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Fourier multipliers on weighted L^p spaces

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The paper provides a complement to the classical results on Fourier multipliers on L^p spaces. In particular, we prove that if $q \in (1, 2)$ and a function $m : \mathbb{R} \to \mathbb{C}$ is of bounded q-variation uniformly on the dyadic intervals in $\mathbb{R}$, i.e., $m \in V_q(\mathcal{D})$, then m is a Fourier multiplier on $L^p(\mathbb{R}, w\,dx)$ for every $p \geq q$ and every weight w satisfying Muckenhoupt's $A_{p/q}$-condition. We also obtain a higher-dimensional counterpart of this result as well as of a result by E. Berkson and T.A. Gillespie including the case of the $V_q(\mathcal{D})$ spaces with $q > 2$. New weighted estimates for modified Littlewood–Paley functions are also provided.

1. Introduction and statement of results

For an interval $[a, b]$ in $\mathbb{R}$ and a number $q \in [1, \infty)$ denoted by $V_q([a, b])$ the space of all functions $m : [a, b] \to \mathbb{C}$ of bounded q-variation over $[a, b]$, i.e.,

$$\|m\|_{V_q([a,b])} := \sup_{x \in [a,b]} |m(x)| + \|m\|_{\mathrm{Var}_q([a,b])} < \infty,$$

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where $\|m\|_{\mathrm{Var}_q([a,b])} := \sup\{(\sum_{i=0}^{n-1} |m(t_{i+1}) - m(t_i)|^q)^{1/q}\}$ and the supremum is taken over all finite sequences $a =: t_0 < t_1 < \cdots < t_n := b$ ($n \in \mathbb{N}$). We write $\mathcal{D}$ for the dyadic decomposition of $\mathbb{R}$, i.e., $\mathcal{D} := \{\pm(2^k, 2^{k+1}] : k \in \mathbb{Z}\}$, and set

$$V_q(\mathcal{D}) := \left\{ m : \mathbb{R} \to \mathbb{C} : \quad \sup_{I \in \mathcal{D}} \|m_{|I}\|_{V_q(I)} < \infty \right\} \qquad (q \in [1, \infty)).$$

Moreover, let $A_p(\mathbb{R})$ ($p \in [1, \infty)$) be the class of weights on $\mathbb{R}$ which satisfy the Muckenhoupt A_p condition. Denote by $[w]_{A_p}$ the A_p-constant of $w \in A_p(\mathbb{R})$. If $w \in A_\infty(\mathbb{R}) := \cup_{p \geq 1} A_p(\mathbb{R})$ we write $M_p(\mathbb{R}, w)$ for the class of all multipliers on $L^p(\mathbb{R}, w)$ ($p > 1$), i.e.,

$$M_p(\mathbb{R}, w) := \{m \in L^\infty(\mathbb{R}) : T_m \text{ extends to a bounded operator on } L^p(\mathbb{R}, w)\}.$$

Here T_m stands for the Fourier multiplier with the symbol m, i.e., $\widehat{(T_m f)} = m\widehat{f}$ ($f \in S(\mathbb{R})$). Note that $M_p(\mathbb{R}, w)$ becomes a Banach space under the norm $\|m\|_{M_p(\mathbb{R},w)} := \|T_m\|_{\mathcal{L}(L^p(\mathbb{R},w))}$ ($m \in M_p(\mathbb{R}, w)$).

The main result of the paper is the following complement to results due to Kurtz [17], Coifman *et al.* [8] and Berkson and Gillespie [4].

Theorem A. (i) *Let $q \in (1, 2]$. Then, $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $p \geq q$ and every Muckenhoupt weight $w \in A_{p/q}(\mathbb{R})$.*
(ii) *Let $q > 2$. Then, $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $2 \leq p < (\frac{1}{2} - \frac{1}{q})^{-1}$ and every Muckenhoupt weight $w \in A_{p/2}$ with $s_w > (1 - p(\frac{1}{2} - \frac{1}{q}))^{-1}$.*

Here, for every $w \in A_\infty(\mathbb{R})$, we set $s_w := \sup\{s \geq 1 : w \in RH_s(\mathbb{R})\}$ and we write $w \in RH_s(\mathbb{R})$ if

$$\sup_{a<b} \left(\frac{1}{b-a} \int_a^b w(x)^s dx \right)^{1/s} \left(\frac{1}{b-a} \int_a^b w(x) dx \right)^{-1} < \infty.$$

Recall that, by the reverse Hölder inequality, $s_w \in (1, \infty]$ for every Muckenhoupt weight $w \in A_\infty(\mathbb{R})$.

For the convenience of the reader we repeat the relevant material from the literature, which we also use in the sequel.

Recall first that in [17] Kurtz proved the following weighted variant of the classical Marcinkiewicz multiplier theorem.

Theorem 1 ([17, Theorem 2]). *$V_1(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p(\mathbb{R})$.*

As in the unweighted case, Theorem 1 is equivalent to a weighted variant of the Littlewood–Paley decomposition theorem, which asserts that for the square function $S^{\mathcal{D}}$ corresponding to the dyadic decomposition $\mathcal{D}$ of $\mathbb{R}$, $\|S^{\mathcal{D}} f\|_{p,w} \eqsim \|f\|_{p,w}$ ($f \in L^p(\mathbb{R}, w)$) for every $p \in (1,\infty)$ and $w \in A_p(\mathbb{R})$; see [17, Theorem 1], and also [17, Theorem 3.3]. Here and subsequently, if $\mathcal{I}$ is a family of disjoint intervals in $\mathbb{R}$, we write $S^{\mathcal{I}}$ for the Littlewood–Paley square function corresponding to $\mathcal{I}$, i.e., $S^{\mathcal{I}} f := \left(\sum_{I\in\mathcal{I}} |S_I f|^2\right)^{1/2}$ ($f \in L^2(\mathbb{R})$).

Recall also that in [25], Rubio de Francia proved the following extension of the classical Littlewood–Paley decomposition theorem.

Theorem 2 ([25, Theorem 6.1]). *Let $2 < p < \infty$ and $w \in A_{p/2}(\mathbb{R})$. Then for an arbitrary family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$ the square function $S^{\mathcal{I}}$ is bounded on $L^p(\mathbb{R}, w\,dx)$.*

Applying Rubio de Francia's inequalities, i.e., Theorem 2, Coifman *et al.* [8] proved the following extension and improvement of the classical Marcinkiewicz multiplier theorem. (See Section 2 for the definition of $R_2(\mathcal{D})$.)

Theorem 3 ([8, Théorème 1 and Lemme 5]). *Let $2 \le q < \infty$. Then, $V_q(\mathcal{D}) \subset M_p(\mathbb{R})$ for every $p \in (1,\infty)$ such that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$.*

Furthermore, $R_2(\mathcal{D}) \subset M_2(\mathbb{R}, w)$ for every $w \in A_1(\mathbb{R})$.

Subsequently, a weighted variant of Theorem 3 was given by Berkson and Gillespie in [4]. According to our notation their result can be formulated as follows.

Theorem 4 ([4, Theorem 1.2]). *Suppose that $2 \le p < \infty$ and $w \in A_{p/2}(\mathbb{R})$. Then, there is a real number $s > 2$, depending only on p and $[w]_{A_{p/2}}$, such that $\frac{1}{s} > |\frac{1}{2} - \frac{1}{p}|$ and $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for all $1 \le q < s$.*

Note that the part (i) of Theorem A fills a gap which occurs in Theorem 1 and the weighted part of Theorem 3. The part (ii) identifies the constant s in Berkson–Gillespie's result, i.e., Theorem 4, as $(\frac{1}{2} - \frac{1}{s'_w p})^{-1}$, where $s'_w := \frac{s_w}{s_w - 1}$, and in general, this constant is best possible.

Except for some details, the proofs given below reproduce well-known arguments from the Littlewood–Paley theory; in particular, ideas which have been presented in [8, 17, 25, 28]. A new point of our approach is the following result on weighted estimates for modified Littlewood–Paley functions $S_q^{\mathcal{I}}(\cdot) := (\sum_{I\in\mathcal{I}} |S_I(\cdot)|^{q'})^{1/q'}$ ($q \in (1,2]$), which may be of independent interest.

Theorem B. (i) *Let $q \in (1,2)$, $p > q$, and $w \in A_{p/q}(\mathbb{R})$. Then, there exists a constant $C > 0$ such that for any family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$*

$$\|S_q^{\mathcal{I}} f\|_{p,w} \le C\|f\|_{p,w} \qquad (f \in L^p(\mathbb{R}, w\,dx)).$$

Moreover, for every $q \in (1,2)$, $p > q$ and $\mathcal{V} \subset A_{p/q}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_{p/q}} < \infty$

$$\sup\{\|S_q^{\mathcal{I}} f\|_{p,w} : w \in \mathcal{V},\ \mathcal{I} \text{ a family of disjoint intervals in } \mathbb{R},\ \|f\|_{p,w} = 1\} < \infty.$$

(ii) *For any family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$ and every Muckenhoupt weight $w \in A_1(\mathbb{R})$, the operator $S_2^{\mathcal{I}}$ maps $L^2(\mathbb{R}, w\,dx)$ into weak-$L^2(\mathbb{R}, w\,dx)$, and*

$$\sup\{\|S_2^{\mathcal{I}} f\|_{L^{2,\infty}_w} : w \in \mathcal{V},\ \mathcal{I} \text{ a family of disjoint intervals in } \mathbb{R},\ \|f\|_{L^2_w} = 1\} < \infty$$

for every $\mathcal{V} \subset A_1(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_1} < \infty$.

