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de Rham and Dolbeault cohomology of solvmanifolds with local systems

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Let G be a simply connected solvable Lie group with a lattice Γ and the Lie algebra $\mathfrak{g}$ and a representation $\rho : G \to GL(V_\rho)$ whose restriction on the nilradical is unipotent. Consider the flat bundle E_ρ given by ρ. By using "many" characters $\{\alpha\}$ of G and "many" flat line bundles $\{E_\alpha\}$ over G/Γ, we show that an isomorphism

$$\bigoplus_{\{\alpha\}} H^*(\mathfrak{g}, V_\alpha \otimes V_\rho) \cong \bigoplus_{\{E_\alpha\}} H^*(G/\Gamma, E_\alpha \otimes E_\rho)$$

holds. This isomorphism is a generalization of the well-known fact: "If G is nilpotent and ρ is unipotent then, the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ holds". By this result, we construct an explicit finite-dimensional cochain complex which compute the cohomology $H^*(G/\Gamma, E_\rho)$ of solvmanifolds even if the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ does not hold. For Dolbeault cohomology of complex parallelizable solvmanifolds, we also prove an analogue of the above isomorphism result which is a generalization of computations of Dolbeault cohomology of complex parallelizable nilmanifolds. By this isomorphism, we construct an explicit finite-dimensional cochain complex which compute the Dolbeault cohomology of complex parallelizable solvmanifolds.

1. Background and main results

1.1. Background

We have nice theorem for de Rham cohomology of nilmanifolds with local systems.

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Theorem 1.1 (due to [12] or [15]). *Let N be a simply connected real nilpotent Lie group and $\mathfrak{n}$ the Lie algebra of N. Suppose N has a lattice Γ. Let $\rho : N \to GL(V_\rho)$ be a finite-dimensional unipotent representation. We define the flat bundle $E_\rho = (N \times V_\rho)/\Gamma$ given by the equivalent relation $(\gamma g, \rho(\gamma)v) \cong (g, v)$ for $g \in N$, $v \in V_\rho$, $\gamma \in \Gamma$. Consider the cochain complex $\bigwedge \mathfrak{n}^*_{\mathbb{C}} \otimes V_\rho$ of Lie algebra (see [13]) and the canonical inclusion*

$$\bigwedge \mathfrak{n}^*_{\mathbb{C}} \otimes V_\rho \to A^*(N/\Gamma, E_\rho).$$

Then this inclusion induces a cohomology isomorphism

$$H^*(\mathfrak{n}, V_\rho) \cong H^*(N/\Gamma, E_\rho).$$

Some researchers tried to extend Theorem 1.1 for solvmanifolds. In fact, it is proved that for a simply connected solvable Lie group G with the Lie algebra $\mathfrak{g}$ admitting a lattice Γ and a representation $\rho : G \to GL(V_\rho)$, if:

(H) ([6]) The representation $\rho \oplus \mathrm{Ad}$ is triangular or,

(M) ([11]) The two images $(\rho \oplus \mathrm{Ad})(G)$ and $(\rho \oplus \mathrm{Ad})(\Gamma)$ have same Zariski-closure in $GL(V_\rho) \times \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$,

then the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ holds. However, in general the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(N/\Gamma, E_\rho)$ does not hold.

As an Analogue of Theorem 1.1 we have the following theorem for Dolbeault cohomology of complex parallelizable nilmanifolds.

Theorem 1.2 (due to [16]). *Let N be a simply connected complex nilpotent Lie group and $\mathfrak{n}$ the Lie algebra (as a complex Lie algebra) of N. Suppose N has a lattice Γ. Let $\sigma : N \to GL(V_\sigma)$ be a finite-dimensional holomorphic unipotent representation. We also consider the anti-holomorphic representation $\bar{\sigma} : N \to GL(V_{\bar{\sigma}})$. Define the flat holomorphic vector bundle $L_{\bar{\sigma}} = (N \times V_{\bar{\sigma}})/\Gamma$ over G/Γ given by the equivalent relation $(\gamma g, \bar{\sigma}(\gamma)v) \cong (g, v)$ for $g \in N$, $v \in V_\sigma$, $\gamma \in \Gamma$. We consider the Dobeault complex $(A^{*,*}(N/\Gamma, L_{\bar{\sigma}}), \bar{\partial})$. We regard $\bigwedge \mathfrak{n}^* \otimes V_\sigma$ as the subcomplex of $(A^{0,*}(N/\Gamma, L_{\bar{\sigma}}), \bar{\partial})$ which consists of the left-invariant "anti"-holomorphic forms with values in $L_{\bar{\sigma}}$. Then the inclusion*

$$\bigwedge \mathfrak{n}^* \otimes V_\sigma \to A^{0,*}(N/\Gamma, L_{\bar{\sigma}})$$

induces a cohomology isomorphism

$$H^*(\mathfrak{n}, V_\sigma) \cong H^{0,*}_{\bar{\partial}}(N, L_{\bar{\sigma}}).$$

Hence since N/Γ is complex parallelizable, we have an isomorphism

$$\bigwedge \mathbb{C}^{\dim N} \otimes H^*(\mathfrak{n}, V_\sigma) \cong H^{*,*}_{\bar\partial}(N, L_{\bar\sigma}).$$

It is desired that Theorems 1.1 and 1.2 are generalized for solvmanifolds and we can compute the de Rham and Dolbeault cohomology of solvmanifolds even if the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ (resp. $\bigwedge \mathbb{C}^{\dim N} \otimes H^*(\mathfrak{n}, V_\sigma) \cong H^{*,*}_{\bar\partial}(N, L_{\bar\sigma})$) does not holds.

1.2. Main results

The first purpose of this paper is to show new-type cohomology isomorphism theorems for solvmanifolds which are generalizations of Theorems 1.1 and 1.2. These analogous each other. We consider the "many" characters of G and "many" line bundles over G/Γ. In this paper, we prove:

Theorem 1.3. *Let G be a simply connected real solvable Lie group with a lattice Γ and $\mathfrak{g}$ the Lie algebra of G. Let N be the nilradical (i.e., maximal connected nilpotent normal subgroup) of G. Let $\mathcal{A}_{(G,N)} = \{\alpha \in \mathrm{Hom}(G, \mathbb{C}^*) \mid \alpha_{|_N} = 1\}$ and $\mathcal{A}_{(G,N)}(\Gamma)$ the set $\{E_\alpha\}$ of all the isomorphism classes of flat line bundles given by $\{V_\alpha\}_{\alpha\in\mathcal{A}_{(G,N)}}$. Let $\rho : G \to GL(V_\rho)$ be a representation. For the nilradical N of G, we assume that the restriction $\rho_{|_N}$ is a unipotent representation. We consider the direct sum*

$$\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho$$

of the Lie algebra cochain complexes. We also consider the direct sum

$$\bigoplus_{E_\alpha\in\mathcal{A}_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).$$

Then the inclusion

$$\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho \to \bigoplus_{E_\alpha\in\mathcal{A}_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho)$$

induces a cohomology isomorphism

$$\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}} H^*(\mathfrak{g}, V_\alpha \otimes V_\rho) \cong \bigoplus_{E_\alpha\in\mathcal{A}_{(G,N)}(\Gamma)} H^*(G/\Gamma, E_\alpha \otimes E_\rho).$$

