beta}}\left(\dfrac{\partial^2\Phi}{\partial z^\alpha \partial \overline{z}^\beta} - 2\Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}}\dfrac{\partial \Phi}{\partial \overline{z}^\gamma}\right), \\ \Delta_{\partial}\Phi = \partial^*\partial\Phi = -h^{\alpha\overline{\beta}}\left(\dfrac{\partial^2\Phi}{\partial z^\alpha \partial \overline{z}^\beta} - 2\Gamma_{\overline{\beta}\alpha}^{\gamma}\dfrac{\partial \Phi}{\partial z^\gamma}\right), \\ \Delta_d\Phi = d^*d\Phi = \Delta_{\overline{\partial}}\Phi + \Delta_\partial\Phi, \end{cases} \tag{3.13}$$

where

$$\Gamma_{\alpha\overline{\beta}}^{\gamma} = \frac{1}{2}h^{\gamma\overline{\delta}}\left(\frac{\partial h_{\alpha\overline{\delta}}}{\partial \overline{z}^\beta} - \frac{\partial h_{\alpha\overline{\beta}}}{\partial \overline{z}^\delta}\right) = \Gamma_{\overline{\beta}\alpha}^{\gamma} = \overline{\Gamma_{\beta\overline{\alpha}}^{\overline{\gamma}}}, \tag{3.14}$$
$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}h^{\gamma\overline{\delta}}\left(\frac{\partial h_{\alpha\overline{\delta}}}{\partial z^\beta} + \frac{\partial h_{\beta\overline{\delta}}}{\partial z^\alpha}\right).$$

Therefore, by Lemma 2.1

$$\begin{aligned} \overline{\partial}_E^*\overline{\partial} f &= \left(\overline{\partial}^*\overline{\partial} f^i - h^{\alpha\overline{\beta}}\Gamma_{jk}^i \frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i \\ &= -h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial \overline{z}^\beta} - 2\Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}}\frac{\partial f^i}{\partial \overline{z}^\gamma} + \Gamma_{jk}^i\frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \partial_E^*\partial f &= \left(\partial^*\partial f^i - h^{\alpha\overline{\beta}}\Gamma_{jk}^i \frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i \\ &= -h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial \overline{z}^\beta} - 2\Gamma_{\overline{\beta}\alpha}^{\gamma}\frac{\partial f^i}{\partial z^\gamma} + \Gamma_{jk}^i\frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i. \end{aligned} \tag{3.16}$$

For more details about the computations, see e.g., [19].

We clarify and summarize the definitions of various harmonic maps in the following.

Definition 3.2. Let (M,h) be a compact Hermitian manifold and (N,g) a Kähler manifold. Let $f:(M,h)\to(N,g)$ be a smooth map and $E=f^*(T^{1,0}N)$.

(1) f is called *$\overline{\partial}$-harmonic* if it is a critical point of $\overline{\partial}$-energy, i.e., $\overline{\partial}_E^*\overline{\partial}f=0$;

(2) f is called *∂-harmonic* if it is a critical point of ∂-energy, i.e., $\partial_E^*\partial f=0$;

(3) f is called *harmonic* if it is a critical point of d-energy, i.e., $\overline{\partial}_E^*\overline{\partial}+\partial_E^*\partial f=0$, i.e.,

$$h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}-\Gamma_{\alpha\overline{\beta}}^{\overline{\gamma}}\frac{\partial f^i}{\partial\overline{z}^\gamma}-\Gamma_{\overline{\beta}\alpha}^{\gamma}\frac{\partial f^i}{\partial z^\gamma}+\Gamma_{jk}^i\frac{\partial f^j}{\partial\overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i=0; \tag{3.17}$$

(4) f is called *Hermitian harmonic* if it satisfies

$$-h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}+\Gamma_{jk}^i\frac{\partial f^j}{\partial\overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i=0; \tag{3.18}$$

(5) f is called *pluri-harmonic* if it satisfies $\partial_E\overline{\partial}f=0$, i.e.,

$$\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}+\Gamma_{jk}^i\frac{\partial f^j}{\partial\overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}=0 \tag{3.19}$$

for any α,β and i.

Remark 3.3. (1) The Hermitian harmonic equation (3.18) was firstly introduced by Jost and Yau in [14]. For more generalizations, see [16, 17] and also references therein;

(2) The harmonic map equation (3.17) is the same as classical harmonic equation (3.2) by using the background Riemmanian metrics;

(3) Pluri-harmonic maps are Hermitian harmonic;

(4) Pluri-harmonic maps are not necessarily ∂-harmonic or $\overline{\partial}$-harmonic;

(5) $\overline{\partial}_E\partial f=0$ and $\partial_E\overline{\partial}f=0$ are equivalent;

(6) For another type of Hermitian harmonic maps between Hermitian manifolds defined using Chern connections, we refer the reader to [34].

Lemma 3.4. *For any smooth map $f:(M,h)\to(N,g)$ from a Hermitian manifold (M,h) to a Kähler manifold (N,g), we have $\overline{\partial}_E\overline{\partial}f=0$ and $\partial_E\partial f=0$.*

Proof. It is easy to see that

$$\begin{aligned}\overline{\partial}_E\overline{\partial} f &= \overline{\partial}_E\left(\frac{\partial f^i}{\partial \overline{z}^\alpha}d\overline{z}^\alpha\otimes e_i\right)\\ &= \frac{\partial^2 f^i}{\partial\overline{z}^\beta\partial\overline{z}^\alpha}d\overline{z}^\beta\wedge d\overline{z}^\alpha\otimes e_i - \frac{\partial f^i}{\partial\overline{z}^\alpha}d\overline{z}^\alpha\wedge\Gamma^j_{ik}\frac{\partial f^k}{\partial\overline{z}^\beta}d\overline{z}^\beta\otimes e_j = 0\end{aligned}$$

since $\Gamma^j_{ik}=\Gamma^j_{ki}$ when (N,h) is Kähler. The proof of $\partial_E\partial f=0$ is similar. $\square$

Lemma 3.5. *Let $f:(M,h)\to(N,g)$ be a smooth map from a compact Hermitian manifold (M,h) to a compact Kähler manifold (N,g). The Hermitian harmonic map equation (3.18) is equivalent to*

$$(3.20)\quad \left(\overline{\partial}_E-2\sqrt{-1}\partial^*\omega_h\right)^*\left(\overline{\partial}f\right)=0\quad \textit{or}\quad \left(\partial_E+2\sqrt{-1}\overline{\partial}^*\omega_h\right)^*(\partial f)=0.$$

Proof. On a compact Hermitian manifold (M,h) with $\omega_h=\frac{\sqrt{-1}}{2}h_{\alpha\overline{\beta}}dz^\alpha\wedge d\overline{z}^\beta$, we have ([19, Lemma A.6])

(3.21)
$$\overline{\partial}^*\omega_h=\sqrt{-1}\Gamma^{\overline{\beta}}_{\overline{\gamma\beta}}dz^\gamma\quad\text{and}\quad -2\sqrt{-1}(\overline{\partial}^*\omega_h)^*(\overline{\partial}f)=-2h^{\alpha\overline{\beta}}\Gamma^\gamma_{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^\gamma}\otimes e_i.$$

The equivalence is derived from (3.21), (3.16) and (3.15). $\square$

Definition 3.6. A compact Hermitian manifold (M,h) is call *balanced* if the fundamental form ω_h is co-closed, i.e., $d^*\omega_h=0$.