Moreover, if $q \in (1,2)$, then for any well-distributed family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$ and every Muckenhoupt weight $w \in A_1(\mathbb{R})$, the operator $S_q^{\mathcal{I}}$ maps $L^q(\mathbb{R}, wdx)$ into weak-$L^q(\mathbb{R}, wdx)$.

Recall that a family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$ is *well-distributed* if there exists $\lambda > 1$ such that $\sup_{x\in\mathbb{R}} \sum_{I\in\mathcal{I}} \chi_{\lambda I}(x) < \infty$, where λI denotes the interval with the same center as I and length λ times that of I.

Note that the validity of the A_1-weighted L^2-estimates for square function $S^{\mathcal{I}} = S_2^{\mathcal{I}}$ corresponding to an arbitrary family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$, i.e.,

$$\|S_2^{\mathcal{I}} f\|_{2,w} \le C_w\|f\|_{2,w} \quad (f \in L^2(\mathbb{R}, w\,dx),\ w \in A_1(\mathbb{R}))$$

is conjectured by Rubio de Francia in [25, Section 6, p. 10]; see also [12, Section 8.2, p. 187]. Theorem B(ii), in particular, provides the validity of the weak variant of Rubio de Francia's conjecture. Notice that in contrast to the square function operators $S_2^{\mathcal{I}}$, in general, operators $S_q^{\mathcal{I}}$ $(q \in [1,2))$ are not bounded on (unweighted) $L^q(\mathbb{R})$; see [9]. Moreover, in [23], Quek proved that if $\mathcal{I}$ is a well-distributed family of disjoint intervals in $\mathbb{R}$, then the operator $S_q^{\mathcal{I}}$ maps $L^q(\mathbb{R})$ into $L^{q,q'}(\mathbb{R})$ for every $q \in (1,2)$. Note that this result is in a sharp sense, i.e., $L^{q,q'}(\mathbb{R})$ cannot be replaced by $L^{q,s}(\mathbb{R})$ for any $s < q'$, see [23, Remark 3.2]. Therefore, Theorem B provides also a weighted variant of this line of researches. See also relevant results given by Kisliakov in [16].

Furthermore, as a consequence of our approach we also get a higher-dimensional analogue of Theorem A, see Theorem C in Section 4, which extends earlier results by Xu [28]; see also Lacey [18, Chapter 4]. Since the formulation of Theorem C is more involved and its proof is essentially the iteration of one-dimensional arguments, we refer the reader to Section 4 for more information.

The part (ii) of Theorem A is a quantitative improvement of [4, Theorem 1.2] due to Berkson and Gillespie. Furthermore, we present an alternative approach based on a version of the Rubio de Francia extrapolation theorem that holds for limited ranges of p which was recently given in [1].

The organization of the paper is well reflected by the titles of the following sections. However, we conclude with an additional comment. The proof of Theorem A is based on weighted estimates from the part (i) of Theorem B. To keep the pattern of the proof of the main result of the paper, Theorem A, more transparent, we postpone the proof of Theorem B (ii) to Section 3.

2. Proofs of Theorems B(i) and A

We first introduce auxiliary spaces which are useful in the proof of Theorem A. Let $q \in [1,\infty)$. If I is an interval in $\mathbb{R}$ we denote by $\mathcal{E}(I)$ the family of all step functions from I into $\mathbb{C}$. If $m := \sum_{J\in\mathcal{I}} a_J \chi_J$, where $\mathcal{I}$ is a decomposition of I into subintervals and $(a_J) \subset \mathbb{C}$, write $[m]_q := (\sum_{J\in\mathcal{I}} |a_J|^q)^{1/q}$. Set $\mathcal{R}_q(I) := \{m \in \mathcal{E}(I) : [m]_q \le 1\}$ and

$$\mathcal{R}_q(\mathcal{D}) := \left\{m : \mathbb{R} \to \mathbb{C} :\ m_{|I} \in \mathcal{R}_q(I) \text{ for every } I \in \mathcal{D}\right\}.$$

Moreover, let

$$R_q(I) := \left\{ \sum_j \lambda_j m_j \,:\, m_j \in \mathcal{R}_q(I),\ \sum_j |\lambda_j| < \infty \right\}$$

and

$$\|m\|_{R_q(I)} := \inf\left\{\sum_j |\lambda_j| : m = \sum_j \lambda_j m_j,\ m_i \in \mathcal{R}_q(I)\right\} \quad (m \in R_q(I)).$$

Note that $\big(R_q(I), \|\cdot\|_{R_q(I)}\big)$ is a Banach space. Set

$$R_q(\mathcal{D}) := \left\{m : \mathbb{R} \to \mathbb{C} : \sup_{I\in\mathcal{D}} \|m_{|I}\|_{R_q(I)} < \infty\right\} \qquad (q \in [1,\infty)).$$

In the sequel, if $\mathcal{I}$ is a family of disjoint intervals in $\mathbb{R}$, we write $S_1^{\mathcal{I}} f := \sup_{I\in\mathcal{I}} |S_I f|$ $(f \in L^1(\mathbb{R}))$ and $S_r^{\mathcal{I}} f := (\sum_{I\in\mathcal{I}} |S_I(f)|^{r'})^{1/r'}$ $(r \in (1,2]$, $f \in L^r(\mathbb{R}))$.

We next collect main ingredients of the proof of Theorem B(i), which provides crucial vector-valued estimates for weighted multipliers in the proof of Theorem A; see, e.g., (3).

Lemma 5 is a special version of the result on weighted inequalities for Carleson's operator given by Rubio de Francia *et al.* in [24]; see also [24, Remarks 2.2, Part III].

Lemma 5 ([24, Theorem 2.1, Part III]). *Let $s \in (1,\infty)$ and $w \in A_s(\mathbb{R})$. Then, there exists a constant $C > 0$ such that for any family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$*

$$\|S_1^{\mathcal{I}} f\|_{s,w} \le C\|f\|_{s,w} \quad (f \in L^s(\mathbb{R}, w\,dx)).$$

Moreover, for every $s > 1$ and every set $\mathcal{V} \subset A_s(\mathbb{R})$ with $\sup_{w\in\mathcal{V}} [w]_{A_s} < \infty$

$$\sup\big\{\|S_1^{\mathcal{I}}\|_{s,w} : w \in \mathcal{V},\ \mathcal{I} \textit{ a family of disjoint intervals in } \mathbb{R}\big\} < \infty.$$

Remark 6. The second statement of Lemma 5 can be obtained from a detailed analysis of the constants involved in the results which are used in the proof of [24, Theorem 2.1(a)⇒(b), Part III], i.e., the weighted version of the Fefferman–Stein inequality and the reverse Hölder inequality.

Recall the weighted version of the Fefferman–Stein inequality, which in particular says that for every $p \in (1,\infty)$ and every Muckenhoupt weight

$w \in A_p(\mathbb{R})$ there exists a constant $C_{p,w} > 0$, which depends only on p and $[w]_{A_p}$, such that

(1)
$$\int_{\mathbb{R}} Mf(t)^p\, w(t)\; dt \leq C_{p,w} \int_{\mathbb{R}} M^{\#} f(t)^p\, w(t)\; dt \quad (f \in L^p(\mathbb{R}) \cap L^p(\mathbb{R}, w)),$$

where M and $M^{\sharp}$ denote the Hardy–Littlewood maximal operator and the Fefferman–Stein sharp maximal operator, respectively; see [15, Theorem, p. 41], or [14, Theorem 2.20, Chapter IV]. We emphasize here that the constant $C_{p,w}$ on the right-hand side of this inequality is not given explicitly in the literature, but it can be obtained from a detailed analysis of the constants involved in the results which are used in the proof of (1), $\sup_{w\in\mathcal{V}} C_{p,w} < \infty$ for every subset $\mathcal{V} \subset A_p(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_p} < \infty$.

Furthermore, it should be noted that if $\mathcal{V} \subset A_p(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_p} < \infty$, then there exists $\epsilon > 0$ such that $\mathcal{V} \subset A_{p-\epsilon}(\mathbb{R})$ and $\sup_{w\in\mathcal{V}}[w]_{A_{p-\epsilon}} < \infty$. It can be directly obtained from a detailed analysis of the constants involved in main ingredients of the proof of the reverse Hölder inequality. See, e.g., [19, Lemma 2.3].