We also prove:

Theorem 1.4. *Let G be a simply connected complex solvable Lie group with a lattice Γ and $\mathfrak{g}$ the Lie algebra (as a complex Lie algebra) of G. Let N be the nilradical of G. Let $\mathcal{B}_{(G,N)} = \{\alpha \in \mathrm{Hom}_{\mathrm{hol}}(G, \mathbb{C}^*) | \alpha_{|_N} = 1\}$ and $\mathcal{B}_{(G,N)}(\Gamma)$ the set $\{L_{\bar{\alpha}}\}$ of all the isomorphism classes of holomorphic line bundles given by $\{V_{\bar{\alpha}}\}_{\alpha \in \mathcal{B}_{(G,N)}}$. Let $\sigma : G \to GL(V_\sigma)$ be a holomorphic representation. For the nilradical N of G, we assume that the restriction $\sigma_{|_N}$ is a unipotent representation. We consider the direct sum*

$$\bigoplus_{\alpha \in \mathcal{B}_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma$$

of the Lie algebra cochain complexes. We also consider the direct sum

$$\bigoplus_{L_{\bar{\alpha}} \in \mathcal{B}_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}})$$

of Dolbeault complexes.

Then the inclusion

$$\bigoplus_{\alpha \in \mathcal{B}_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma \to \bigoplus_{L_{\bar{\alpha}} \in \mathcal{B}_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}})$$

induces a cohomology isomorphism

$$\bigoplus_{\alpha \in \mathcal{B}_{(G,N)}} H^*(\mathfrak{g}, V_\alpha \otimes V_\sigma) \cong \bigoplus_{L_{\bar{\alpha}} \in \mathcal{B}_{(G,N)}(\Gamma)} H^{0,*}(G/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}}).$$

Remark 1. The correspondence $\mathcal{A}_{(G,N)} \to \mathcal{A}_{(G,N)}(\Gamma)$ (resp. $\mathcal{B}_{(G,N)} \to \mathcal{B}_{(G,N)}(\Gamma)$) is not 1 to 1. This remark is very important for the case the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ (resp. $H^*(\mathfrak{g}, V_\sigma) \cong H^{0,*}(G/\Gamma, L_{\bar{\sigma}})$) does not hold.

The second purpose of this paper is to construct a explicit finite- dimensional cochain complex which compute the de Rham cohomology $H^*(G/\Gamma, E_\rho)$ and the Dolbeault cohomology $H^{0,*}(G/\Gamma, L_{\bar{\sigma}})$ using Theorems 1.3 and 1.4. We prove:

Theorem 1.5. *Let G be a simply connected real (resp complex) solvable Lie group and $\mathfrak{g}$ the Lie algebra of G. Define $\mathcal{A}_{(G,N)}$ (resp. $\mathcal{B}_{(G,N)}$) as in Theorem* 1.3 *(resp Theorem 1.4). Let $\rho : G \to GL(V_\rho)$ (resp. $\sigma : G \to GL(V_\sigma)$) be a* *representation with the assumption of Theorem* 1.3 *(resp. Theorem* 1.4*). We consider the direct sum*

$$\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}$$

(resp.

$$\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}$$

) of the Lie algebra cochain complexes.

Then there exists a finite-dimensional subcomplex

$$A^{*}\subset\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}$$

(resp.

$$B^{*}\subset\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}$$

) such that the inclusion induces a cohomology isomorphism.

By Theorem 1.3 (resp. Theorem 1.4), we have the inclusion

$$\iota : A^{*}\to\bigoplus_{E_{\alpha}\in\mathcal{A}_{(G,N)}(\Gamma)}A^{*}(G/\Gamma,E_{\alpha}\otimes E_{\rho})$$

(resp.

$$\iota : B^{*}\to\bigoplus_{L_{\bar\alpha}\in\mathcal{A}_{(G,N)}(\Gamma)}A^{0,*}(G/\Gamma,L_{\bar\alpha}\otimes L_{\bar\sigma})$$

) inducing a cohomology isomorphism. Hence we have:

Corollary 1.6. *Let* $A^{*}_{\Gamma}=\iota^{-1}(A^{*}(G/\Gamma,E_{\rho}))$ *(resp.* $B^{*}_{\Gamma}=\iota^{-1}(A^{0,*}(G/\Gamma,L_{\bar\sigma}))$*). Then we have an isomorphism*

$$H^{*}(A^{*}_{\Gamma})\cong H^{*}(G/\Gamma,E_{\rho})$$

(resp.

$$H^{*}(B^{*}_{\Gamma})\cong H^{0,*}(G/\Gamma,L_{\bar\sigma})$$

and hence $\bigwedge\mathbb{C}^{\dim G}\otimes H^{*}(B^{*}_{\Gamma})\cong H^{*,*}(G/\Gamma,L_{\bar\sigma})$*).*

Consider the adjoint representation Ad. Then the restriction $\mathrm{Ad}_{|_N}$ is unipotent. Hence by the above cochain complex A^*_Γ, we can compute the cohomology $H^*(G/\Gamma, E_{\mathrm{Ad}})$ on general solvmanifolds. The cohomology

$$H^*(G/\Gamma, E_{\mathrm{Ad}}) \cong H^*(\Gamma, \mathrm{Ad})$$

is important for studying the deformation of lattice Γ in G.

2. Preliminary: Jordan decompositions of representations

Let $A \in GL_n(\mathbb{C})$. We denote by A_s (resp. A_u) the semi-simple (resp. unipotent) part of A for the Jordan decomposition (see [8] for the definition). We will use the following facts.

Lemma 2.1. *Let N be a simply connected nilpotent Lie group and $\varphi : N \to GL(V_\varphi)$ a representation. Then the map $\varphi' : N \ni g \to (\varphi(g))_s$ is also a representation (see* [2]*). Since $\varphi'(N)$ is connected nilpotent group and consists of semi-simple elements, the Zariski-closure of $\varphi'(N)$ is an algebraic torus (see* [8, *Section* 19]*) and hence φ' is diagonalizable.*

3. Proof of Theorem 1.3

3.1. Cohomology of tori

Let A be a simply connected real abelian Lie group with a lattice Γ and $\mathfrak{a}$ the Lie algebra of A.

Lemma 3.1. *Let $\rho : A \to GL(V_\rho)$ be a representation. Suppose $\rho = \beta \otimes \phi$ such that β is a character of A and ϕ is a unipotent representation. Then we have:*

If β is non-trivial, then we have

$$H^*(\mathfrak{a}, V_\rho) = 0.$$

If the flat line bundle E_β is non-trivial, then we have

$$H^*(A/\Gamma, E_\rho) = 0.$$

Proof. Suppose $\dim V_\rho = 1$. Then if β is non-trivial, we can show $H^*(\mathfrak{a}, V_\rho) = H^*(\mathfrak{a}, V_\beta) = 0$ by simple computation and if E_β is non-trivial, then we have

$$H^*(A/\Gamma, E_\rho) = H^*(\Gamma, \beta) = 0$$

by [10, Lemma 2.1].