Proposition 3.7. *Let (M,h) be a compact balanced Hermitian manifold and (N,g) a Kähler manifold. The E', E'' and E-critical points coincide. Moreover, they satisfy the Hermitian harmonic equation (3.18). That is, $\overline{\partial}$-harmonic, ∂-harmonic, Hermitian harmonic and harmonic maps are the same if the domain (M,h) is a balanced manifold.*

Proof. The balanced condition $d^*\omega_h=0$ is equivalent to $\partial^*\omega_h=0$ or $\overline{\partial}^*\omega_h=0$ or $h^{\alpha\overline{\beta}}\Gamma^\gamma_{\alpha\overline{\beta}}=0$ for $\gamma=1,\ldots,m$. By formulas (3.16) and (3.15), we obtain $\overline{\partial}^*_E\overline{\partial}f=\partial^*_E\partial f$. The second statement follows by Lemma 3.5. $\square$

Proposition 3.8. *Let (M,h) be a compact Hermitian manifold and (N,g) a Kähler manifold. If $f:(M,h)\to(N,g)$ is totally geodesic and (M,h) is Kähler, then f is pluri-harmonic.*

Proof. Considering the complexified connection ∇^F on the vector bundle $F = T^*M \otimes f^*(TN)$, we have

$$\begin{aligned}
\nabla^F \overline{\partial} f &= \nabla^F \left(\frac{\partial f^i}{\partial \overline{z}^\beta} d\overline{z}^\beta \otimes e_i \right) \\
&= \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} - \frac{\partial f^i}{\partial \overline{z}^\gamma} \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} + \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \Gamma^i_{jk} \right) dz^\alpha \otimes d\overline{z}^\beta \otimes e_i \\
&\quad + \left(\frac{\partial^2 f^i}{\partial \overline{z}^\gamma \partial \overline{z}^\delta} - \frac{\partial f^i}{\partial \overline{z}^\alpha} \Gamma^{\overline{\alpha}}_{\overline{\gamma}\overline{\delta}} + \frac{\partial f^j}{\partial \overline{z}^\gamma} \frac{\partial f^k}{\partial \overline{z}^\delta} \Gamma^i_{jk} \right) d\overline{z}^\gamma \otimes d\overline{z}^\delta \otimes e_i \\
&\quad - \frac{\partial f^i}{\partial \overline{z}^\alpha} \Gamma^{\overline{\alpha}}_{\overline{\beta}\lambda} d\overline{z}^\beta \otimes dz^\lambda \otimes e_i.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nabla^F \partial f &= \nabla^F \left(\frac{\partial f^i}{\partial z^\alpha} dz^\alpha \otimes e_i \right) \\
&= \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} - \frac{\partial f^i}{\partial z^\gamma} \Gamma^{\gamma}_{\overline{\beta}\alpha} + \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \Gamma^i_{jk} \right) d\overline{z}^\beta \otimes dz^\alpha \otimes e_i \\
&\quad + \left(\frac{\partial^2 f^i}{\partial z^\gamma \partial z^\delta} - \frac{\partial f^i}{\partial z^\alpha} \Gamma^{\alpha}_{\gamma\delta} + \frac{\partial f^j}{\partial z^\gamma} \frac{\partial f^k}{\partial z^\delta} \Gamma^i_{jk} \right) dz^\gamma \otimes dz^\delta \otimes e_i \\
&\quad - \frac{\partial f^i}{\partial z^\alpha} \Gamma^{\alpha}_{\lambda\overline{\beta}} dz^\lambda \otimes d\overline{z}^\beta \otimes e_i.
\end{aligned}$$

That is,

$$\begin{aligned}
\nabla^F df &= \nabla^F \overline{\partial} f + \nabla^F \partial f \\
&= \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} - \frac{\partial f^i}{\partial \overline{z}^\gamma} \Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}} - \frac{\partial f^i}{\partial z^\lambda} \Gamma^{\lambda}_{\alpha\overline{\beta}} + \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \Gamma^i_{jk} \right) dz^\alpha \otimes d\overline{z}^\beta \otimes e_i \\
&\quad + \left(\frac{\partial^2 f^i}{\partial \overline{z}^\gamma \partial \overline{z}^\delta} - \frac{\partial f^i}{\partial \overline{z}^\alpha} \Gamma^{\overline{\alpha}}_{\overline{\gamma}\overline{\delta}} + \frac{\partial f^j}{\partial \overline{z}^\gamma} \frac{\partial f^k}{\partial \overline{z}^\delta} \Gamma^i_{jk} \right) d\overline{z}^\gamma \otimes d\overline{z}^\delta \otimes e_i \\
&\quad + \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} - \frac{\partial f^i}{\partial z^\gamma} \Gamma^{\gamma}_{\overline{\beta}\alpha} - \frac{\partial f^i}{\partial \overline{z}^\delta} \Gamma^{\overline{\delta}}_{\overline{\beta}\alpha} + \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \Gamma^i_{jk} \right) d\overline{z}^\beta \otimes dz^\alpha \otimes e_i \\
&\quad + \left(\frac{\partial^2 f^i}{\partial z^\gamma \partial z^\delta} - \frac{\partial f^i}{\partial z^\alpha} \Gamma^{\alpha}_{\gamma\delta} + \frac{\partial f^j}{\partial z^\gamma} \frac{\partial f^k}{\partial z^\delta} \Gamma^i_{jk} \right) dz^\gamma \otimes dz^\delta \otimes e_i.
\end{aligned}$$

If f is totally geodesic and (M, h) is Kähler, then f is pluri-harmonic by degree reasons. □

Remark 3.9. It is easy to see that pluri-harmonic maps are not necessarily totally geodesic.