We refer the reader to [14, Chapter IV], [12, Chapter 7] for recent expositions of the results involved in the proof of the reverse Hölder inequality and the Fefferman–Stein inequality, which originally come from [7, 21, 22].

The next lemma is a special variant of Rubio de Francia's extrapolation theorem; see [25, Theorem 3]. For the convenience of the reader we rephrase [25, Theorem 3] here in the context of Muckenhoupt weights merely.

Lemma 7 ([26, Theorem 3]). *Let λ and r be fixed with $1 \leq \lambda \leq r < \infty$, and let $\mathcal{S}$ be a family of sublinear operators which is uniformly bounded in $L^r(\mathbb{R}, w\,dx)$ for each $w \in A_{r/\lambda}(\mathbb{R})$, i.e.,*

$$\int |Sf|^r w\, dx \leq C_{r,w} \int |f|^r w\, dx \quad (S \in \mathcal{S},\ w \in A_{r/\lambda}(\mathbb{R})).$$

If $\lambda < p, \alpha < \infty$ and $w \in A_{p/\lambda}(\mathbb{R})$, then $\mathcal{S}$ is uniformly bounded in $L^p(\mathbb{R}, w\,dx)$ and even more:

$$\int \left(\sum_j |S_j f_j|^\alpha \right)^{p/q} w\, dx \leq C_{p,\alpha,w} \int \left(\sum_j |f_j|^\alpha \right)^{p/q} w\, dx$$

for every $f_j \in L^p(\mathbb{R}, w\,dx)$ and $S_j \in \mathcal{S}$.

Combining Lemma 5 with Theorem 2 we get the intermediate weighted estimates for operators $S_q^{\mathcal{I}}$ ($q \in (1,2)$) stated in Theorem B(i).

For the background on the interpolation theory we refer the reader to [3]; in particular, see [3, Chapter 4 and Section 5.5].

Proof of Theorem B(i). Fix $q \in (1,2)$ and $w \in A_{2/q}(\mathbb{R})$. By the reverse Hölder inequality, $w \in A_{2/r}(\mathbb{R})$ for some $r \in (q,2)$. Note that there exist $p \in (2,q')$ and $s > 1$ such that $\frac{p}{q'}\frac{1}{p} + (1 - \frac{p}{q'})\frac{1}{s} = \frac{1}{r}$. Therefore, combining Theorem 2 with Lemma 5, by complex interpolation, the operator $S^{\mathcal{I}}_{(2q'/p)'}$ is bounded on $L^r(\mathbb{R}, v)$ for every $v \in A_1(\mathbb{R})$. Since $p > 2$, the same conclusion holds for $S_q^{\mathcal{I}}$.

By Rubio de Francia's extrapolation theorem, Lemma 7, we get that $S_q^{\mathcal{I}}$ is bounded on $L^2(\mathbb{R}, v)$ for every $v \in A_{2/r}(\mathbb{R})$. According to our choice of r, we get the boundedness of $S_q^{\mathcal{I}}$ on $L^2(\mathbb{R}, w)$.

Since the weight w was taken arbitrarily, we can again apply Rubio de Francia's extrapolation theorem, Lemma 7, to complete the proof of the first statement.

The second statement follows easily from a detailed analysis of the first one. For a discussion on the character of the dependence of constants in Rubio de Francia's iteration algorithm, we refer the reader to [11], or [10, Section 3.4]. See also the comment on the reverse Hölder inequality in Remark 6. □

Note that $R_q(I) \subsetneq V_q(I)$ for every interval I in $\mathbb{R}$ and $q \in [1,\infty)$. However, the following reverse inclusions hold for these classes.

Lemma 8 ([8, Lemme 2]). *Let $1 \le q < p < \infty$. For every interval I in $\mathbb{R}$, $V_q(I) \subset R_p(I)$ with the inclusion norm bounded by a constant independent of I.*

The patterns of the proofs of the parts (i) and (ii) of Theorem A are essentially the same. Therefore, we sketch the proof of the part (ii) below.

Proof of Theorem A. (i) We only give the proof for the more involved case $q \in (1,2)$; the case $q = 2$ follows simply from Theorem 3 and interpolation arguments presented below; see also Remark 9 below.

Fix $q \in (1,2)$. We first show that for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ such that $\sup_{w\in\mathcal{V}}[w]_{A_{2/q}} < \infty$ we have

$$\sup\left\{\|T_{m\chi_I}\|_{2,w} : m \in R_q(\mathcal{D}), \|m\|_{R_q(\mathcal{D})} \leq 1, w \in \mathcal{V}, I \in \mathcal{D}\right\} < \infty.$$

Fix $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_{2/q}} < \infty$. Note that, by the definition of the R_q-classes, it is sufficient to prove the claim with $R_q(\mathcal{D})$ replaced by $\mathcal{R}_q(\mathcal{D})$. Fix $m \in \mathcal{R}_q(\mathcal{D})$ and set $m\chi_I =: \sum_{J\in\mathcal{I}_I} a_{I,J}\chi_J$ for every $I \in \mathcal{D}$, where $\mathcal{I}_I = \mathcal{I}_{I,m}$ is a decomposition of I and $(a_{I,J})_{J\in\mathcal{I}_I} \subset \mathbb{C}$ is a sequence with $\sum_{J\in\mathcal{I}_I} |a_{I,J}|^q \leq 1$. Note that $T_{m\chi_I} f = \sum_J a_{I,J} S_J f$ and $\|T_{m\chi_I} f\|_{2,w} \leq \|S_q^{\mathcal{I}_I} f\|_{2,w}$ for every $I \in \mathcal{D}$, $w \in \mathcal{V}$ and $f \in L^2(\mathbb{R}, w)$. Therefore, by Lemma 5, our claim holds.

By interpolation argument, we next sharpen this claim and prove that for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_{2/q}} < \infty$ there exists $\alpha = \alpha(q, \mathcal{V}) > 1$ such that

$$\sup\left\{\|T_{m\chi_I}\|_{2,w} : m \in R_{\alpha q}(\mathcal{D}), \|m\|_{R_{\alpha q}(\mathcal{D})} \leq 1, w \in \mathcal{V}, I \in \mathcal{D}\right\} < \infty. \tag{2}$$

Note that, by the reverse Hölder inequality, see also Remark 6, there exists $\alpha > 1$ such that $w^\alpha \in A_{2/q}(\mathbb{R})$ $(w \in \mathcal{V})$ and $\sup_{w\in\mathcal{V}}[w^\alpha]_{A_{2/q}} < \infty$. From what has already been proved and Plancherel's theorem, for every $I \in \mathcal{D}$ and $w \in \mathcal{V}$ the bilinear operators

$$R_q(I) \times L^2(\mathbb{R}, w^\alpha dx) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R}, w^\alpha dx)$$
$$L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R})$$

are well defined and bounded uniformly with respect to $w \in \mathcal{V}$ and $I \in \mathcal{D}$. Therefore, by complex interpolation, $(R_q(I), L^\infty(\mathbb{R}))_{[\frac{1}{\alpha}]} \subset M_2(\mathbb{R}, w)$. However, it is easy to check that $R_{\alpha q}(I) \subset (R_q(I), L^\infty(\mathbb{R}))_{[\frac{1}{\alpha}]}$ with the inclusion norm bounded by a constant independent of $I \in \mathcal{D}$. We thus get (2).

In consequence, by Lemma 8, it follows that

$$\sup\left\{\|T_{m\chi_I}\|_{2,w} : m \in V_q(\mathcal{D}),\ \|m\|_{V_q(\mathcal{D})} \leq 1,\ w \in \mathcal{V}, I \in \mathcal{D}\right\} < \infty \tag{3}$$

for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_{2/q}} < \infty$.

Hence, we can apply a truncation argument based on Kurtz' weighted variant of Littlewood–Paley's inequality. Namely, fix $w \in A_{2/q}(\mathbb{R})$, $m \in V_q(\mathcal{D})$ with $\|m\|_{V_q(\mathcal{D})} \leq 1$, and $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, w)$, $g \in L^2(\mathbb{R}, w) \cap L^2(\mathbb{R}, w^{-1})$. Note that $gw \in L^2(\mathbb{R})$ and $A_{2/q}(\mathbb{R}) \subset A_2(\mathbb{R})$. Therefore, combining the

Cauchy–Schwarz inequality and Kurtz' result, [17, Theorem 1], we get

$$\begin{aligned}|(T_m f, g)_{L^2(\mathbb{R},w)}| &= \left|\sum_{I\in\mathcal{D}}\int_{\mathbb{R}} S_I(T_m f) S_I(gw)dx\right| \\ &\le C\left\|\left(\sum_{I\in\mathcal{D}}|T_{m\chi_I}S_I f|^2\right)^{1/2}\right\|_{2,w}\left\|\left(\sum_{I\in\mathcal{D}}|S_I(gw)|^2\right)^{1/2}\right\|_{2,w^{-1}} \\ &\le C\|f\|_{2,w}\|g\|_{2,w},\end{aligned}$$

where C is an absolute constant independent of m, f and g. Now the converse of Hölder inequality and a density argument show that $m \in M_2(\mathbb{R}, w)$.