In case $\dim V_\sigma = n > 1$, by the triangulation of ρ, we have a $(n-1)$-dimensional A-submodule $V_{\rho'}$ such that $V_\rho/V_{\rho'} = V_\beta$. Then by the long exact sequence of cohomology of Lie algebra or group (see [13]), the lemma follows inductively. □

Lemma 3.2. *Let $\rho : A \to GL(V_\rho)$ be a representation. Then we have a basis of V_ρ such that ρ is represented by*

$$\rho = \bigoplus_{i=1}^{k} \alpha_i \otimes \phi_i$$

for characters α_i of G and unipotent representations ϕ_i of G.

Proof. For a character α, we denote by W_α the subspace of V_ρ consisting of the elements $w \in V_\rho$ such that for some positive integer n we have $(\rho(a) - \alpha(a)I)^n w = 0$ for any $a \in A$. Since A is abelian, we have a decomposition

$$V_\rho = W_{\alpha_1} \oplus \cdots \oplus W_{\alpha_k}$$

by generalized eigenspace decomposition of $\rho(a)$ for all $a \in A$. Let $\rho_i(a) = (\rho(a))_{|W_{\alpha_i}}$. Then we have $\rho = \rho_1 \oplus \cdots \oplus \rho_k$. We have $(\rho_i(a))_s = \alpha_i I$. Let $\phi_i(a) = (\rho_i(a))_u$. By Lemma 2.1, ϕ_i is a unipotent representation and we have $\rho_i(a) = (\rho_i(a))_s(\rho_i(a))_u = (\alpha_i \otimes \phi_i)(a)$. Hence the Lemma follows. □

Let $\{V_\alpha\}_{\alpha \in \mathrm{Hom}(A,\mathbb{C}^*)}$ be the set of all one-dimensional representations of A and $\mathcal{H}(A/\Gamma) = \{E_\beta\}$ the set of all the isomorphism classes of flat line bundles given by $\{V_\alpha\}_{\alpha \in \mathrm{Hom}(A,\mathbb{C}^*)}$. We notice that the correspondence $\{V_\alpha\}_{\alpha \in \mathrm{Hom}(A,\mathbb{C}^*)} \to \mathcal{H}(A/\Gamma)$ is not injective. We consider the direct sums

$$\bigoplus_{\alpha \in \mathrm{Hom}(A,\mathbb{C}^*)} \bigwedge \mathfrak{a}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho$$

and

$$\bigoplus_{E_\alpha \in \mathcal{H}(A/\Gamma)} A^*(A/\Gamma, E_\alpha \otimes E_\rho).$$

Proposition 3.3. *The inclusion*

$$\bigoplus_{\alpha\in \mathrm{Hom}(A,\mathbb{C}^{*})}\bigwedge \mathfrak{a}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\to \bigoplus_{E_{\alpha}\in \mathcal{H}(A/\Gamma)} A^{*}(A/\Gamma, E_{\alpha}\otimes E_{\rho})$$

induces a cohomology isomorphism.

Proof. Consider the decomposition

$$\sigma=\bigoplus_{i=1}^{k}\alpha_{i}\otimes\phi_{i}$$

as the above lemma. Then we have

$$\bigoplus_{\alpha\in \mathrm{Hom}(A,\mathbb{C}^{*})}\bigwedge \mathfrak{a}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}=\bigoplus_{\alpha\in \mathrm{Hom}(A,\mathbb{C}^{*})}\bigwedge \mathfrak{a}^{*}\otimes \bigoplus_{i=1}^{k} V_{\alpha\alpha_{i}}\otimes V_{\phi_{i}}$$

and

$$\bigoplus_{E_{\alpha}\in \mathcal{H}(A/\Gamma)} A^{*}(A/\Gamma, E_{\alpha}\otimes E_{\rho})=\bigoplus_{E_{\alpha}\in \mathcal{H}(A/\Gamma)} A^{*}(A/\Gamma, \bigoplus_{i=1}^{k} E_{\alpha}\otimes E_{\alpha_{i}}\otimes E_{\phi_{i}}).$$

By Theorem 1.1 and Lemma 3.1, we have

$$\begin{aligned} H^{*}\left(\bigoplus_{E_{\alpha}\in \mathcal{H}(A/\Gamma)} A^{*}(A/\Gamma, E_{\alpha}\otimes E_{\rho})\right) &\cong H^{*}\left(A/\Gamma, \bigoplus_{i=1}^{k} E_{\phi_{i}}\right)\\ &\cong H^{*}\left(\mathfrak{a}, \bigoplus_{i=1}^{k} V_{\phi_{i}}\right). \end{aligned}$$

By Lemma 3.1 we have

$$H^{*}\left(\bigoplus_{\alpha\in \mathrm{Hom}(A,\mathbb{C}^{*})}\bigwedge \mathfrak{a}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right)\cong H^{*}\left(\mathfrak{a}, \bigoplus_{i=1}^{k} V_{\phi_{i}}\right).$$

Hence the proposition follows. □

3.2. Mostow bundle and spectral sequence

Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. Let N be the nilradical of G. It is known that $\Gamma\cap N$

is a lattice of N and $\Gamma/\Gamma\cap N$ is a lattice of the abelian Lie group G/N (see [15]). The solvmanifold G/Γ is a fibre bundle

$$N/\Gamma\cap N = N\Gamma/\Gamma \longrightarrow G/\Gamma \longrightarrow G/N\Gamma = (G/N)/(\Gamma/\Gamma\cap N)$$

over a torus with a nilmanifold $N/\Gamma\cap N$ as fibre. We call this fibre bundle the Mostow bundle of G/Γ. The structure group is $N\Gamma/\Gamma_0$ as left translations where Γ_0 is the largest normal subgroup of Γ which is normal in $N\Gamma$ (see [17]).

Let $\rho : G \to GL(V_\rho)$ be a representation such that the restriction $\rho_{|_N}$ is a unipotent representation. For the Mostow bundle $p : G/\Gamma \to (G/N)/(\Gamma/\Gamma\cap N)$, we define the vector bundle

$$\mathbf{H}^q(N/\Gamma\cap N) = \sqcup_{x\in(G/N)/(\Gamma/\Gamma\cap N)} H^q(p^{-1}(x), E_\rho)$$

over the torus $(G/N)/(\Gamma/\Gamma\cap N)$. By Theorem 1.1, we have $H^q(p^{-1}(x), E_\rho) \cong H^q(\mathfrak{n}, V_\rho)$. Hence let $\Lambda_q : G/N \to GL(H^q(\mathfrak{n}, V_\rho))$ be the representation induced by the extension $1 \to N \to G \to G/N \to 1$, then we can regard $\mathbf{H}^q(N/\Gamma\cap N)$ as the flat bundle E_{Λ_q}. We consider the filtration

$$F^p \bigwedge^{p+q} \mathfrak{g}^*_{\mathbb{C}} = \left\{\omega \in \bigwedge^{p+q} \mathfrak{g}^*_{\mathbb{C}} | \omega(X_1,\dots,X_{p+q}) = 0 \text{ for } X_1,\dots,X_{p+1} \in \mathfrak{n}_{\mathbb{C}}\right\}.$$

This filtration gives the filtration of the cochain complex $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho$ and the filtration of the de Rham complex $A^*(G/\Gamma, E_\rho)$. We consider the spectral sequence $E^{*,*}_*(\mathfrak{g})$ of $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho$ and the spectral sequence $E^{*,*}_*(G/\Gamma)$ of $A^*(G/\Gamma, E_\rho)$. Set $G/N = A$ and $\Gamma/\Gamma\cap N = \Delta$ and $\mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. Then we have the commutative diagram

$$\begin{array}{ccc} E_1^{*,q}(\mathfrak{g}) & \longrightarrow & E_1^{*,q}(G/\Gamma) \\ \downarrow\cong & & \downarrow\cong \\ \bigwedge \mathfrak{a}^*_{\mathbb{C}} \otimes V_{\Lambda_q} & \longrightarrow & A^*(A/\Delta, E_{\Lambda_q}) \end{array}$$

(see [6], [15, Section 7]).