Lemma 3.10. *Let f be a pluri-harmonic map from a complex manifold M to a Kähler manifold (N,g). Then the real $(1,1)$ forms*

$$\omega_0 = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial \overline{f}^j}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \partial f^i \wedge \overline{\partial f}^j, \tag{3.22}$$

and

$$\omega_1 = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \frac{\partial \overline{f}^j}{\partial z^\alpha} \frac{\partial f^i}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \partial \overline{f}^i \wedge \overline{\partial} f^j \tag{3.23}$$

are all d-closed, i.e.,

$$d\omega_0 = \partial\omega_0 = \overline{\partial}\omega_0 = 0, \quad \textit{and} \quad d\omega_1 = \partial\omega_1 = \overline{\partial}\omega_1 = 0.$$

Proof. By definition, we see

$$\begin{aligned}
\partial\omega_0 &= -\frac{\sqrt{-1}}{2}\partial f^i \wedge \partial\left(g_{i\bar{j}}\overline{\partial f}^j\right)\\
&= -\frac{\sqrt{-1}}{2}\partial f^i \wedge \left(\frac{\partial g_{i\bar{j}}}{\partial z^k}\cdot \partial f^k \wedge \overline{\partial f}^j + \frac{\partial g_{i\bar{j}}}{\partial \overline{z}^\ell}\cdot \partial \overline{f}^\ell \wedge \overline{\partial f}^j + g_{i\bar{j}}\partial\overline{\partial f}^j\right)\\
(g \text{ is Kähler}) &= -\frac{\sqrt{-1}}{2}\partial f^i \wedge \left(\frac{\partial g_{i\bar{j}}}{\partial \overline{z}^\ell}\cdot \partial \overline{f}^\ell \wedge \overline{\partial f}^j + g_{i\bar{j}}\partial\overline{\partial f}^j\right)\\
&= -\frac{\sqrt{-1}}{2}\partial f^i \wedge g_{i\bar{s}}\left(\partial\overline{\partial f}^s + g^{p\bar{s}}\frac{\partial g_{p\bar{q}}}{\partial \overline{z}^\ell}\cdot \partial \overline{f}^\ell \wedge \overline{\partial f}^q\right)\\
&= -\frac{\sqrt{-1}}{2}\partial f^i \wedge g_{i\bar{s}}\left(\partial\overline{\partial f}^s + \overline{\Gamma^s_{q\ell}}\cdot \partial \overline{f}^\ell \wedge \overline{\partial f}^q\right)\\
&= 0,
\end{aligned}$$

where the last step follows from the definition equation (3.19) of pluri-harmonic maps. Hence, we obtain $d\omega_0 = 0$. The proof of $d\omega_1 = 0$ is similar. □

3.3. Harmonic maps between Hermitian manifolds

Let (M,h) and (N,g) be two compact Hermitian manifolds. Using the same notation as in the previous subsection, we can define $\overline{\partial}$-harmonic (resp. ∂-harmonic, harmonic) map $f:(M,h)\to(N,g)$ by using the critical point of the Euler–Lagrange equation of $E''(f)$ (resp. $E'(f)$, $E(f)$). In this case, the harmonic equations have the same second-order parts, but the torsion parts are different. For example, the $\overline{\partial}$-harmonic equation is

$$\left(\Delta_{\overline{\partial}} f^i + T^i(f)\right) \otimes e_i = 0, \tag{3.24}$$

where $T(f)$ is a quadratic function in df and the coefficients are the Christoffel symbols of (N, g). One can see it clearly from the proof of Lemma 3.1.

3.4. Harmonic maps from Hermitian manifolds to Riemannian manifolds

Let (M, h) be a compact Hermitian manifold, (N, g) a Riemannian manifold and $E = f^*(TN)$ with the induced Levi–Civita connection. As similar as in the Kähler target manifold case, we can define the $\overline{\partial}$-energy of $f : (M, h) \to (N, g)$

$$E''(f) = \int_M |\overline{\partial} f|^2 \frac{\omega_h^m}{m!} = \int_M g_{ij} h^{\alpha\overline{\beta}} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^j}{\partial \overline{z}^\beta} \frac{\omega_h^m}{m!}. \tag{3.25}$$

It is easy to see that the Euler–Lagrange equation of (3.25) is

$$\overline{\partial}_E^* \overline{\partial} f = \Delta_{\overline{\partial}} f^i - h^{\alpha\overline{\beta}} \Gamma_{jk}^i \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} = 0. \tag{3.26}$$

Similarly, we can define $E'(f)$ and get its Euler–Lagrange equation

$$\partial_E^* \partial f = \Delta_{\partial} f^i - h^{\alpha\overline{\beta}} \Gamma_{jk}^i \frac{\partial f^j}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} = 0. \tag{3.27}$$

The Euler–Lagrange equation of $E(f)$ is $\overline{\partial}_E^* \overline{\partial} f + \partial_E^* \partial f = 0$.

Definition 3.11. Let (M, h) be a compact Hermitian manifold and (N, g) a Riemannian manifold. Let $f : (M, h) \to (N, g)$ be a smooth map and $E = f^*(TN)$.

(1) f is called *$\overline{\partial}$-harmonic* if it is a critical point of $\overline{\partial}$-energy, i.e., $\overline{\partial}_E^* \overline{\partial} f = 0$;

(2) f is called *∂-harmonic* if it is a critical point of ∂-energy, i.e., $\partial_E^* \partial f = 0$;

(3) f is called *harmonic* if it is a critical point of d-energy, i.e., $\overline{\partial}_E^* \overline{\partial} + \partial_E^* \partial f = 0$;

(4) f is called *Hermitian harmonic* if it satisfies

$$-h^{\alpha\overline{\beta}} \left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} + \Gamma_{jk}^i \frac{\partial f^j}{\partial \overline{z}^\beta} \frac{\partial f^k}{\partial z^\alpha} \right) \otimes e_i = 0; \tag{3.28}$$

(5) f is called *pluri-harmonic* if it satisfies $\partial_E\overline{\partial} f = 0$, i.e.,

$$\left(\frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta} + \Gamma^i_{jk}\frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right) dz^\alpha \wedge d\overline{z}^\beta \otimes e_i = 0. \tag{3.29}$$

As similar as Proposition 3.7, we have

Corollary 3.12. *Let $f : (M, h) \to (N, g)$ be a smooth map from a compact Hermitian manifold (M, h) to a Riemannian manifold (N, g). If (M, h) is a balanced Hermitian manifold, i.e., $d^*\omega_h = 0$, then ∂-harmonic map, $\overline{\partial}$-harmonic map, Hermitian harmonic map and harmonic map coincide.*

4. Manifolds with non-degenerate curvatures

4.1. Curvatures of Kähler manifolds

Let (N, g) be a Kähler manifold. In the local holomorphic coordinates $(w^1, \dots, w^n)$ of N, the curvature tensor components are

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial w^i \partial \overline{w^j}} + g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial w^i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{w^j}}. \tag{4.1}$$

In [28], Siu introduced the following definition: the curvature tensor $R_{i\bar{j}k\bar{\ell}}$ is said to be *strongly negative* (resp. *strongly positive*) if

$$\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}}(A^i\overline{B}^j - C^i\overline{D}^j)\overline{(A^\ell\overline{B}^k - C^\ell\overline{D}^k)} < 0 \quad (\text{resp.} >0) \tag{4.2}$$

for any nonzero $n \times n$ complex matrix $(A^i\overline{B}^j - C^i\overline{D}^j)_{i,j}$.

Definition 4.1. Let (N, g) be a Kähler manifold. The curvature tensor $R_{i\bar{j}k\bar{\ell}}$ is called *non-degenerate* if it satisfies the condition that

$$\sum_{i,j,k,\ell} R_{i\bar{j}k\bar{\ell}}(A^i\overline{B}^j - C^i\overline{D}^j)\overline{(A^\ell\overline{B}^k - C^\ell\overline{D}^k)} = 0 \tag{4.3}$$

if and only if $A^i\overline{B}^j - C^i\overline{D}^j = 0$ for any i, j.