Consequently, $V_q(\mathcal{D}) \subset M_2(\mathbb{R}, w)$, and Rubio de Francia's extrapolation theorem, Lemma 7, yields $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $p > q$ and every Muckenhoupt weight $w \in A_{p/q}(\mathbb{R})$.

It remains to prove that $V_q(\mathcal{D}) \subset M_q(\mathbb{R}, w)$ for every $w \in A_1(\mathbb{R})$. Fix $m \in V_q(\mathcal{D})$ and $w \in A_1(\mathbb{R})$. Then, by Theorem 3 (see also Remark 9), T_m is bounded on $L^r(\mathbb{R})$ for every $r \in (1,\infty)$. From what has already been proved, T_m is bounded on $L^r(\mathbb{R}, w)$ for every $r > q$. Therefore, the boundedness of T_m on $L^q(\mathbb{R}, w)$ follows by the reverse Hölder inequality for w and a similar interpolation argument as before. This completes the proof of the part (i).

(ii) Fix $q > 2$ and $s > \frac{q}{2}$. Let $\mathcal{V}_s := \{w \in A_1(\mathbb{R}) : w \in RH_s(\mathbb{R})\}$. Note that there exists $r = r_s > q$ such that $\frac{1}{s}\frac{1}{2} + \frac{1}{s'}\frac{1}{r} < \frac{1}{q}$.

Fix $w \in \mathcal{V}_s$. By Theorem 3, the bilinear operators

$$\begin{gathered}R_r(\mathcal{D}) \times L^2(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R}), \\ R_2(\mathcal{D}) \times L^2(\mathbb{R}, w^s) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R}, w^s)\end{gathered}$$

are well defined and bounded. By interpolation, it follows that

$$M_2(\mathbb{R}, w) \supset (R_2(\mathcal{D}), R_r(\mathcal{D}))_{[\frac{1}{s}]} \supset R_{\alpha q}(I)$$

uniformly with respect to $I \in \mathcal{D}$, where $\alpha = \alpha_s := (\frac{1}{2s} + \frac{1}{s'r})^{-1}/q > 1$.

As in the corresponding part of the proof of (i), by truncation and duality arguments, we get $R_{\alpha q}(\mathcal{D}) \subset M_2(\mathbb{R}, w)$.

Consequently, since $\alpha_s > 1$ for every $s > \frac{q}{2}$, by Lemma 8,

$$V_q(\mathcal{D}) \subset M_2(\mathbb{R}, w) \qquad \text{for every } w \in \bigcup_{s>\frac{q}{2}} \mathcal{V}_s(\mathbb{R}). \tag{4}$$

Note that this is precisely the assertion of (ii) for $p = 2$.

We can now proceed by extrapolation. Since for every $s > \frac{q}{2}$ we can rephrase $\mathcal{V}_s$ as $A_{\frac{2}{2}}(\mathbb{R}) \cap RH_{(\frac{2s'}{2})'}(\mathbb{R})$, by Cruz–Uribe et al. [10, Theorem 3.31], we get

$$(5) \qquad V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w) \quad \text{for every } s > \frac{q}{2},\ 2 < p < 2s',\ \text{and} \quad w \in A_{\frac{p}{2}}(\mathbb{R}) \cap RH_{(\frac{2s'}{p})'}(\mathbb{R})\,.$$

Finally, it is easy to see that for every $2 \le p < \frac{1}{2} - \frac{1}{q} = 2(\frac{q}{2})'$ and $w \in A_{p/2}(\mathbb{R})$ with $s_w > (1 - p(\frac{1}{2} - \frac{1}{q}))^{-1} = (\frac{2}{p}(\frac{q}{2})')'$ there exists $s = s_{p,w} > \frac{q}{2}$ such that $p < 2s'$ and $w \in RH_{(\frac{2s'}{p})'}$. Therefore, (5) completes the proof of (ii). □

Remark 9. In the proof of Theorem A we use Theorem 3 due to Coifman, Rubio de Francia and Semmes. Note that the patterns of all proofs are essentially the same.

Indeed, we can rephrase the proof of [8, Théorème 1] as follows. First recall that $M_p(\mathbb{R}) = M_{p'}(\mathbb{R})$ for every $p \in (1, \infty)$. Let $r \geq 2$. By the Littlewood–Paley decomposition theorem, Rubio de Francia's inequalities, and Plancherel's theorem, the bilinear operators

$$R_2(\mathcal{D}) \times L^r(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^r(\mathbb{R}),$$
$$L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R})$$

are well defined and bounded. Therefore, by interpolation, $(R_2(\mathcal{D}), L^\infty(\mathbb{R}))_{[\theta(r)]} \subset M_p(\mathbb{R})$, where $\theta(r) \in (0,1)$ and p such that $\frac{1}{p} = \theta(r)\frac{1}{r} + (1 - \theta(r))\frac{1}{2}$.

Note that if $p \geq 2$ and q satisfies $\frac{1}{q} > \frac{1}{2} - \frac{1}{p}$, then there exists $r > 2$ such that $R_{\alpha q}(I) \subset (R_2(\mathcal{D}), L^\infty(\mathbb{R}))_{[\theta(r)]}$ for an appropriate $\alpha > 1$ and uniformly with respect to $I \in \mathcal{D}$. Indeed, $\frac{1}{2}\theta(r) \searrow \frac{1}{2} - \frac{1}{p}$ as $r \to \infty$. Therefore, Lemma 8 completes the proof of Theorem 3(i).

3. Proof of Theorem B(ii)

We obtain the proof of Theorem B(ii) by means of a Banach function space analogue of Kurtz' weighted variant of Littlewood–Paley inequalities and the Fefferman–Stein inequality; see Lemma 10 below.

Note that without loss of generality in the proof of Theorem B(ii) one can consider only families consisting of bounded intervals in $\mathbb{R}$. For a bounded interval $I \in \mathcal{I}$ we write $\mathcal{W}_I$ for Whitney's decomposition of I (see [25, Section 2] for the definition). Note also that each decomposition $\mathcal{W}_I$, $i \in \mathcal{I}$, is of dyadic type. Furthermore, the family $\mathcal{W}^{\mathcal{I}} := \bigcup_{I\in\mathcal{I}} \mathcal{W}_I$ is well distributed, i.e.,

$$\sup_{x\in\mathbb{R}} \sum_{I\in\mathcal{W}^{\mathcal{I}}} \chi_{2I}(x) \leq 5.$$

We refer the reader primarily to [2] for the background on function spaces. In the sequel, let $\mathbb{E}$ denote a rearrangement invariant Banach function space over $(\mathbb{R}, dx)$. Recall that, by Luxemburg's representation theorem [2, Theorem 4.10, p. 62], there exists a rearrangement invariant Banach function space $\overline{\mathbb{E}}$ over $(\mathbb{R}_+, dt)$ such that for every scalar, measurable function f on $\mathbb{R}$, $f \in \mathbb{E}$ if and only if $f^* \in \overline{\mathbb{E}}$, where f^* stands for the decreasing rearrangement of f. In this case $\|f\|_{\mathbb{E}} = \|f^*\|_{\overline{\mathbb{E}}}$ for every $f \in \mathbb{E}$.

Following [20], we define the *lower* and *upper Boyd indices,* respectively, by

$$p_{\mathbb{E}} := \lim_{t\to\infty} \frac{\log t}{\log h_{\mathbb{E}}(t)} \qquad \text{and} \qquad q_{\mathbb{E}} := \lim_{t\to 0+} \frac{\log t}{\log h_{\mathbb{E}}(t)},$$

where $h_{\mathbb{E}}(t) = \|D_t\|_{\mathcal{L}(\overline{\mathbb{E}})}$ and $D_t : \overline{\mathbb{E}} \to \overline{\mathbb{E}}$ $(t > 0)$ is the *dilation operator* defined by

$$D_t f(s) = f(s/t), \qquad 0 < t < \infty, \quad f \in \overline{\mathbb{E}}.$$

One always has $1 \leq p_{\mathbb{E}} \leq q_{\mathbb{E}} \leq \infty$, see for example [2, Proposition 5.13, p. 149], where the Boyd indices are defined as the reciprocals with respect to our definitions.

Let w be a weight in $A_\infty(\mathbb{R})$. Then we can associate with $\mathbb{E}$ and w a rearrangement invariant Banach function space over $(\mathbb{R}, w\,dx)$ as follows:

$$\mathbb{E}_w = \{f : \mathbb{R} \to \mathbb{C} \ \text{ measurable} : f_w^* \in \overline{\mathbb{E}}\},$$

and its norm is $\|f\|_{\mathbb{E}_w} = \|f_w^*\|_{\overline{\mathbb{E}}}$, where f_w^* denotes the decreasing rearrangement of f with respect to $w\,dx$.