3.3. Proof of Theorem 1.3

Proof. Consider the spectral sequence $E_*^{*,*}(\mathfrak{g})$ of

$$\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha \otimes V_\rho$$

and the spectral sequence $E_*^{*,*}(G/\Gamma)$ of

$$\bigoplus_{E_\alpha\in\mathcal{A}_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha\otimes E_\rho).$$

Set $A = G/N$ and $\Delta = \Gamma/\Gamma\cap N$ and $\mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. Since we can identify $\mathcal{A}_{(G,N)}$ (resp. $\mathcal{A}_{(G,N)}(\Gamma)$) with $\mathrm{Hom}(A,\mathbb{C}^*)$ (resp. $\mathcal{H}(A/\Delta)$), we have the commutative diagram

$$\begin{array}{ccc} E_1^{*,q}(\mathfrak{g}) & \longrightarrow & E_1^{*,q}(G/\Gamma) \\ \downarrow\cong & & \downarrow\cong \\ \bigoplus_{\alpha\in\mathrm{Hom}(A,\mathbb{C}^*)} \bigwedge \mathfrak{a}_{\mathbb{C}}^*\otimes V_\alpha\otimes V_{\Lambda_q} & \longrightarrow & \bigoplus_{E_\alpha\in\mathcal{H}(A/\Delta)} A^*(A/\Delta, E_\alpha\otimes E_{\Lambda_q}). \end{array}$$

By Proposition 3.3, the homomorphism $E_1^{*,*}(\mathfrak{g})\to E_1^{*,*}(G/\Gamma)$ induces a cohomology isomorphism and hence we have an isomorphism $E_2^{*,*}(\mathfrak{g})\cong E_2^{*,*}(G/\Gamma)$. Hence the theorem follows. □

4. Proof of Theorem 1.4

4.1. Dolbeault cohomology of tori

First we prove Theorem 1.2 by Sakane's Theorem [16].

Proof of Theorem 1.2. In case $\dim V_\sigma = 1$, σ is trivial and the theorem follows from Sakane's Theorem [16].

In case $\dim V_\sigma = n > 1$, since σ is unipotent, we have a $(n-1)$-dimensional G-submodule $V_{\sigma'}\subset V_\sigma$ such that $V_\sigma/V_{\sigma'}$ is the trivial submodule. Then we have the spectral sequences

$$0\longrightarrow \bigwedge \mathfrak{g}^*\otimes V_{\sigma'}\longrightarrow \bigwedge \mathfrak{g}^*\otimes V_\sigma\longrightarrow \bigwedge \mathfrak{g}^*\otimes V_\sigma/V_{\sigma'}\longrightarrow 0$$

and

$$0\longrightarrow A^{0,*}(G/\Gamma, L_{\bar\sigma'})\longrightarrow A^{0,*}(G/\Gamma, L_{\bar\sigma})\longrightarrow A^{0,*}(G/\Gamma, L_{\bar\sigma}/L_{\bar\sigma'})\longrightarrow 0.$$

We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge \mathfrak{g}^* \otimes V_{\sigma'} & \longrightarrow & \bigwedge \mathfrak{g}^* \otimes V_{\sigma} & \longrightarrow & \bigwedge \mathfrak{g}^* \otimes V_{\sigma}/V_{\sigma'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^{0,*}(G/\Gamma, L_{\bar{\sigma}'}) & \longrightarrow & A^{0,*}(G/\Gamma, L_{\bar{\sigma}}) & \longrightarrow & A^{0,*}(G/\Gamma, L_{\bar{\sigma}}/L_{\bar{\sigma}'}) & \longrightarrow & 0. \end{array}$$

Considering the long exact sequence of cohomologies, by the five lemma, the theorem follows inductively. □

Let A be a simply connected complex abelian group with a lattice Γ and $\mathfrak{a}$ the Lie algebra of A.

Lemma 4.1. *Let $\sigma : A \to GL(V_\sigma)$ be a holomorphic representation. Suppose $\sigma = \beta \otimes \phi$ such that β is a character of A and ϕ is a unipotent representation. Then we have:*

If β is non-trivial, then we have

$$H^*(\mathfrak{a}, V_\sigma) = 0.$$

If the holomorphic line bundle $L_{\bar{\beta}}$ is non-trivial, then we have

$$H^{0,*}(A/\Gamma, L_{\bar{\sigma}}) = 0$$

Proof. In case $\dim V_\sigma = 1$, the lemma is proved in [10].

In case $\dim V_\sigma = n > 1$, by the triangulation of σ, we have a $(n-1)$-dimensional A-submodule $V_{\sigma'}$ such that $V_\sigma/V_{\sigma'} = V_\beta$. Then we have the exact sequence

$$0 \longrightarrow A^{0,*}(A/\Gamma, L_{\bar{\sigma}'}) \longrightarrow A^{0,*}(A/\Gamma, L_{\bar{\sigma}}) \longrightarrow A^{0,*}(A/\Gamma, L_{\bar{\sigma}}/L_{\bar{\sigma}'}) \longrightarrow 0.$$

Considering the long exact sequence of cohomologies, the lemma follows inductively. □

By similar proof of Lemma 3.2, we have the following lemma.

Lemma 4.2. *Let $\sigma : A \to GL(V_\sigma)$ be a holomorphic representation. Then we have a basis of V_σ such that σ is represented by*

$$\sigma = \bigoplus_{i=1}^{k} \alpha_i \otimes \phi_i$$

for holomorphic characters α_i and holomorphic unipotent representations ϕ_i.

Let $\{V_\alpha\}_{\alpha\in \mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)}$ be the set of all one-dimensional holomorphic representations of A and $\mathcal{H}_{\mathrm{hol}}(A/\Gamma)=\{L_{\bar\alpha}\}$ the set of all the isomorphism classes of holomorphic line bundles given by $\{V_\alpha\}_{\alpha\in \mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)}$. We notice that the correspondence $\{V_\alpha\}_{\alpha\in \mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)}\to \mathcal{H}(A/\Gamma)$ is not injective. We consider the direct sums

$$\bigoplus_{\alpha\in \mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)} \bigwedge \mathfrak{a}^*\otimes V_\alpha\otimes V_\sigma$$

and

$$\bigoplus_{L_{\bar\alpha}\in \mathcal{H}_{\mathrm{hol}}(A/\Gamma)} A^{0,*}(A/\Gamma, L_{\bar\alpha}\otimes L_{\bar\sigma}).$$

Proposition 4.3. *The inclusion*

$$\bigoplus_{\alpha\in \mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)} \bigwedge \mathfrak{a}^*\otimes V_\alpha\otimes V_\sigma \to \bigoplus_{L_{\bar\alpha}\in \mathcal{H}_{\mathrm{hol}}(A/\Gamma)} A^{0,*}(A/\Gamma, L_{\bar\alpha}\otimes L_{\bar\sigma})$$

induces a cohomology isomorphism.