It is easy to see that both manifolds with strongly positive curvatures and manifolds with strongly negative curvatures have non-degenerate curvatures.

4.2. Curvatures of Riemannian manifolds

Let (M, g) be a Riemannian manifold and ∇ the Levi–Civita connection. The curvature tensor R is defined by

$$R(X, Y, Z, W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W).$$

To illustrate our computation rules on Riemannain manifolds, for example, the Riemannian curvature tensor components of S^n induced by the canonical metric of $\mathbb{R}^{n+1}$ are $R_{ijk\ell} = g_{i\ell} g_{jk} - g_{ik} g_{j\ell}$. The Ricci curvature tensor components are $R_{jk} = g^{i\ell} R_{ijk\ell} = (n-1) g_{jk}$.

As similar as Siu's definition, Sampson [26] proposed the following definition:

Definition 4.2. Let (M, g) be a compact Riemannian manifold.

(1) The curvature tensor R of (M, g) is said to be *Hermitian-positive* (resp. *Hermitian-negative*) if

$$R_{ijk\ell} A^{i\overline{\ell}} A^{j\overline{k}} \geq 0 \,(\text{resp. } \leq 0) \tag{4.4}$$

for any Hermitian semi-positive matrix $A = (A^{i\overline{\ell}})$. R is called *strongly Hermitian-positive* (resp. *strongly Hermitian-negative*) if R is Hermitian-positive (resp. Hermitian-negative) and the equality in (4.4) holds only for Hermitian semi-positive matrix A with complex rank ≤ 1.

(2) R is said to be *Hermitian non-degenerate at some point* $p \in M$ if

$$R_{ijk\ell}(p) A^{i\overline{\ell}} A^{j\overline{k}} = 0 \tag{4.5}$$

for some Hermitian semi-positive matrix $A = (A^{i\overline{j}})$ implies A has rank ≤ 1. R is said to be *Hermitian non-degenerate* if it is Hermitian non-degenerate everywhere.

Note that any rank one Hermitian matrix can be written as $A^{i\overline{j}} = a^i b^{\overline{j}}$ and so for any curvature tensor $R_{ijk\ell}$, one has

$$R_{ijk\ell} A^{i\overline{\ell}} A^{j\overline{k}} = 0.$$

On the other hand, it is easy to see that both manifolds with strongly Hermitian-positive curvatures and manifolds with strongly Hermitian-negative curvatures are Hermitian non-degenerate.

Lemma 4.3 ([26]). *If (M, g) has positive (resp. negative) constant sectional curvature, then the curvature tensor is strongly Hermitian positive (resp. negative). In particular, it is Hermitian non-degenerate.*

Proof. Let $R_{ijk\ell} = \kappa(g_{i\ell}g_{jk} - g_{ik}g_{j\ell})$. Then

$$R_{ijk\ell}A^{i\bar{\ell}}A^{j\bar{k}} = \kappa\left((TrA)^2 - Tr(A^2)\right). \tag{4.6}$$

The results follow by this identity easily. □

Remark 4.4. In [18], we give a complete list on the curvature relations of a Kähler manifold (M, g):

(1) semi-dual-Nakano-negative;

(2) non-positive Riemannian curvature operator;

(3) strongly non-positive in the sense of siu;

(4) non-positive complex sectional curvature;

(5) non-positive Riemannian sectional curvature;

(6) non-positive holomorphic bisectional curvature; and

(7) non-positive isotropic curvature.

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (5) \Longrightarrow (6)$$
$$(1) \Longrightarrow (3) \Longleftrightarrow (4) \Longrightarrow (7).$$

So far, it is not clear to the authors whether one of them can imply (Sampson's) Hermitian negativity. However, it is easy to see that the Poincaré disks and projective spaces have Hermitian-negative and Hermitian-positive curvatures, respectively. It is hopeful that semi-dual-Nakano-negative curvatures can imply Hermitian-negative curvatures (in the sense of Sampson). We will go back to this topic later.

5. Existence of various harmonic maps

In their pioneering work [5], Eells–Sampson have proposed the heat flow method to study the existence of harmonic maps. In this section, we will consider a similar setting. Let $f : (M, h) \to (N, g)$ be a continuous map from a compact Hermitian manifold to a compact Riemannian manifold. In the paper [14] of Jost and Yau, they considered the heat flow for the Hermitian harmonic equation, i.e.,

$$\begin{cases} h^{\alpha\overline{\beta}}\left(\dfrac{\partial^2 f^i(z,t)}{\partial z^\alpha \partial \overline{z}^\beta} + \Gamma^i_{jk}\dfrac{\partial f^j(z,t)}{\partial \overline{z}^\beta}\dfrac{\partial f^k(z,t)}{\partial z^\alpha}\right) - \dfrac{\partial f^i(z,t)}{\partial t} = 0, \\ f_0 = f, \end{cases} \tag{5.1}$$

where Γ^i_{jk} are Christoffel symbols of the Riemannian manifold (N, g).

Lemma 5.1 (Jost–Yau). *If (N, g) has non-positive Riemannian sectional curvature, then a solution of (5.1) exists for all $t \geq 0$.*

Similarly, we can consider the following parabolic system for the $\overline{\partial}$-energy of a smooth map f from a compact Hermitian manifolds (M, h) to a Riemannian manifold (N, g),

$$\begin{cases} \dfrac{df_t}{dt} = -\overline{\partial}^*_E \overline{\partial} f_t \\ f_0 = f. \end{cases} \tag{5.2}$$

Locally, the parabolic equation (5.2) is

$$h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i(z,t)}{\partial z^\alpha \partial \overline{z}^\beta} - 2\Gamma^{\overline{\gamma}}_{\alpha\overline{\beta}}\frac{\partial f^i(z,t)}{\partial \overline{z}^\gamma} + \Gamma^i_{jk}\frac{\partial f^j(z,t)}{\partial \overline{z}^\beta}\frac{\partial f^k(z,t)}{\partial z^\alpha}\right) - \frac{\partial f^i(z,t)}{\partial t} = 0. \tag{5.3}$$

The difference between (5.3) and (5.1) are the first-order derivative terms of f. By the theory of parabolic partial differential equations, if (N, g) has non-positive sectional curvature, the solution of (5.3) exists for all $t \geq 0$ following the adapted methods in [5, 14]. Let

$$e(f) = h^{\alpha\overline{\beta}} g_{ij} \frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial \overline{z}^\beta}$$

be the energy density. By differentiating equation (5.3), we obtain

$$\left(\Delta_c - \frac{\partial}{\partial t}\right) e(f) \geq \frac{1}{2}|\nabla^2 f|^2 - Ce(f) \tag{5.4}$$

if (N,g) has non-positive sectional curvature where $C = C(M,h)$ is a positive constant only depending on (M,h), and Δ_c is the canonical Laplacian $\Delta_c = h^{\alpha\overline{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \overline{z}^\beta}$. The extra first-order terms in f are absorbed in $|\nabla^2 f|$ using the Schwarz inequality. As analogous to the existence results of Eells–Sampson ([5], harmonic maps) and Jost–Yau ([14], Hermitian harmonic maps), we obtain