For further purposes, recall also that examples of rearrangement Banach function spaces are the Lorentz spaces $L^{p,q}$ $(1 \leq p, q \leq \infty)$. Note that $L_w^{p,\infty} =$ weak $- L^p(\mathbb{R}, w)$ for every $p \in (1, \infty)$ and $w \in A_\infty(\mathbb{R})$. The Boyd indices can be computed explicitly for many examples of concrete rearrangement invariant Banach function spaces, see, e.g., [2, Chapter 4]. In particular, we have $p_{\mathbb{E}} = q_{\mathbb{E}} = p$ for $\mathbb{E} := L^{p,q}$ $(1 < p < \infty,\ 1 \leq q \leq \infty)$; see [2, Theorem 4.6].

Lemma 10. *Let $\mathbb{E}$ be a rearrangement invariant Banach function space on $(\mathbb{R}, dx)$ such that $1 < p_{\mathbb{E}}, q_{\mathbb{E}} < \infty$. Then the following statements hold:*

(i) *For every Muckenhoupt weight $w \in A_{p_{\mathbb{E}}}(\mathbb{R})$ there exists a constant $C_{w,\mathbb{E}}$ such that for any family $\mathcal{I}$ of disjoint bounded intervals in $\mathbb{R}$*

$$C_{\mathbb{E},w}^{-1} \|S^{\mathcal{I}} f\|_{\mathbb{E}_w} \leq \|S^{\mathcal{W}^{\mathcal{I}}} f\|_{\mathbb{E}_w} \leq C_{\mathbb{E},w} \|S^{\mathcal{I}} f\|_{\mathbb{E}_w} \tag{6}$$

and

$$\|Mf\|_{\mathbb{E}_w} \leq C_{\mathbb{E},w} \|M^{\sharp} f\|_{\mathbb{E}_w} \tag{7}$$

for every $f \in \mathbb{E}_w$.
Moreover, if $\mathcal{V} \subset A_{p_{\mathbb{E}}}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}} [w]_{A_{p_{\mathbb{E}}}} < \infty$, then $\sup_{w\in\mathcal{V}} C_{\mathbb{E},w} < \infty$.

(ii) *For every $r \in (1,\infty)$ and every Muckenhoupt weight $w \in A_{p_{\mathbb{E}}}(\mathbb{R})$ there exists a constant $C_{r,\mathbb{E},w}$ such that for any family $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$*

$$\left\| \left(\sum_{I\in\mathcal{I}} |S_I f_I|^r \right)^{1/r} \right\|_{\mathbb{E}_w} \leq C_{r,\mathbb{E},w} \left\| \left(\sum_{I\in\mathcal{I}} |f_I|^r \right)^{1/r} \right\|_{\mathbb{E}_w} \tag{8}$$

for every $(f_I)_{I\in\mathcal{I}} \subset \mathbb{E}_w(l^r(\mathcal{I}))$.

The proof follows the idea of the proof of [25, Lemma 6.3], i.e., it is based on the iteration algorithm of the Rubio de Francia extrapolation theory. We refer the reader to [10] for a recent account of this theory; in particular, see the proofs of [10, Theorems 3.9 and 4.10]. We provide below main supplementary observations which should be made.

Proof of Lemma 10. Note that we can restrict ourself to finite families $\mathcal{I}$ of disjoint bounded intervals in $\mathbb{R}$. The final estimates obtained below are independent of $\mathcal{I}$, and a standard limiting argument proves the result in the general case.

According to [17, Theorem 3.1], for every Muckenhoupt weight $w \in A_2(\mathbb{R})$ there exists a constant $C_{2,w}$ such that

$$C_{2,w}^{-1} \|S^{\mathcal{I}} f\|_{L^2(\mathbb{R},w)} \leq \|S^{\mathcal{W}^{\mathcal{I}}} f\|_{L^2(\mathbb{R},w)} \leq C_{2,w} \|S^{\mathcal{I}} f\|_{L^2(\mathbb{R},w)} \tag{9}$$
$$\text{for every } f \in L^2(\mathbb{R}, w).$$

Moreover, one can show that $\sup_{w\in\mathcal{V}} C_{2,w} < \infty$ for every subset $\mathcal{V} \subset A_2(\mathbb{R})$ with $\sup_{w\in\mathcal{V}} [w]_{A_2} < \infty$.

Therefore, we are in a position to adapt the extrapolation techniques from A_2 weights; see for example the proof of [10, Theorem 4.10, p. 76]. Fix $\mathbb{E}$ and $w \in A_{p_{\mathbb{E}}}(\mathbb{R})$ as in the assumption. Let $\mathbb{E}'_w$ be the associate space of $\mathbb{E}_w$, see [2, Definition 2.3, p. 9]. Let $\mathcal{R} = \mathcal{R}_w : \mathbb{E}_w \to \mathbb{E}_w$ and $\mathcal{R}' = \mathcal{R}'_w : \mathbb{E}'_w \to \mathbb{E}'_w$ be defined by

$$\mathcal{R}h(t) = \sum_{j=0}^{\infty} \frac{M^j h(t)}{2^j \|M\|_{\mathbb{E}_w}^j}, \quad 0 \le h \in \mathbb{E}_w,$$
$$\mathcal{R}'h(t) = \sum_{j=0}^{\infty} \frac{S^j h(t)}{2^j \|S\|_{\mathbb{E}'_w}^j}, \quad 0 \le h \in \mathbb{E}'_w,$$

where $Sh := M(hw)/w$ for $h \in \mathbb{E}'_w$. As in the proof of [10, Theorem 4.10, p. 76] the following statements are easily verified:

(a) For every positive $h \in \mathbb{E}_w$ one has

$$h \le \mathcal{R}h \text{ and } \|\mathcal{R}h\|_{\mathbb{E}_w} \le 2\|h\|_{\mathbb{E}_w},$$
$$\mathcal{R}h \in A_1 \text{ with } [\mathcal{R}h]_{A_1} \le 2\|M\|_{\mathbb{E}_w}.$$

(b) For every positive $h \in \mathbb{E}'_w$ one has

$$h \le \mathbb{R}'h \text{ and } \|\mathcal{R}'h\|_{\mathbb{E}'_w} \le 2\|h\|_{\mathbb{E}'_w},$$
$$(\mathcal{R}'h)w \in A_1 \text{ with } [(\mathcal{R}'h)w]_{A_1} \le 2\|S\|_{\mathbb{E}'_w}.$$

The last lines in (a) and (b) follow from the estimates $M(\mathcal{R}h) \le 2\|M\|_{\mathbb{E}_w}\mathcal{R}h$ and $M((\mathcal{R}'h)w) \le 2\|S\|_{\mathbb{E}'_w}(\mathcal{R}'h)w)$, respectively, which in turn follow from the definitions of $\mathcal{R}$ and $\mathcal{R}'$.

Note that $f \in L^2(\mathbb{R}, w_{|f|,h})$ for every $f \in \mathbb{E}_w$ and every positive $h \in \mathbb{E}'_w$, where $w_{g,h} := (\mathcal{R}g)^{-1}(\mathcal{R}'h)w$ for every $0 \le g \in \mathbb{E}_w$ and $0 \le h \in \mathbb{E}'_w$. Moreover, by Boyd's interpolation theorem, the Hilbert transform is bounded on $\mathbb{E}_w$. Therefore, by the well-known identity relating partial sum operators S_I and the Hilbert transform, since $\mathcal{I}$ is finite, we get that $S^{\mathcal{I}}f \in \mathbb{E}_w$ for every $f \in \mathbb{E}_w$. Similarly, combining Kurtz' inequalities, [17, Theorem 3.1], with Boyd's interpolation theorem, we conclude that $S^{\mathcal{W}_I}f \in \mathbb{E}_w$ $(I \in \mathcal{I})$, and consequently $S^{\mathcal{W}^{\mathcal{I}}}f \in \mathbb{E}_w$ for every $f \in \mathbb{E}_w$.

Finally, a close analysis of the proof of [10, Theorem 4.10] shows that we can take

$$C_{\mathbb{E},w} := 4\sup\{C_{2,w_{g,h}} : 0 \le g \in \mathbb{E}_w, 0 \le h \in \mathbb{E}'_w, \|g\|_{\mathbb{E}_w} \le 2, \|h\|_{\mathbb{E}'_w} = 1\}.$$

Recall that for every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that $\|M\|_{L^p_w} \le C_p[w]_{A_p}^{p'/p}$ for every Muckenhoupt weight $w \in A_p(\mathbb{R})$; see [5]. A detailed analysis of Boyd's interpolation theorem shows that $\sup_{w\in\mathcal{V}} \max(\|M\|_{\mathbb{E}_w}, \|S\|_{\mathbb{E}_w}) < \infty$ for every $\mathcal{V} \subset A_{p_{\mathbb{E}}}(\mathbb{R})$ with $\sup_{w\in\mathcal{V}}[w]_{A_{p_{\mathbb{E}}}} < \infty$. By the so-called reverse factorization (or by Hölder's inequality; see, e.g., [12, Proposition 7.2]), and by properties (a) and (b), we obtain that $w_{g,h} \in A_2(\mathbb{R})$ and

$$[w_{g,h}]_{A_2} \le [\mathcal{R}g]_{A_1}[(\mathcal{R}'h)w]_{A_1} \le 4\|M\|_{\mathbb{E}_w}\|S\|_{\mathbb{E}'_w}$$

for every $0 \le g \in \mathbb{E}_w$ and $0 \le h \in \mathbb{E}'_w$. Therefore, on account of the remark on the constants $C_{2,w}$ in (9), we get the desired boundedness property of constants $C_{\mathbb{E},w}$. This completes the proof of (6).