Proof. Using Theorem 1.2 and Lemmas 4.1 and 4.2, we can prove the proposition by similar argument of the proof of Proposition 3.3 □

4.2. Mostow bundle and spectral sequence

Let G be a simply connected complex solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. Then the Mostow bundle

$$N/\Gamma\cap N = N\Gamma/\Gamma \longrightarrow G/\Gamma \longrightarrow G/N\Gamma = (G/N)/(\Gamma/\Gamma\cap N)$$

is holomorphic.

Let $\sigma : G\to GL(V_\sigma)$ be a representation such that the restriction $\sigma_{|_N}$ is a unipotent representation. For the Mostow bundle $p: G/\Gamma\to (G/N)/(\Gamma/\Gamma\cap N)$, we define the vector bundle

$$\mathbf{H}^{0,q}(N/\Gamma\cap N)=\sqcup_{x\in (G/N)/(\Gamma/\Gamma\cap N)} H^{0,q}(p^{-1}(x), L_{\bar\sigma})$$

over the torus $(G/N)/(\Gamma/\Gamma\cap N)$. By Theorem 1.2, we have $H^{0,q}(p^{-1}(x), L_{\bar\sigma}) \cong H^q(\mathfrak{n}, V_\sigma)$. Hence let $\Lambda_q : G/N\to GL(H^q(\mathfrak{n}, V_\sigma))$ be the representation induced by the extension $1\to N\to G\to G/N\to 1$, then we can regard

$\mathbf{H}^{0,q}(N/\Gamma\cap N)$ as the flat holomorphic bundle $L_{\bar{\Lambda}}$. We consider the filtration

$$F^p\bigwedge^{p+q}\mathfrak{g}^* = \left\{\omega\in\bigwedge^{p+q}\mathfrak{g}^* | \omega(X_1,\dots,X_{p+q})=0 \text{ for } X_1,\dots,X_{p+1}\in\mathfrak{n}\right\}.$$

This filtration gives the filtration of the cochain complex $\bigwedge\mathfrak{g}^*\otimes V_\sigma$ and the filtration of the Dolbeault complex $A^{0,*}(G/\Gamma,L_{\bar{\sigma}})=C^\infty(G/\Gamma,L_{\bar{\sigma}})\otimes\bigwedge\mathfrak{g}^*$. We consider the spectral sequence ${}_{\mathrm{Dol}}E_*^{*,*}(\mathfrak{g})$ of $\bigwedge\mathfrak{g}^*\otimes V_\sigma$ and the spectral sequence ${}_{\mathrm{Dol}}E_*^{*,*}(G/\Gamma)$ of $A^{0,*}(G/\Gamma,L_{\bar{\sigma}})$. Set $G/N=A$ and $\Gamma/\Gamma\cap N=\Delta$ and $\mathfrak{a}=\mathfrak{g}/\mathfrak{n}$. By Borel's result [7, Appendix 2], we have the commutative diagram

$$\begin{array}{ccc}
{}_{\mathrm{Dol}}E_1^{*,q}(\mathfrak{g}) & \longrightarrow & {}_{\mathrm{Dol}}E_1^{*,q}(G/\Gamma)\\
\downarrow\cong & & \downarrow\cong\\
\bigwedge\mathfrak{a}^*\otimes V_{\Lambda_q} & \longrightarrow & A^{0,*}(A/\Delta,L_{\bar{\Lambda}_q}).
\end{array}$$

4.3. Proof of theorem

Proof. Consider the spectral sequence ${}_{\mathrm{Dol}}E_*^{*,*}(\mathfrak{g})$ of

$$\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\sigma$$

and the spectral sequence ${}_{\mathrm{Dol}}E_*^{*,*}(G/\Gamma)$ of

$$\bigoplus_{L_{\bar{\alpha}}\in\mathcal{B}_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma,L_{\bar{\alpha}}\otimes L_{\bar{\sigma}}).$$

Set $A=G/N$ and $\Delta=\Gamma/\Gamma\cap N$ and $\mathfrak{a}=\mathfrak{g}/\mathfrak{n}$. Since we can identify $\mathcal{B}_{(G,N)}$ (resp. $\mathcal{B}_{(G,N)}(\Gamma)$) with $\mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)$ (resp. $\mathcal{H}_{\mathrm{hol}}(A/\Delta)$ as Section 4.1), we have the commutative diagram

$$\begin{array}{ccc}
{}_{\mathrm{Dol}}E_1^{*,*}(\mathfrak{g}) & \longrightarrow & {}_{\mathrm{Dol}}E_1^{*,*}(G/\Gamma)\\
\downarrow\cong & & \downarrow\cong\\
\bigoplus_{\alpha\in\mathrm{Hom}_{\mathrm{hol}}(A,\mathbb{C}^*)}\bigwedge\mathfrak{a}^*\otimes V_\alpha\otimes V_\Lambda & \longrightarrow & \bigoplus_{L_\alpha\in\mathcal{H}_{\mathrm{hol}}(A/\Delta)} A^{0,*}(A/\Delta,L_{\bar{\alpha}}\otimes L_{\bar{\Lambda}}).
\end{array}$$

By Proposition 4.3, the homomorphism ${}_{\mathrm{Dol}}E_1^{*,*}(\mathfrak{g})\to{}_{\mathrm{Dol}}E_1^{*,*}(G/\Gamma)$ induces a cohomology isomorphism and hence we have an isomorphism ${}_{\mathrm{Dol}}E_2^{*,*}(\mathfrak{g})\cong{}_{\mathrm{Dol}}E_2^{*,*}(G/\Gamma)$. Hence the theorem follows. $\square$

5. Construction of finite cochain complex (de Rham case)

We will use the following proposition.

Proposition 5.1 ([2, Proposition 3.3]). *Let G be a simply connected solvable Lie group G and N the nilradical of G. Then we have a simply connected nilpotent subgroup $C \subset G$ such that $G = C \cdot N$.*

Remark 2. This proposition is given by the decomposition (not necessarily direct sum) $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$ (see [3, Theorem 2.2]). Since this decomposition is compatible with any field (see [3, Theorem 2.2]), if G is complex Lie group we can take a subgroup C also complex.

Let G be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of G. Let N be the nilradical of G. Let $\rho : G \to GL(V_\rho)$ be a representation. Suppose the restriction $\rho_{|_N}$ is unipotent. We consider the direct sum

$$\bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho.$$

Then we have the G-action on this cochain complex via $\bigoplus \mathrm{Ad} \otimes \alpha \otimes \rho$. Since this action is extension of the Lie derivation, the induced action on the cohomology is trivial. Consider the semi-simple part

$$\left(\left(\bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} \mathrm{Ad} \otimes \alpha \otimes \rho \right) (g) \right)_s = \bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} (\mathrm{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s.$$

Take a simply connected nilpotent subgroup $C \subset G$ as Proposition 5.1. Since C is nilpotent, the map

$$\Phi : C \ni c \mapsto \bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} (\mathrm{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s$$
$$\in \mathrm{Aut} \left(\bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho \right)$$

is a homomorphism.We denote by

$$\left(\bigoplus_{\alpha \in \mathcal{A}_{(G,N)}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho \right)^{\Phi(C)}$$

the subcomplex consisting of the $\Phi(C)$-invariant elements.