Theorem 5.2. *Let (M,h) be a compact Hermitian manifold and (N,g) a compact Riemannian manifold of negative Riemannian sectional curvature. Let $\varphi : M \to N$ be continuous, and suppose that φ is not homotopic to a map onto a closed geodesic of N. Then there exists a $\overline{\partial}$-harmonic (resp. ∂-harmonic) map which is homotopic to φ.*

Theorem 5.3. *Let (M,h) be a compact Hermitian manifold (N,g) a compact Riemannian manifold of negative Riemannian sectional curvature. Let $\varphi : M \to N$ be a continuous map with $e(\varphi^*(TN)) \neq 0$ where e is the Euler class. Then there exists a $\overline{\partial}$-harmonic (resp. ∂-harmonic) map which is homotopic to φ.*

As a special case, we have

Corollary 5.4. *Let (M,h) be a compact Hermitian manifold and (N,g) be a compact Kähler manifold of strongly negative curvature. Let $\varphi : M \to N$ be a continuous map and suppose that φ is not homotopic to a map onto a closed geodesic of N. Then there exists a $\overline{\partial}$-harmonic (resp. ∂-harmonic) map which is homotopic to φ.*

Proof. It follows from the fact that if a Kähler manifold has strongly negative curvature, then the background Riemannain metric has negative sectional curvature. □

Finally, we need to point out that, along the same line, one can easily obtain similar existence results for various harmonic maps into a Hermitian target manifold (N,g) if the background Riemannian metric on N has non-positive Riemannian sectional curvature. For more details about the existence and uniqueness results on various harmonic maps in the Hermitian context, we refer the reader to [4–7, 10, 11, 14, 15, 20, 23, 33] and also references therein.

6. The complex analyticity of harmonic maps

6.1. Harmonic maps from Hermitian manifolds to Kähler manifolds

Let $f : (M, h) \to (N, g)$ be a smooth map from a Hermitian manifold (M, h) to a Kähler manifold. Let $E = f^*(T^{1,0}N)$. In the local coordinates $\{z^\alpha\}$ on M, and $\{w^i\}$ on N, one can get

$$
\begin{aligned}
(6.1)\quad Q := {} & \sqrt{-1}\left\langle [R^E, \Lambda]\overline{\partial} f, \overline{\partial} f \right\rangle \\
= {} & -\frac{1}{2}\sum_{\alpha,\gamma} R_{i\overline{j}k\overline{\ell}} \left(\frac{\partial f^i}{\partial z^\alpha}\frac{\partial \overline{f}^j}{\partial z^\gamma} - \frac{\partial f^i}{\partial z^\gamma}\frac{\partial \overline{f}^j}{\partial z^\alpha}\right) \overline{\left(\frac{\partial f^\ell}{\partial z^\alpha}\frac{\partial \overline{f}^k}{\partial z^\gamma} - \frac{\partial f^\ell}{\partial z^\gamma}\frac{\partial \overline{f}^k}{\partial z^\alpha}\right)}
\end{aligned}
$$

in the local normal coordinates $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$ centered at a point $p \in M$ where R^E is the $(1,1)$ component of the curvature tensor of E and $R_{i\overline{j}k\overline{\ell}}$ are components of the curvature tensor of (N, g). If Q is zero, (N, g) has *non-degenerate curvature*, N is compact and $\mathrm{rank}_{\mathbb{R}} df \geq 4$, one can show $\partial f = 0$ or $\overline{\partial} f = 0$ (cf. [28, Siu]).

Now let us recall Siu's $\partial\overline{\partial}$ trick [28, 29] in the Hermitian setting (cf. [14]). Let $f : (M, h) \to (N, g)$ be a smooth map between Hermitian manifolds and $E = f^*(T^{1,0}N)$.

Lemma 6.1. *We have the following formula:*

$$
\partial\overline{\partial}\{\overline{\partial} f, \overline{\partial} f\} = -\{\partial_E \overline{\partial} f, \partial_E \overline{\partial} f\} + \{\overline{\partial} f, R^E \overline{\partial} f\}. \tag{6.2}
$$

Lemma 6.2. *Let E be any Hermitian vector bundle over a Hermitian manifold (M, ω), and φ a smooth E-valued $(1,1)$-form on M. One has*

$$
-\{\varphi, \varphi\}\frac{\omega^{m-2}}{(m-2)!} = 4\left(|\varphi|^2 - |Tr_\omega \varphi|^2\right)\frac{\omega^m}{m!}. \tag{6.3}
$$

Proof. Without loss of generality, we can assume that E is a trivial bundle, and $h_{i\overline{j}} = \delta_{i\overline{j}}$ at a fixed point $p \in M$, then for $\varphi = \varphi_{p\overline{q}} dz^p \wedge d\overline{z}^q$.

$$
\begin{aligned}
& \left(\frac{\sqrt{-1}}{2}\right)^2 \{\varphi, \varphi\}\frac{\omega^{m-2}}{(m-2)!} \\
& = \left(\frac{\sqrt{-1}}{2}\right)^2 \left(\sum_{p,q,s,t} \varphi_{p\overline{q}} dz^p \wedge d\overline{z}^q \cdot \overline{\varphi}_{s\overline{t}} d\overline{z}^s \wedge dz^t\right)\frac{\omega^{m-2}}{(m-2)!}
\end{aligned}
$$

$$= \left(\sum_{1\le p<q\le m} 2|\varphi_{p\bar{q}}|^2 - 2\sum_{1\le s<t\le m} \varphi_{s\bar{s}}\overline{\varphi_{t\bar{t}}}\right)\frac{\omega^m}{m!}$$
$$= \left(\sum_{p,q} |\varphi_{p\bar{q}}|^2 - \sum_{s,t} \varphi_{s\bar{s}}\overline{\varphi_{t\bar{t}}}\right)\frac{\omega^m}{m!}$$
$$= \left(|\varphi|^2 - |Tr_\omega\varphi|^2\right)\frac{\omega^m}{m!}.$$

□

Remark 6.3. The right-hand side of (6.3) is not positive in general. When φ is primitive, i.e., $Tr_\omega\varphi = 0$, (6.3) is the Riemann–Hodge bilinear relation for primitive $(1,1)$ forms (e.g., [13, Corollary 1.2.36] or [32, Proposition 6.29]).

Lemma 6.4. *We have the following formula for any smooth map f from a Hermitian manifold (M,h) to a Kähler manifold (N,g).*

$$\{\overline{\partial} f, R^E\overline{\partial} f\}\frac{\omega_h^{m-2}}{(m-2)!} = 4Q\cdot\frac{\omega_h^m}{m!}, \tag{6.4}$$

where Q is defined in (6.1).