Note that, by the weighted Fefferman–Stein inequality, see Remark 6, and the basic inequality $M^\sharp f \le 2Mf$ $(f \in L^1_{\mathrm{loc}}(\mathbb{R}))$, the analogous reasoning as before yields (7).

For the proof of the part (ii), for fixed $r \in (1,\infty)$ it is sufficient to apply Rubio de Francia's extrapolation algorithm from A_r weights in the same manner as above. □

Let $\mathcal{W}$ be a well-distributed family of disjoint intervals in $\mathbb{R}$, i.e., there exists $\lambda > 1$ such that $\sup_{x\in\mathbb{R}} \sum_{I\in\mathcal{I}} \chi_{\lambda I}(x) < \infty$.

Following [25, Section 3], consider the smooth version of $S^{\mathcal{W}^{\mathcal{I}}}$, $G = G^{\mathcal{W}}$, defined as follows: let ϕ be an even, decreasing, smooth function such that $\hat\phi(\xi) = 1$ on $\xi \in [-\frac{1}{2}, \frac{1}{2}]$ and $\operatorname{supp}\hat\phi \subset [-\lambda/2, \lambda/2]$. Let $\phi_I(x) := e^{2\pi i c_I\cdot x}|I|\phi(|I|x)$ $(x \in \mathbb{R})$, where c_I stands for the centre of an interval $I \in \mathcal{W}$ and $|I|$ for its length. Then,

$$Gf := G^{\mathcal{W}}f := \left(\sum_{I\in\mathcal{W}} |\phi_I \star f|^2\right)^{1/2} \qquad (f \in L^2(\mathbb{R})).$$

Since $\widehat{\phi_I}(\xi) = 1$ for $\xi \in I$, and $\widehat{\phi_I}(\xi) = 0$ for $\xi \notin \lambda I$, by Plancherel's theorem, G is bounded on $L^2(\mathbb{R})$.

Recall that the crucial step of the proof of [25, Theorem 6.1] consists in showing that the Hilbert space-valued kernel related with G satisfies weak-(D'_2) condition (see [25, Part IV(E)] for the definition). This leads to the following pointwise estimates for G:

$$M^\sharp(Gf)(x) \le CM(|f|^2)(x)^{1/2} \qquad (\text{ a.e. } x \in \mathbb{R}) \tag{10}$$

for every $f \in L^\infty(\mathbb{R})$ with compact support, and a constant C depending only on λ. In particular, G is bounded on $L^p(\mathbb{R}, w)$ for every $p > 2$ and every Muckenhoupt weight $w \in A_{p/2}(\mathbb{R})$.

Proof of Theorem B(ii). We can assume that $\mathcal{I}$ is a finite family of bounded intervals in $\mathbb{R}$. By a standard limiting arguments we easily get the general case.

We start with the proof of the statement of Theorem B(ii) for $q = 2$. Recall that $p_{\mathbb{E}} = q_{\mathbb{E}} = 2$ for $\mathbb{E} := L^{2,\infty}$; see [2, Theorem 4.6]. Fix $w \in A_1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$ with compact support. Note that the classical Littlewood–Paley theory shows that $G^{\mathcal{W}_I}$ is bounded on L^2_w for every $I \in \mathcal{I}$. Consequently, $G = G^{\mathcal{W}^{\mathcal{I}}}$ maps L^2_w into itself.

Therefore, combining Lemma 10, Lebesgue's differentiation theorem and (10) we get

$$\begin{aligned}\|S^{\mathcal{I}} f\|_{L^{2,\infty}_w} &\le C_w \|S^{\mathcal{W}^{\mathcal{I}}} f\|_{L^{2,\infty}_w} \le C_w \|Gf\|_{L^{2,\infty}_w} \le C_w \|M(Gf)\|_{L^{2,\infty}_w} \\ &\le C_w \|M^\sharp(Gf)\|_{L^{2,\infty}_w} \le C_w \|M(|f|^2)^{1/2}\|_{L^{2,\infty}_w} = C_w \|M(|f|^2)\|^{1/2}_{L^{1,\infty}_w} \\ &\le C_w \|f\|_{L^2_w},\end{aligned}$$

where C_w is an absolute constant independent of $\mathcal{I}$ and f. The last inequality follows from the fact that the Hardy–Littlewood maximal operator M is of weak $(1,1)$ type. Furthermore, one can show that for every subset $\mathcal{V} \subset A_1(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} [w]_{A_1} < \infty$ we have $\sup_{w \in \mathcal{V}} C_w < \infty$. Since $S^{\mathcal{I}}$ is continuous on L^2_w and the space of all functions in $L^\infty(\mathbb{R})$ with compact support is dense in L^2_w we get the desired boundedness for $S^{\mathcal{I}}$. This completes the proof of the statement of Theorem B(ii) for $q = 2$.

We now proceed by interpolation to show the case of $q \in (1,2)$. Let $\mathcal{W}$ be a well-distributed family of disjoint intervals in $\mathbb{R}$, and G denote the corresponding smooth varsion of $\mathcal{S}^{\mathcal{W}}$. First, it is easily seen that $|\phi_I \star f| \le (\int \phi \, dx) Mf$ for every $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $I \in \mathcal{W}$. Moreover, analysis similar to the above shows that G maps $L^2_w(\mathbb{R})$ into $L^{2,\infty}_w(\mathbb{R})$ for every $w \in A_1(\mathbb{R})$. Therefore, for every $w \in A_1(\mathbb{R})$ the operators

$$\begin{aligned} L^1_w \ni f &\mapsto (\phi_I \star f)_{I \in \mathcal{W}} \in L^{1,\infty}_w(l^\infty), \\ L^2_w \ni f &\mapsto (\phi_I \star f)_{I \in \mathcal{W}} \in L^{2,\infty}_w(l^2) \end{aligned}$$

are bounded. Fix $q \in (1,2)$. By interpolation arguments, we conclude that the operator

$$L^q_w \ni f \mapsto (\phi_I \star f)_{I \in \mathcal{W}} \in L^{q,\infty}_w(l^{q'})$$

is well defined and bounded. To show it one can proceed analogously to the proof of a relevant result [23, Lemma 3.1]. Therefore, we omit details here.

Since $p_{\mathbb{E}} = q_{\mathbb{E}} = q$ for $\mathbb{E} := L^{q,\infty}$, see [2, Theorem 4.6], by Lemma 10 (ii), for every $w \in A_1(\mathbb{R})$ we get

$$\begin{aligned}\|S_q^{\mathcal{W}} f\|_{L_w^{q,\infty}} &= \left\| \left(\sum_{I\in\mathcal{W}} |S_I(\phi_I \star f)|^{q'} \right)^{1/q'} \right\|_{L_w^{q,\infty}} \\ &\le C_{q,w} \left\| \left(\sum_{I\in\mathcal{W}} |\phi_I \star f|^{q'} \right)^{1/q'} \right\|_{L_w^{q,\infty}} \le C_{q,w}\|f\|_{L_w^q},\end{aligned}$$

where $C_{q,w}$ is an absolute constant. This completes the proof. □

Remark 11. We conclude with the relevant result on A_2-weighted L^2-estimates for square functions $S^{\mathcal{I}}$ corresponding to arbitrary families $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$, i.e., $\|S^{\mathcal{I}}\|_{2,w} \le C\|f\|_{2,w}$ $(f \in L_w^2)$. According to [24, Part IV(E)(ii)], these weighted endpoint estimates can be reached by interpolation provided that $\mathcal{I}$ is a family such that $S^{\mathcal{I}}$ admits an extension to a bounded operator on (unweighted) $L^p(\mathbb{R})$ for some $p < 2$. This observation leads to a natural question: for which partitions $\mathcal{I}$ of $\mathbb{R}$ do there exist *local* variants of the Littlewood–Paley decomposition theorem, i.e., there exists $r \ge 2$ such that $S^{\mathcal{I}}$ is bounded on $L^p(\mathbb{R})$ for all $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{r}$.