Lemma 5.2. *The inclusion*

$$\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right)^{\Phi(C)}\subset\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}$$

induces a cohomology isomorphism.

Proof. Since the induced G-action on the cohomology $H^{*}(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho})$ is trivial and $\Phi(C)$-action is semi-simple part of G-action, the induced $\Phi(C)$-action on the cohomology $H^{*}(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho})$ is also trivial and hence

$$H^{*}\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right)^{\Phi(C)}=H^{*}\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right).$$

Since Φ is diagonalizable, we have

$$\begin{aligned}&H^{*}\left(\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right)^{\Phi(C)}\right)\\&=H^{*}\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho}\right)^{\Phi(C)}.\end{aligned}$$

Hence the lemma follows. □

The subcomplex $(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}_{\mathbb{C}}\otimes V_{\alpha}\otimes V_{\rho})^{\Phi(C)}$ is desired subcomplex A^{*} as in Theorem 1.5. Using certain basis, we see that this complex is finite-dimensional and write down the subcomplex A^{*}_{Γ} as Corollary 1.6 explicitly.

We have a basis $X_1,\dots,X_n$ of $\mathfrak{g}_{\mathbb{C}}$ such that $(\mathrm{Ad}_c)_s=\mathrm{diag}(\alpha_1(c),\dots,\alpha_n(c))$ for $c\in C$. Let $x_1,\dots,x_n$ be the basis of $\mathfrak{g}^{*}_{\mathbb{C}}$ which is dual to $X_1,\dots,X_n$. We have a basis $v_1,\dots,v_m$ of V_{ρ} such that $(\rho(c))_s=\mathrm{diag}(\alpha'_1(c),\dots,\alpha'_m(c))$ for any $c\in C$. Let v_{α} be a basis of V_{α} for each character $\alpha\in\mathcal{A}_{(G,N)}$. By $G=C\cdot N$, we have $G/N=C/C\cap N$ and hence we have $\mathcal{A}_{(G,N)}=\mathcal{A}_{C,C\cap N}=\{\alpha\in\mathrm{Hom}(C,\mathbb{C}^{*})|\alpha_{|_{C\cap N}}=1\}$.

For a multi-index $I = \{i_1, \dots, i_p\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. We consider the basis

$$\{x_I \otimes v_\alpha \otimes v_k\}_{I \subset \{1,\dots,n\}, \alpha \in \mathcal{A}_{C, C\cap N}, k \in \{1,\dots,m\}}$$

of $\bigoplus_{\alpha \in \mathcal{A}_{C,C\cap N}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho$. Since the action

$$\Phi : C \to \mathrm{Aut}\left(\bigoplus_\alpha \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\rho\right)$$

is the semi-simple part of $(\bigoplus \mathrm{Ad} \otimes \alpha \otimes \rho)_{|C}$, we have

$$\Phi(a)(x_I \otimes v_\alpha \otimes v_k) = \alpha_I^{-1} \alpha \alpha'_k x_I \otimes v_\alpha \otimes v_k.$$

Hence we have

$$\begin{aligned} \left(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho\right)^{\Phi(C)} &= \langle x_I \otimes v_{\alpha_I \alpha_k'^{-1}} \otimes v_k \otimes \rangle_{I \subset \{1,\dots,n\}, k \in \{1,\dots,m\}} \\ &= \bigwedge \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle \\ &\quad \otimes \langle v_{\alpha_1'^{-1}} \otimes v_1, \dots, v_{\alpha_m'^{-1}} \otimes v_m \rangle. \end{aligned}$$

Finally we construct a finite-dimensional complex A^*_Γ which computes the de Rham cohomology $H^*(G/\Gamma, E_\rho)$.

Corollary 5.3. *Let A^*_Γ be the subcomplex of $(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho)^{\Phi(C)}$ defined as*

$$A^*_\Gamma = \langle x_I \otimes v_{\alpha_I \alpha_k'^{-1}} \otimes v_k | (\alpha_I \alpha_k'^{-1})_{|\Gamma} = 1 \rangle.$$

Then we have an isomorphism

$$H^*(A^*_\Gamma) \cong H^*(G/\Gamma, E_\rho).$$

Proof. Consider the inclusion

$$\iota : \left(\bigoplus_{\alpha \in \mathcal{A}_{G,N}} \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha \otimes V_\rho\right)^{\Phi(C)} \to \bigoplus_{E_\alpha \in \mathcal{A}_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).$$

$\iota(x_I \otimes v_{\alpha_I \alpha_k'^{-1}} \otimes v_k) \in A^*(G/\Gamma, E_\rho)$ if and only if $(\alpha_I \alpha_k'^{-1} \rho)_{|\Gamma} = \rho_{|\Gamma}$. Hence we have $\iota^{-1}(A^*(G/\Gamma, E_\rho)) = A^*_\Gamma$. □

Corollary 5.4. *We consider the following conditions:*

($\diamondsuit_1$) *For each multi-index $I = \{i_1, \dots, i_p\}$ and $k \in \{1, \dots, m\}$, the character $\alpha_I \alpha_k'^{-1}$ is trivial if and only if the restriction $(\alpha_I \alpha_k'^{-1})_{|\Gamma}$ is trivial.*

($\diamondsuit_2$)*For each multi-index $I = \{i_1, \dots, i_p\}$ and $k \in \{1, \dots, m\}$, the character $\alpha_I \alpha_k'^{-1}$ is trivial or non-unitary.*

If the condition ($\diamondsuit_1$) or ($\diamondsuit_2$) holds, then we have an isomorphism

$$H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho).$$

Proof. If the condition ($\diamondsuit_1$) holds, then we have $A^*_\Gamma = (\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho)^{\Phi(C)}$. Hence we have

$$H^*(G/\Gamma, E_\rho) \cong H\left(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho\right)^{\Phi(C)} \cong H^*(\mathfrak{g}, V_\rho).$$

The condition ($\diamondsuit_2$) is special case of the condition ($\diamondsuit_1$). Hence the corollary follows. □

Remark 3. For a representation $\rho : G \to GL(V_\rho)$ such that the restriction $\rho_{|_N}$ is trivial, the condition (M) (resp. (H)) in Section 1 is a special case of the condition ($\diamondsuit_1$) (resp. ($\diamondsuit_2$))