Lemma 6.5. *Let f be any* ***smooth map*** *from a compact Hermitian manifold (M,h) to a Kähler manifold (N,g). We have the following identity:*

$$\int_M \partial\overline{\partial}\{\overline{\partial} f, \overline{\partial} f\}\frac{\omega_h^{m-2}}{(m-2)!} = 4\int_M \left(|\partial_E\overline{\partial} f|^2 - |Tr_\omega\partial_E\overline{\partial} f|^2\right)\frac{\omega_h^m}{m!} + \int_M 4Q\cdot\frac{\omega_h^m}{m!}. \tag{6.5}$$

Proof. It follows by formula (6.2)–(6.4). □

Now one can get the following generalization of Siu's result ([28]):

Corollary 6.6 ([14, Jost–Yau]). *Let (N,g) be a compact Kähler manifold, and (M,h) a compact Hermitian manifold with $\partial\overline{\partial}\omega_h^{m-2} = 0$ where $m = \dim_{\mathbb{C}} M$. Let $f : (M,h) \to (N,g)$ be a Hermitian harmonic map. Then f is holomorphic or anti-holomorphic if (N,g) has strongly negative curvature and* $\mathrm{rank}_{\mathbb{R}} df \ge 4$.

6.2. Harmonic maps from Hermitian manifolds to Riemannian manifolds

In this subsection, we shall apply similar ideas in Section 6.1 to harmonic maps from Hermitian manifolds to Riemannian manifolds. Let $f : (M, h) \to (N, g)$ be a smooth map from a compact Hermitian manifold (M, h) to a Riemannian manifold (N, g).

Lemma 6.7. *The* $(1,1)$*-part of the curvature tensor of* $E = f^*(TN)$ *is*

$$R_{1,1}^{f^*(TN)} = 2R_{ijk}^{\ell} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^j}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta \otimes e^k \otimes e_\ell. \tag{6.6}$$

Proof. Since the curvature tensor of the real vector bundle TN is

$$R^{TN} = R_{ijk}^{\ell} dx^i \wedge dx^j \otimes \left(dx^k \otimes \frac{\partial}{\partial x^\ell} \right) \in \Gamma(N, \Lambda^2 T^* N \otimes End(TN)), \tag{6.7}$$

we get the full curvature tensor of the pullback vector bundle $E = f^*(TN)$,

$$f^* \left(R^{TN} \right) = R_{ijk}^{\ell} df^i \wedge df^j \otimes e^k \otimes e_\ell \in \Gamma(N, \Lambda^2 T^* M \otimes End(E)). \tag{6.8}$$

The $(1,1)$ part of it is

$$\begin{aligned} R_{1,1}^{f^*(TN)} &= R_{ijk}^{\ell} \left(\frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^j}{\partial \overline{z}^\beta} - \frac{\partial f^i}{\partial \overline{z}^\beta} \frac{\partial f^j}{\partial z^\alpha} \right) dz^\alpha \wedge d\overline{z}^\beta \otimes e^k \otimes e_\ell \\ &= 2R_{ijk}^{\ell} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^j}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta \otimes e^k \otimes e_\ell, \end{aligned}$$

since $R_{ijk}^{\ell} = -R_{jik}^{\ell}$. □

Lemma 6.8. *We have*

$$\left\langle \sqrt{-1}[R_{1,1}^{f^*(TN)}, \Lambda] \partial f, \partial f \right\rangle = 2h^{\alpha\overline{\delta}} h^{\gamma\overline{\beta}} R_{ijk\ell} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \frac{\partial f^j}{\partial z^\gamma} \frac{\partial f^\ell}{\partial \overline{z}^\delta} \tag{6.9}$$

and

(6.10)
$$Q_0 := \left\langle \sqrt{-1}[R_{1,1}^{f^*(TN)}, \Lambda] \overline{\partial} f, \overline{\partial} f \right\rangle = -2h^{\alpha\overline{\delta}} h^{\gamma\overline{\beta}} R_{ijk\ell} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial f^k}{\partial \overline{z}^\beta} \frac{\partial f^j}{\partial z^\gamma} \frac{\partial f^\ell}{\partial \overline{z}^\delta}.$$

Proof. It is easy to see that the identity (6.10) is the complex conjugate of (6.9). By Lemma 6.7,

$$\begin{aligned}\sqrt{-1}[R_{1,1}^{f^*(TN)},\Lambda]\partial f &= -\sqrt{-1}\Lambda R^E\partial f \\ &= 2h^{\alpha\overline{\beta}}\left(-R_{ijk}^{\ell}+R_{kji}^{\ell}\right)\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\gamma}dz^\gamma\otimes e_\ell \\ &= 2h^{\alpha\overline{\beta}}R_{kij}^{\ell}\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial \overline{z}^\beta}\frac{\partial f^k}{\partial z^\gamma}dz^\gamma\otimes e_\ell,\end{aligned}$$

where the last step follows by Bianchi identity. Therefore

$$\left\langle\sqrt{-1}[R_{1,1}^{f^*(TN)},\Lambda]\partial f,\partial f\right\rangle = -2R_{ijk\ell}\left(h^{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^\ell}{\partial \overline{z}^\beta}\right)\left(h^{\gamma\overline{\delta}}\frac{\partial f^j}{\partial z^\gamma}\frac{\partial f^k}{\partial \overline{z}^\delta}\right).$$

□

Theorem 6.9 ([26, Sampson]). *Let $f:(M,h)\to(N,g)$ be a harmonic map from a compact Kähler manifold (M,h) to a Riemannian manifold (N,g). Then $\mathrm{rank}_{\mathbb{R}}df\le 2$ if (N,g) has strongly Hermitian-negative curvature.*

Proof. By formula (2.9) for the vector bundle $E=f^*(TN)$ when (M,h) is Kähler,

$$\Delta_{\overline{\partial}_E}\overline{\partial}f = \Delta_{\partial_E}\overline{\partial}f+\sqrt{-1}[R^E,\Lambda]\overline{\partial}f. \tag{6.11}$$

If f is harmonic, i.e., $\overline{\partial}_E^*\overline{\partial}f=0$, we obtain, $\Delta_{\overline{\partial}_E}\overline{\partial}f=0$. That is,

$$0=\|\partial_E\overline{\partial}f\|^2+\int_M Q_0\frac{\omega_h^m}{m!}. \tag{6.12}$$

If (N,g) has strongly Hermitian-negative curvature, i.e., $Q_0\ge 0$ pointwisely, then $Q_0=0$. Hence we get $\mathrm{rank}_{\mathbb{R}}df\le 2$. □

Now we go back to work on the Hermitian (domain) manifold (M,h). As similar as Lemma 6.5, we obtain