Recall that Carleson, who first noted the possible extension of the classical Littlewood–Paley inequality for other types of partitions of $\mathbb{R}$, proved in the special case $\mathcal{I} := \{[n, n+1) : n \in \mathbb{Z}\}$ that the corresponding square function $S^{\mathcal{I}}$ is bounded on $L^p(\mathbb{R})$ only if $p \ge 2$; see [6]. Moreover, it should be noted that such lack of the boundedness of the square function $S^{\mathcal{I}}$ on $L^p(\mathbb{R})$ for some $p < 2$ occurs in the case of decompositions of $\mathbb{R}$ determined by sequences which are in a sense not too different from lacunary ones. Indeed, applying the ideas from [13, Section 8.5], we show below that even in the case of the decomposition $\mathcal{I}$ of $\mathbb{R}$ determined by a sequence $(a_j)_{j=0}^{\infty} \subset (0,\infty)$ such that $a_{j+1} - a_j \sim \lambda^{\phi(j)j}$, where $\lambda > 1$ and $\phi(j) \to 0^+$ arbitrary slowly as $j \to \infty$, the square function $S^{\mathcal{I}}$ is not bounded on $L^p(\mathbb{R})$ for every $p < 2$.

If I is a bounded interval in $\mathbb{R}$, set f_I for the function with $\widehat{f_I} = \chi_I$. Then, $|f_I| = \left|\frac{\sin(|I|\pi\cdot)}{\pi(\cdot)}\right|$, and for every $p > 2$ and every $\epsilon > 0$ there exists $c > 0$ such that

$$\frac{1}{c}|I|^{1/p'} \le \|f_I\|_p \le c|I|^{1/p'}$$

for all intervals I with $|I| > \epsilon$. This simply observation allows one to express [13, Theorem 8.5.4] for decompositions of $\mathbb{R}$ instead of $\mathbb{Z}$. Namely, if $a = (a_j)_{j=0}^{\infty} \subset (0, \infty)$ is an increasing sequence such that $a_j - a_{j-1} \to \infty$ as $j \to \infty$, and $\mathcal{I}_a := \{(-a_0, a_0)\} \cup \{\pm[a_{j-1}, a_j)\}_{j\geq 1}$, then the boundedness of $S^{\mathcal{I}_a}$ on $L^{p'}(\mathbb{R})$ for some $p > 2$, $1/p' + 1/p = 1$, implies that there exists a constant $C_p > 0$ such that

$$a_k^{2/p'} \leq C_p \sum_{j=1}^{k} (a_j - a_{j-1})^{2/p'} \quad (k \geq 1). \tag{11}$$

Moreover, it is straightforward to adapt the idea of the proof of [13, Corollary 8.5.5] to give the following generalization.

Let $a = (a_j)_{j=0}^{\infty} \subset (0, \infty)$ be an increasing sequence such that $a_{j+1} - a_j \sim \lambda^{\psi(j)}$, where $\lambda > 1$, the function $\psi \in \mathcal{C}^1([0, \infty))$ is increasing and satisfies the condition: $\psi(s)/s \to 0$ and $\psi'(s) \to 0$ as $s \to \infty$. If the square function $S^{\mathcal{I}_a}$ was bounded on $L^{p'}(\mathbb{R})$ for some $p > 2$, then (11) yields

$$\left(\int_0^{k-1} \lambda^{\psi(s)} ds \right)^{2/p'} \leq C_p \int_0^{k+1} \lambda^{\psi(s) 2/p'} ds \quad (k \geq 1).$$

However, this leads to a contradiction with the assumptions on ψ.

4. Higher-dimensional analogue of Theorem A

The higher-dimensional extension of the results due to Coifman *et al.* [8] was established essentially by Xu in [28]; see also Lacey [18, Chapter 4].

We start with higher-dimensional counterparts of some notions from previous sections. Here and subsequently, we consider only bounded intervals with sides parallel to the axes.

Let $q \geq 1$ and $d \in \mathbb{N}$. For $h > 0$ and $1 \leq k \leq d$ we write $\Delta_h^{(k)}$ for the difference operator, i.e.,

$$\left(\Delta_h^{(k)} m \right)(x) := m(x + he_k) - m(x) \qquad (x \in \mathbb{R}^d)$$

for any function $m : \mathbb{R}^d \to \mathbb{C}$, where e_k is the kth coordinate vector. Suppose that J is an interval in $\mathbb{R}^d$ and set $\overline{J} =: \Pi_{i=1}^d [a_i, a_i + h_i]$ with $h_i > 0$ $(1 \leq i \leq d)$. We write

$$(\Delta_J m) := \left(\Delta_{h_1}^{(1)} \dots \Delta_{h_d}^{(d)} m \right)(a),$$

where $a := (a_1, \dots, a_d)$ and $m : \mathbb{R} \to \mathbb{C}$. Moreover, for an interval I in $\mathbb{R}^d$ and a function $m : \mathbb{R}^d \to \mathbb{C}$ we set

$$\|m\|_{\mathrm{Var}_q(I)} := \sup_{\mathcal{J}} \left(\sum_{J \in \mathcal{J}} |\Delta_J m|^q \right)^{1/q},$$

where $\mathcal{J}$ ranges over all decompositions of I into subintervals.

Following Xu [28], see also [18, Section 4.2], the spaces $V_q(I)$ for intervals in $\mathbb{R}^d$ are defined inductively as follows.

The definition of $V_q(I)$ ($q \in [1, \infty)$) for one-dimensional intervals is introduced in Section 1. Suppose now that $d \in \mathbb{N} \setminus \{1\}$ and fix an interval $I = I_1 \times \cdots \times I_d$ in $\mathbb{R}^d$. For a function $m : \mathbb{R}^d \to \mathbb{C}$, we write $m \in V_q(I)$ if

$$\|m\|_{V_q(I)} := \sup_{x \in I} |m(x)| + \sup_{x_1 \in I_1} \|m(x_1, \cdot)\|_{V_q(I_2 \times \cdots \times I_d)} + \|m\|_{\mathrm{Var}_q(I)} < \infty.$$

Subsequently, $\mathcal{D}^d$ stands for the family of the dyadic intervals in $\mathbb{R}^d$. The definition of the spaces $(V_q(\mathcal{D}^d), \|\cdot\|_{V_q(\mathcal{D}^d)})$ ($d \geq 2,\ q \in [1, \infty)$) is quite analogous to the corresponding ones in the case of $d = 1$ from Section 1.

For a Banach space X, an interval I in $\mathbb{R}$ and $q \geq 1$, we consider below the vector-valued variants $V_q(I; X)$, $\mathcal{R}_q(I; X)$, and $R_q(I; X)$ of the spaces $V(I)$, $\mathcal{R}_q(I)$, and $R(I)$, respectively. Note that $V_q(I; X) \subset R_p(I; X)$ for any $1 \leq q < p$ and any interval I in $\mathbb{R}$ with the inclusion norm bounded by a constant depending only on p and q; see [28, Lemma 2]. Moreover, higher-dimensional counterparts of these spaces we define inductively as follows: let $I := \Pi_{i=1}^d I_i$ be a closed interval in $\mathbb{R}^d$ ($d \geq 2$). Set $\widetilde{R}_q(I) := R_q(I_1; \widetilde{R}_q(I_2 \times \cdots \times I_d))$ and $\widetilde{V}_q(I) := V_q(I_1; \widetilde{V}_q(I_2 \times \cdots \times I_d))$, where $\widetilde{R}_q(I_d) := R(I_d)$ and $\widetilde{V}_q(I_d) := V_q(I_d)$. Recall also that for any $1 \leq q < p$ and any interval I in $\mathbb{R}^d$ ($d \geq 1$) we have

$$V_q(I) \subset \widetilde{V}_q(I) \subset \widetilde{R}_p(I) \tag{12}$$

with the inclusion norm bounded by a constant independent of I.

Finally, we denote by $A_p^*(\mathbb{R}^d)$ ($p \in [1, \infty)$) the class of weights on $\mathbb{R}^d$ which satisfy the strong Muckenhoupt A_p condition. Note that, in the case of $d = 1$, $A_p^*(\mathbb{R})$ is the classical Muckenhoupt $A_p(\mathbb{R})$ class ($p \in [1, \infty)$). We refer the reader, e.g., to [17] or [14, Chapter IV.6] for the background on A_p^*-weights.

The following complement to [28, Theorem (i)] is the main result of this section.

Theorem C. *Let $d \geq 2$ and $q \in (1,2]$. Then, $V_q(\mathcal{D}^d) \subset M_p(\mathbb{R}^d, w)$ for every $p \geq q$ and every weight $w \in A^*_{p/q}(\mathbb{R}^d)$.*
(ii) *Let $d \geq 2$ and $q > 2$. Then, $V_q(\mathcal{D}^d) \subset M_p(\mathbb{R}^d, w)$ for every $2 \leq p < (\frac{1}{2} - \frac{1}{q})^{-1}$ and every weight $w \in A^*_{p/2}(\mathbb{R}^d)$ with $s_w > (1 - p(\frac{1}{2} - \frac{1}{q}))^{-1}$.*

Lemma 12. *For every $d \in \mathbb{N}$, $q \in (1,2]$, $p > q$, and every subset $\mathcal{V} \subset A^*_{p/q}(\mathbb{R}^d)$ with $\sup_{w \in \mathcal{V}} [w]_{A^*_{p/q}(\mathbb{R}^d)} < \infty$ we have $\widetilde{R}_q(\mathcal{D}^d) \subset M_p(\mathbb{R}^d, w)$ $(w \in \mathcal{V})$ and*

$$\sup\left\{ \|T_{m\chi_I}\|_{p,w} : m \in \widetilde{R}_q(\mathcal{D}^d),\ \|m\|_{\widetilde{R}_q(\mathcal{D}^d)} \leq 1,\ w \in \mathcal{V},\ I \in \mathcal{D}^d \right\} < \infty.$$

Here $\widetilde{R}_q(\mathcal{D}^d)$ $(q \geq 1)$ stands for the space of all functions m defined on $\mathbb{R}^d$ such that $m\chi_I \in \widetilde{R}_q(I)$ for every $I \in \mathcal{D}^d$ and $\sup_{I \in \mathcal{D}^d} \|m\chi_I\|_{\widetilde{R}_q(I)} < \infty$. Define $\widetilde{V}_q(\mathcal{D}^d)$ similarly.