Remark 4. Let $\mathfrak{c}$ be the Lie algebra of C. Take a subvector $V \subset \mathfrak{c}$ (not necessarily Lie algebra) such that $\mathfrak{g} = V \oplus \mathfrak{n}$. Then we define the map

$$\mathrm{ad}_s : \mathfrak{g} = V \oplus \mathfrak{n} \ni A + X \mapsto (\mathrm{ad}_A)_s \in D(\mathfrak{g}),$$

where $(\mathrm{ad}_A)_s$ is the semi-simple part of ad_A and $D(\mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$. This map is a Lie algebra homomorphism and a diagonalizable representation (see [4, 9]). Let $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ be the extension of ad_s. Then this map is identified with the map

$$G = C \cdot N \ni c \cdot n \mapsto (\mathrm{Ad}_c) \in \mathrm{Aut}(\mathfrak{g}).$$

We define the Lie algebra $\mathfrak{u}_G \subset D(\mathfrak{g}) \ltimes \mathfrak{g}$ as

$$\mathfrak{u}_G = \{X - \mathrm{ad}_{sX} | X \in \mathfrak{g}\}.$$

Consider the above basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}^*_{\mathbb{C}}$. Then in [9] the author showed that we have an isomorphism

$$\bigwedge \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle \cong \bigwedge (\mathfrak{u}_G \otimes \mathbb{C})^*.$$

(This fact gives the new developments of de Rham homotopy theory on solvmanifolds. See [9].) Hence we can regard

$$\left(\bigoplus_{\alpha\in\mathcal{A}_{G,N}}\bigwedge\mathfrak{g}^*_{\mathbb{C}}\otimes V_\alpha\otimes V_\rho\right)^{\Phi(C)}=\bigwedge\langle x_1\otimes v_{\alpha_1},\dots,x_n\otimes v_{\alpha_n}\rangle$$
$$\otimes\,\langle v_{\alpha_1'^{-1}}\otimes v_1,\dots,v_{\alpha_m'^{-1}}\otimes v_m\rangle$$

as the cochain complex of the nilpotent Lie algebra $\mathfrak{u}_G$ with values in some representation.

6. Construction of finite cochain complex (Dolbeault case)

In this case, we can say almost same argument for the de Rham case without difficulties. Let G be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of G. Let N be the nilradical of G. Let $\sigma : G \to GL(V_\sigma)$ be a holomorphic representation. Suppose the restriction $\sigma_{|_N}$ is unipotent. We consider the direct sum

$$\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\rho.$$

Then we have the G-action on this cochain complex via $\bigoplus \mathrm{Ad}\otimes\alpha\otimes\rho$. Consider the semi-simple part

$$\left(\left(\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\mathrm{Ad}\otimes\alpha\otimes\rho\right)(g)\right)_s=\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}(\mathrm{Ad}_g)_s\otimes\alpha(g)\otimes(\rho(g))_s.$$

Take a simply connected complex nilpotent subgroup $C\subset G$ as Proposition 5.1 and Remark 2. Since C is nilpotent, the map

$$\Phi: C\ni c\mapsto\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}(\mathrm{Ad}_g)_s\otimes\alpha(g)\otimes(\rho(g))_s$$
$$\in\mathrm{Aut}\left(\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\sigma\right)$$

is a homomorphism. We denote by

$$\left(\bigoplus_{\alpha\in\mathcal{A}_{(G,N)}}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}\right)^{\Phi(C)}$$

the subcomplex consisting of the $\Phi(C)$-invariant elements. By similar proof of Lemma 5.2, we have:

Lemma 6.1. *The inclusion*

$$\left(\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}\right)^{\Phi(C)}\subset\bigoplus_{\alpha\in\mathcal{B}_{(G,N)}}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}$$

induces a cohomology isomorphism.

We have a basis $X_1,\dots,X_n$ of $\mathfrak{g}$ such that $(\mathrm{Ad}_c)_s=\mathrm{diag}(\alpha_1(c),\dots,\alpha_n(c))$ for $c\in C$. Let $x_1,\dots,x_n$ be the basis of $\mathfrak{g}^*$ which is dual to $X_1,\dots,X_n$. We have a basis $v_1,\dots,v_m$ of V_σ such that $(\sigma(c))_s=\mathrm{diag}(\alpha'_1(c),\dots,\alpha'_m(c))$ for any $c\in C$. Let v_α be a basis of V_α for each character $\alpha\in\mathcal{B}_{(G,N)}$. By $G=C\cdot N$, we have $G/N=C/C\cap N$ and hence we have $\mathcal{B}_{(G,N)}=\mathcal{B}_{C,C\cap N}=\{\alpha\in\mathrm{Hom}_{\mathrm{hol}}(C,\mathbb{C}^*)|\alpha_{|_{C\cap N}}=1\}$.

For a multi-index $I=\{i_1,\dots,i_p\}$ we write $x_I=x_{i_1}\wedge\cdots\wedge x_{i_p}$, and $\alpha_I=\alpha_{i_1}\cdots\alpha_{i_p}$. We consider the basis

$$\{x_I\otimes v_\alpha\otimes v_k\}_{I\subset\{1,\dots,n\},\alpha\in\mathcal{A}_{C,C\cap N},k\in\{1,\dots,m\}}$$

of $\bigoplus_{\alpha\in\mathcal{B}_{C,C\cap N}}\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\sigma$. Since the action

$$\Phi:C\to\mathrm{Aut}\left(\bigoplus_{\alpha}\bigwedge\mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}\right)$$

is the semi-simple part of $(\bigoplus\mathrm{Ad}\otimes\alpha\otimes\sigma)_{|_C}$, we have

$$\Phi(a)(x_I\otimes v_\alpha\otimes v_k)=\alpha_I^{-1}\alpha\alpha'_k x_I\otimes v_\alpha\otimes v_k.$$

Hence we have

$$\left(\bigoplus_{\alpha}\bigwedge \mathfrak{g}^{*}\otimes V_{\alpha}\otimes V_{\sigma}\right)^{\Phi(C)} = \langle x_I\otimes v_{\alpha_I\alpha_k^{\prime -1}}\otimes v_k\rangle_{I\subset\{1,\dots,n\},k\in\{1,\dots,m\}}$$
$$= \bigwedge\langle x_1\otimes v_{\alpha_1},\dots,x_n\otimes v_{\alpha_n}\rangle \otimes\langle v_{\alpha_1^{\prime -1}}\otimes v_1,\dots,v_{\alpha_m^{\prime -1}}\otimes v_m\rangle.$$

Corollary 6.2. *Let* B^*_Γ *be the subcomplex of* $\langle x_I\otimes v_{\alpha_I\alpha_k^{\prime -1}}\otimes v_k\rangle_{I\subset\{1,\dots,n\},k\in\{1,\dots,m\}}$ *defined as*

$$B^*_\Gamma = \left\langle x_I\otimes v_{\alpha_I\alpha_k^{\prime -1}}\otimes v_k \,\middle|\, \left(\frac{\bar\alpha_I\bar\alpha_k^{\prime -1}}{\alpha_I\alpha_k^{\prime -1}}\right)_{|\Gamma} = 1\right\rangle.$$