Lemma 6.10. *Let $f : (M,h) \to (N,g)$ be a smooth map from a compact Hermitian manifold (M,h) to a Riemannian manifold (N,g). Then*

$$\int_M \partial\overline{\partial}\{\overline{\partial} f, \overline{\partial} f\} \frac{\omega_h^{m-2}}{(m-2)!} = 4\int_M \left(|\partial_E \overline{\partial} f|^2 - |Tr_\omega \partial_E \overline{\partial} f|^2\right) \frac{\omega_h^m}{m!} + \int_M 4Q_0 \cdot \frac{\omega_h^m}{m!}. \tag{6.13}$$

Theorem 6.11. *Let (M,h) be a compact Hermitian manifold with $\partial\overline{\partial}\omega_h^{m-2} = 0$ and (N,g) a Riemannian manifold. Let $f : (M,h) \to (N,g)$ be a Hermitian harmonic map, then $\mathrm{rank}_{\mathbb{R}} df \leq 2$ if (N,g) has strongly Hermitian-negative curvature. In particular, if $\dim_{\mathbb{C}} M > 1$, there is no Hermitian harmonic immersion of M into Riemannian manifolds of constant negative curvature.*

Proof. If f is Hermitian harmonic, i.e., $Tr_\omega \partial_E \overline{\partial} f = 0$, by formula (6.13),

$$\int_M \partial\overline{\partial}\{\overline{\partial} f, \overline{\partial} f\} \frac{\omega_h^{m-2}}{(m-2)!} = 4\int_M |\partial_E \overline{\partial} f|^2 \frac{\omega_h^m}{m!} + \int_M 4Q_0 \cdot \frac{\omega_h^m}{m!}. \tag{6.14}$$

From integration by parts, we obtain

$$4\int_M |\partial_E \overline{\partial} f|^2 \frac{\omega_h^m}{m!} + \int_M 4Q_0 \cdot \frac{\omega_h^m}{m!} = 0.$$

If (N,g) has strongly Hermitian-negative curvature, then $Q_0 = 0$ and so $\mathrm{rank}_{\mathbb{R}} df \leq 2$. □

Corollary 6.12. *Let $M = \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1} (p+q \geq 1)$ be the Calabi–Eckmann manifold. Then there is no Hermitian harmonic immersion of M into manifolds of constant negative curvature.*

Proof. By a result of Matsuo [21], every Calabi–Eckmann manifold has a Hermitian metric ω with $\partial\overline{\partial}\omega^{n-2} = 0$. □

Remark 6.13. (1) If M is Kähler, a Hermitian harmonic immersion is also minimal.

(2) By Proposition 3.7, if the manifold (M,h) is balanced, then Hermitian harmonic map is harmonic. However, if ω_h is balanced (i.e., $d^*\omega_h = 0$) and also $\partial\overline{\partial}\omega_h^{m-2} = 0$, then ω_h must be Kähler ([22]).

7. Rigidity of pluri-harmonic maps

7.1. Pluri-harmonic maps from complex manifolds to Kähler manifolds

Let $f : M \to (N, g)$ be a pluri-harmonic map from the compact complex manifold M to the compact Kähler manifold (N, g). From the definition formula (3.19), the pluri-harmonicity of f is independent of the background metric on the domain manifold M and so we do not impose a metric there.

When the domain manifold M is Kähler, there is a number of results on the complex analyticity and rigidity of the pluri-harmonic f, mainly due to Ohinta, Udagawa and also Burns–Burstall–Barttolomeis (e.g., [1, 24, 25, 30, 31] and references therein). The common feature in their results is that they need even more properties of the Kähler manifold M, for example, $c_1(M) > 0$, or $b_2(M) = 1$.

Now we present our main results in this section.

Theorem 7.1. *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a compact Kähler manifold (N, g). Then it is holomorphic or anti-holomorphic if (N, g) has non-degenerate curvature and* $\mathrm{rank}_{\mathbb{R}} df \geq 4$. *In particular, when (N, g) has strongly negative curvature (in the sense of Siu) and* $\mathrm{rank}_{\mathbb{R}} df \geq 4$, *then f is holomorphic or anti-holomorphic.*

From the proof, we can see that this theorem also holds if the target manifold N is a compact quotient of a bounded symmetric domain and f is a submersion.

Proof. We fix an arbitrary Hermitian metric h on M. Let $E = f^*(T^{1,0}N)$ and R^E be the $(1,1)$-part of the curvature tensor of E. If f is pluri-harmonic, i.e., $\partial_E \overline{\partial} f = 0$, then by the Bochner formula (2.9), the equation

$$\begin{aligned}\Delta_{\overline{\partial}_E} \overline{\partial} f = \Delta_{\partial_E} \overline{\partial} f + \sqrt{-1}[R^E, \Lambda](\overline{\partial} f) + (\tau^* \partial_E + \partial_E \tau^*)(\overline{\partial} f) \\ - (\overline{\tau}^* \overline{\partial}_E + \overline{\partial}_E \overline{\tau}^*)(\overline{\partial} f)\end{aligned}$$

is equivalent to

$$\overline{\partial}_E \overline{\partial}_E^* \overline{\partial} f = \sqrt{-1}[R^E, \Lambda](\overline{\partial} f) - (\overline{\partial}_E \overline{\tau}^*)(\overline{\partial} f). \tag{7.1}$$

On the other hand, by Lemma 2.2, we have the relation $[\Lambda, \partial_E] = \sqrt{-1}(\overline{\partial}_E^* + \overline{\tau}^*)$, and so

$$\overline{\partial}_E(\overline{\partial}_E^* + \overline{\tau}^*)\overline{\partial} f = -\sqrt{-1}\overline{\partial}_E \Lambda \partial_E \overline{\partial} f = 0 \tag{7.2}$$

since f is pluri-harmonic. By (7.1), we get the identity $Q = \langle\sqrt{-1}[R^E, \Lambda](\overline{\partial} f), \overline{\partial} f\rangle = 0$. (Note that we get $Q = 0$ without using the curvature property of (N, g), which is different from the proofs in [28, Siu] and [14, Jost–Yau]!) By formula (6.1) and the assumption that (N, g) has non-degenerate curvature, we obtain $\partial f^i \wedge \partial \overline{f}^j = 0$ for any i and j. If $\text{rank}_{\mathbb{R}}(df) \geq 4$, by Siu's argument ([28]), f is holomorphic or anti-holomorphic. □

Using Theorem 7.1, we can generalize a number of results in [25, 30, 31] to complex (domain) manifolds.

Proposition 7.2. *Let M be an arbitrary m-dimensional $(m \geq 2)$ compact complex manifold, (N, g) a compact Kähler manifold and $f : M \to (N, g)$ a pluri-harmonic map. Suppose M has one of the following properties:*

(1) $\dim_{\mathbb{C}} H^2(M) = 0$; *or*

(2) $\dim_{\mathbb{C}} H^{1,1}(M) = 0$; *or*

(3) $\dim_{\mathbb{C}} H^2(M) = 1$ *and* $H^2(M)$ *has a generator* $[\eta]$ *with* $\int_M \eta^m \neq 0$; *or*

(4) $\dim_{\mathbb{C}} H^{1,1}(M) = 1$ *and* $H^{1,1}(M)$ *has a generator* $[\eta]$ *with* $\int_M \eta^m \neq 0$.