The classes $\widetilde{R}_q(\mathcal{D}^d)$ and $A^*_p(\mathbb{R}^d)$ are well adapted to iterate one-dimensional arguments from the proof of Theorem A(i). Therefore, below we give only main supplementary observations should be made.

Proof of Lemma 12. We proceed by induction on d. The proof of the statement of Lemma 12 for $d = 1$ and $p = 2$ is provided in the proof of Theorem A(i). The general case of $d = 1$ and $p > q$ follows from this special one by means of Rubio de Francia's extrapolation theorem; see Lemma 7.

Assume that the statement holds for $d \geq 1$; we will prove it for $d+1$. Let $m \in \widetilde{R}_q(\mathcal{D}^{d+1})$ with $\|m\|_{\widetilde{R}_q(\mathcal{D}^{d+1})} \leq 1$. By approximation, we can assume that $m_I \in \mathcal{R}_q(I_1; \widetilde{R}_q(I_2 \times \cdots \times I_{d+1}))$ for every $I := I_1 \times \cdots \times I_{d+1} \in \mathcal{D}^{d+1}$. Set $m_I := \sum_{J \in \mathcal{I}_I} \gamma_{I,J} a_{I,J} \chi_J$, where $\gamma_{I,J} \geq 0$ with $\sum_J \gamma^q_{I,J} \leq 1$ and $a_{I,J} \in \widetilde{R}_q(I_2 \times \cdots \times I_{d+1})$ with $\|a_{I,J}\|_{\widetilde{R}_q(I_2 \times \cdots \times I_{d+1})} = 1$ for every $I \in \mathcal{D}^{d+1}$. Here $\mathcal{I}_I$ stands for a decomposition of I_1 corresponding to m_I.

Let $q \in (1,2]$, $p \geq q'$ and $\mathcal{V}_{q,p} \subset A^*_{p/q}(\mathbb{R}^{d+1})$ with $\sup_{w \in \mathcal{V}} [w]_{A^*_{p/q}(\mathbb{R}^{d+1})} < \infty$. By Lebesque's differentiation theorem, for every $w \in A^*_r(\mathbb{R}^{d+1})$ $(r > 1)$ one can easily show that $w(\cdot, y) \in A_{p/q}(\mathbb{R})$, $w(x, \cdot) \in A^*_{p/q}(\mathbb{R}^d)$, and $[w(\cdot, y)]_{A_{p/q}(\mathbb{R})}, [w(x, \cdot)]_{A^*_{p/q}(\mathbb{R}^d)} \leq [w]_{A^*_{p/q}(\mathbb{R}^{d+1})}$ for almost every $y \in R^d$ and $x \in \mathbb{R}$; see, e.g., [17, Lemma 2.2].

Therefore, by induction assumption, for every $q \in (1,2]$ and $p \in [q', \infty) \setminus \{2\}$ there exists a constant $C_{q,p} > 0$ independent of m and $w \in \mathcal{V}_{q,p}$ such that for every $w \in \mathcal{V}_{q,p}$:

$$(13) \quad \sup\left\{ \|T_{a_{I,J}}\|_{p,w(x,\cdot)} : J \in \mathcal{I}_I,\ I \in \mathcal{D}^{d+1} \right\} \leq C_{q,p} \qquad \text{for a.e. } x \in \mathbb{R}.$$

Let $f(x,y) := \phi(x)\rho(y)$ $((x,y)\in\mathbb{R}^{d+1})$, where $\phi\in S(\mathbb{R})$ and $\rho\in S(\mathbb{R}^d)$. Note that the set of functions of this form is dense in $L^{q'}(\mathbb{R}^{d+1},w)$. Indeed, by the strong doubling and open ended properties of A_p^*-weights, we get $(1+|\cdot|)^{-dr}w\in L^1(\mathbb{R}^d)$ $(r>1, w\in A_r^*(\mathbb{R}^d))$; see, e.g., [27, Chapter IX, Proposition 4.5]. Hence, this claim follows from the standard density arguments. Moreover, we have $T_{m_I}f=\sum_J \gamma_{I,J}S_J\phi T_{a_{I,J}}\rho$. In the sequel, we consider the case of $q\in(1,2)$ and $q=2$ separately. For $q\in(1,2)$, by Fubini's theorem, we get

$$\|T_{m_I}f\|_{q',w}^{q'} \le \sum_{J\in\mathcal{I}_I}\int_{\mathbb{R}}|S_J\phi|^{q'}\int_{\mathbb{R}^d}|T_{a_{I,J}}\rho|^{q'}w\,dy\,dx \qquad (w\in\mathcal{V}_{q,q'}, I\in\mathcal{D}^{d+1})$$

Therefore, by Theorem B(i) and (13), we conclude that

$$\sup\Big\{\|T_{m_I}\|_{q',w} : w\in\mathcal{V}_{q,q'},\ m\in\widetilde{R}_q(\mathcal{D}^{d+1}),\ \|m\|_{\widetilde{R}_q(\mathcal{D}^{d+1})}\le 1, \\ I\in\mathcal{D}^{d+1}\Big\}<\infty.$$

Consequently, by Rubio de Francia's extrapolation algorithm, see [26, Theorem 3] or [10, Chapter 3], the same conclusion holds for all $p>q$.

For $q=2$, by Fubini's theorem and Minkowski's inequality, we conclude that

$$\|T_{m_I}f\|_{p,w}^p \le \int_{\mathbb{R}}|S^{\mathcal{I}_I}\phi(x)|^p\left(\sum_{J\in\mathcal{I}_I}\gamma_{I,J}^2\|T_{a_{I,J}}\rho\|_{p,w(x,\cdot)}^2\right)^{\frac{p}{2}}dx \\ (w\in\mathcal{V}_{2,p}, I\in\mathcal{D}^{d+1})$$

for every $p>2$. Hence, by Theorem 2 and (13), we get the statement of Lemma 12 also for $q=2$. □

Proof of Theorem C. Note first that for every $\mathcal{V}\subset A_1^*(\mathbb{R}^d)$ with $\sup_{w\in\mathcal{V}}[w]_{A_1^*}<\infty$, by the reverse Hölder inequality, there exists $s>1$ such that $w^s\in A_1^*(\mathbb{R}^d)$ $(w\in\mathcal{V})$ and $\sup_{p\ge 2, w\in\mathcal{V}}[w^s]_{A_{p/2}^*}<\infty$. Thus, by Lemma 12 and an interpolation argument similar to that in the proof of Theorem A(i), we get

$$\sup\Big\{\|T_{m\chi_I}\|_{2,w} : w\in\mathcal{V},\ m\in\widetilde{R}_2(\mathcal{D}^d),\ \|m\|_{\widetilde{R}_2(\mathcal{D}^d)}\le 1,\ I\in\mathcal{D}^d\Big\}<\infty.$$

Therefore, as in the proof of Theorem A(i), one can show that for every $q\in(1,2]$ and every subset $\mathcal{V}\subset A_{2/q}^*(\mathbb{R}^d)$ with $N:=\sup_{w\in\mathcal{V}}[w]_{A_{2/q}^*}<$

∞, there exists a constant $\alpha = \alpha(d, q, N) > 1$ such that $\widetilde{R}_{\alpha q}(\mathcal{D}^d) \subset M_2(\mathbb{R}^d, w)$ $(w \in \mathcal{V})$ and

$$\sup\left\{\|T_{m\chi_I}\|_{2,w} : m \in \widetilde{R}_{\alpha q}(\mathcal{D}^d),\ \|m\|_{\widetilde{R}_{\alpha q}(\mathcal{D}^d)} \leq 1,\ w \in \mathcal{V},\ I \in \mathcal{D}^d\right\} < \infty.$$

Now, by means of (12), Kurtz' weighted variant of Littlewood–Paley's inequalities, [17, Theorem 1], and Rubio de Francia's extrapolation theorem, [26, Theorem 3], the rest of the proof of (i) runs analogously to the corresponding part of the proof of Theorem A(i).

Consequently, by (i), the proof of the part (ii) follows the lines of the proof of Theorem A(ii). □

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