Then

(1) *f is constant if* $\text{rank}_{\mathbb{R}} df < 2m$. *In particular, if $m > n$, then f is constant.*

(2) *f is holomorphic or anti-holomorphic if (N, g) has non-degenerate curvature. Here, we have no rank restriction on df.*

Proof. If $\text{rank}_{\mathbb{R}} df < 2m$, we can consider the following real $(1, 1)$ form

$$\omega_0 = \frac{\sqrt{-1}}{2} g_{i\overline{j}} \frac{\partial f^i}{\partial z^\alpha} \frac{\partial \overline{f}^j}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta = \frac{\sqrt{-1}}{2} g_{i\overline{j}} \partial f^i \wedge \overline{\partial f}^j. \tag{7.3}$$

If f is pluri-harmonic, by Lemma 3.10, $\partial\omega_0 = \overline{\partial}\omega_0 = 0 = d\omega_0$. On the other hand, when $\text{rank}_{\mathbb{R}} df < 2m$, $\omega_0^m = 0$. If conditions (1) or (3) holds, we obtain $\omega_0 = d\gamma_0$. If conditions (2) or (4) holds, we have $\omega = \overline{\partial}\gamma_1$. In any case, by Stokes' Theorem, $\int_C \omega_0 = 0$ on any closed curve C of M. But ω_0 is a non-negative $(1, 1)$ form on M, we obtain $\omega_0 = 0$. Therefore $\partial f = 0$. Similarly, by using

$$\omega_1 = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \frac{\partial \overline{f}^j}{\partial z^\alpha} \frac{\partial f^i}{\partial \overline{z}^\beta} dz^\alpha \wedge d\overline{z}^\beta = \frac{\sqrt{-1}}{2} g_{i\bar{j}} \partial \overline{f}^i \wedge \overline{\partial} f^j$$

we know $\overline{\partial} f = 0$. Hence f is constant. In particular, if $m > n$, i.e., $\text{rank}_{\mathbb{R}} df < 2m$, f is constant.

Suppose (N, g) has non-degenerate curvature. If $\text{rank}_{\mathbb{R}} df \geq 2m \geq 4$, by Theorem 7.1, then f is holomorphic or anti-holomorphic. If $\text{rank}_{\mathbb{R}} df < 2m$, by the proof above, we see f is constant. □

Corollary 7.3. *Any pluri-harmonic map from the Calabi–Eckmann manifold $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ to the n-dimensional complex space form $N(c)$ is constant if $p + q \geq n$.*

The following result is well known (e.g., [1, 24, 25]).

Corollary 7.4. *Every pluri-harmonic map from $\mathbb{P}^m$ to $\mathbb{P}^n$ is constant if $m > n$.*

7.2. Pluri-harmonic maps from Hermitian manifolds to Riemannian manifolds

In this subsection, we shall use similar ideas as in Section 7.1 to study the rigidity of pluri-harmonic maps from Hermitian manifolds to Riemannian manifolds.

Theorem 7.5. *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold M to a Riemannian manifold (N, g). If (N, g) has non-degenerate Hermitian curvature at some point p, then* $\text{rank}_{\mathbb{R}} df(p) \leq 2$.

Proof. We fix an arbitrary Hermitian metric h on M. If f is pluri-harmonic, i.e., $\partial_E \overline{\partial} f = 0$, then by the Bochner formula (2.9), the equation

$$\begin{aligned} \Delta_{\overline{\partial}_E} \overline{\partial} f = \Delta_{\partial_E} \overline{\partial} f + \sqrt{-1}[R^E, \Lambda](\overline{\partial} f) + (\tau^* \partial_E + \partial_E \tau^*)(\overline{\partial} f) \\ - (\overline{\tau}^* \overline{\partial}_E + \overline{\partial}_E \overline{\tau}^*)(\overline{\partial} f) \end{aligned}$$

is equivalent to $\overline{\partial}_E \overline{\partial}_E^* \overline{\partial} f = \sqrt{-1}[R^E, \Lambda](\overline{\partial} f) - (\overline{\partial}_E \overline{\tau}^*)(\overline{\partial} f)$. By a similar argument as in Theorem 7.1, we obtain $Q_0 = \langle \sqrt{-1}[R^E, \Lambda](\overline{\partial} f), \overline{\partial} f \rangle = 0$.

That is,

$$R_{ijk\ell}\left(h^{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^\ell}{\partial \overline{z}^\beta}\right)\left(h^{\gamma\overline{\delta}}\frac{\partial f^j}{\partial z^\gamma}\frac{\partial f^k}{\partial \overline{z}^\delta}\right)=0. \tag{7.4}$$

If the curvature tensor R of (N,g) is non-degenerate at some point $p \in M$, then the complex rank of the matrix $\left(h^{\alpha\overline{\beta}}\frac{\partial f^i}{\partial z^\alpha}\frac{\partial f^j}{\partial \overline{z}^\beta}\right)$ is ≤ 1, i.e., $\mathrm{rank}_{\mathbb{R}} df(p) \leq 2$. □

Proposition 7.6. *Let M be an arbitrary m-dimensional ($m \geq 2$) compact complex manifold, (N,g) a Riemannian manifold and $f : M \to (N,g)$ a pluri-harmonic map. Suppose M has one of the following properties:*

(1) $\dim_{\mathbb{C}} H^2(M) = 0$*; or*

(2) $\dim_{\mathbb{C}} H^{1,1}(M) = 0$*; or*

(3) $\dim_{\mathbb{C}} H^2(M) = 1$ *and* $H^2(M)$ *has a generator* $[\eta]$ *with* $\int_M \eta^m \neq 0$*; or*

(4) $\dim_{\mathbb{C}} H^{1,1}(M) = 1$ *and* $H^{1,1}(M)$ *has a generator* $[\eta]$ *with* $\int_M \eta^m \neq 0$*;*

then

(1) f *is constant if* $\mathrm{rank}_{\mathbb{R}} df < 2m$*. In particular, if* $m > n$*, then* f *is constant.*

(2) f *is constant if* (N,g) *has non-degenerate curvature.*

Proof. Assume $\mathrm{rank}_{\mathbb{R}} df < 2m$. We can consider $\omega_0 = \frac{\sqrt{-1}}{2} g_{i\overline{j}} \partial f^i \wedge \overline{\partial} f^j$. By a similar proof as in Proposition 7.2, we obtain $\overline{\partial} f = 0$, and so f is a constant. On the other hand, if (N,g) has non-degenerate curvature, then $\mathrm{rank}_{\mathbb{R}} df \leq 2 < 2m$, hence f is constant. □

Corollary 7.7. (1) *Any pluri-harmonic map from the Calabi–Eckmann manifold* $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ *to the real space form* $N(c)$ *is constant if* $p + q \geq 1$*.*

(2) *Any pluri-harmonic map from* $\mathbb{CP}^n$ *to the real space form* $N(c)$ *is constant if* $n \geq 2$*.*

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