Then we have an isomorphism

$$H^*(B^*_\Gamma)\cong H^{0,*}(G/\Gamma, L_{\bar\sigma}).$$

Proof. It is known that we have the $1-1$ correspondence between the isomorphism classes of flat holomorphic line bundles over a complex torus and the unitary characters of its lattice (see [14]). By this, for $\alpha\in\mathcal{B}_{(G,N)}$, considering the unitary character $\frac{\bar\alpha}{\alpha}$, the holomorphic line bundle $L_{\bar\alpha}$ is trivial if and only if the restriction $(\frac{\bar\alpha}{\alpha})_{|\Gamma}$ is trivial. Hence

$$\iota(x_I\otimes v_{\alpha_I\alpha_k^{\prime -1}}\otimes v_k)\in A^*(G/\Gamma, L_{\bar\sigma})$$

if and only if the restriction $(\frac{\bar\alpha_I\bar\alpha_k^{\prime -1}}{\alpha_I\alpha_k^{\prime -1}})_{|\Gamma}$ is trivial. Then we have $\iota^{-1}(A^*(G/\Gamma, L_{\bar\sigma})) = B^*_\Gamma$. □

Corollary 6.3. *We consider the following condition:*

($\star$) *For each multi-index* $I=\{i_1,\dots,i_p\}$ *and* $k\in\{1,\dots,m\}$, *the character* $\alpha_I\alpha_k^{\prime -1}$ *is trivial if and only if the restriction* $(\frac{\bar\alpha_I\bar\alpha_k^{\prime -1}}{\alpha_I\alpha_k^{\prime -1}})_{|\Gamma}$ *is trivial.*

If the condition ($\star$) *holds, then we have an isomorphism*

$$H^*(\mathfrak{g}, V_\sigma)\cong H^{0,*}(G/\Gamma, L_{\bar\sigma}).$$

Proof. Suppose the condition ($\star$) holds. Then we have $B^*_\Gamma = (\bigwedge\mathfrak{g}^*\otimes V_\sigma)^{\Phi(C)}$. Hence we have

$$H^{0,*}(G/\Gamma, L_{\bar\sigma})\cong H^*\left(\bigwedge\mathfrak{g}^*\otimes V_\sigma\right)^{\Phi(C)}\cong H^*(\mathfrak{g}, V_\sigma).$$

□

Remark 5. We define the nilpotent Lie algebra $\mathfrak{u}_G$ as Remark 4. In the complex case, $\mathfrak{u}_G$ is also a complex Lie algebra. As similar to Remark 4, we have

$$\left(\bigoplus_{\alpha}\bigwedge\mathfrak{g}^*\otimes V_\alpha\right)^{\Phi(C)} = \bigwedge\langle x_1\otimes v_{\alpha_1},\dots,x_n\otimes v_{\alpha_n}\rangle \cong \bigwedge\mathfrak{u}_G^*.$$

Suppose G has a lattice Γ. We consider the cochain complex

$$B_\Gamma^* = \left\langle x_I\otimes v_{\alpha_I} \mid \left(\frac{\bar\alpha_I}{\alpha_I}\right)_{|\Gamma} = 1\right\rangle.$$

Then we have an isomorphism $H^{0,*}(B_\Gamma^*)\cong H^{0,*}(G/\Gamma)$ by Corollary 6.2. We consider the following condition.

($\square$) For each $1\le i\le n$, the restriction $(\frac{\bar\alpha_i}{\alpha_i})_{|\Gamma}$ is trivial.
If the condition ($\square$) holds, then we have

$$B_\Gamma^* = \bigwedge\langle x_1\otimes v_{\alpha_1},\dots,x_n\otimes v_{\alpha_n}\rangle \cong \bigwedge\mathfrak{u}_G^*.$$

Let U_G be the simply connected complex Lie group with the Lie algebra $\mathfrak{u}_G$. Then U_G is the nilradical of the semi-simple splitting of G (see [2]). It is known that if G has a lattice, then U_G has a lattice Γ' (see [1]).

Hence we have:

Corollary 6.4. *Let G be a simply connected complex solvable Lie group with a lattice Γ. If the condition ($\square$) holds, then there exists a complex parallelizable nilmanifold U_G/Γ' such that we have an isomorphism*

$$H^{*,*}(G/\Gamma)\cong H^{*,*}(U_G/\Gamma').$$

By this corollary we have some solvmanifolds whose Dolbeault cohomology is isomorphic to the Dolbeault cohomology of nilmanifolds.

7. Example

Let $G=\mathbb{C}\ltimes_\phi\mathbb{C}^2$ such that

$$\phi(z)=\begin{pmatrix} e^z & 0\\ 0 & e^{-z}\end{pmatrix}.$$

Then we have $a+\sqrt{-1}b, c+\sqrt{-1}d\in\mathbb{C}$ such that $\mathbb{Z}(a+\sqrt{-1}b)+\mathbb{Z}(c+\sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a+\sqrt{-1}b)$ and $\phi(c+\sqrt{-1}d)$ are conjugate to elements of $SL(4,\mathbb{Z})$ where we regard $SL(2,\mathbb{C})\subset SL(4,\mathbb{R})$ (see [5]). Hence we have a lattice $\Gamma=(\mathbb{Z}(a+\sqrt{-1}b)+\mathbb{Z}(c+\sqrt{-1}d))\ltimes_\phi\Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$.

7.1. Twisted de Rham cohomology $H^1(G/\Gamma, E_{\mathrm{Ad}})$

For a coordinate $(w,z_1,z_2)\in\mathbb{C}\ltimes_\phi\mathbb{C}^2$ we have the basis $\{v_1,\dots,v_6\}$ of $\mathfrak{g}_\mathbb{C}$ such that

$$\begin{aligned}&v_1=e^w\frac{\partial}{\partial z_1},\quad v_2=e^{\bar w}\frac{\partial}{\partial \bar z_1},\quad v_3=e^{-w}\frac{\partial}{\partial z_2},\quad v_4=e^{-\bar w}\frac{\partial}{\partial \bar z_2},\\ &v_5=\frac{\partial}{\partial w},\quad v_6=\frac{\partial}{\partial \bar w}.\end{aligned}$$

Consider the dual basis

$$e^{-w}dz_1, e^{-\bar w}d\bar z_1, e^w dz_2, e^{\bar w}d\bar z_2, dw, d\bar w.$$

As we consider $\mathfrak{g}_\mathbb{C}$ as a representation of $\mathfrak{g}$ via Ad, we have the cochian complex $\bigwedge\mathfrak{g}^*\otimes\mathfrak{g}_\mathbb{C}$ whose differential is given by

$$\begin{aligned}&dv_1=dw\otimes v_1,\quad dv_2=d\bar w\otimes v_2,\quad dv_3=-dw\otimes v_3,\quad dv_4=-d\bar w\otimes v_4,\\ &dv_5=-e^{-w}dz_1\otimes v_1+e^w dz_2\otimes v_3,\quad dv_6=-e^{\bar w}d\bar z_1\otimes v_2+e^{\bar w}d\bar z_2\otimes v_4.\end{aligned}$$

For $(w,0,0)\in\mathbb{C}$, we have $(\mathrm{Ad}_{(w,0,0)})_s=\mathrm{diag}(e^w,e^{\bar w},e^{-w},e^{-\bar w},1,1)$ for the basis $\{v_1,\dots,v_6\}$. Consider the cochain complex

$$\left(\bigoplus_\alpha\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\rho\right)^{\Phi(C)}$$

as Section 5 where $C=\mathbb{C}$. Then we have

$$\begin{aligned}&\left(\bigoplus_\alpha\bigwedge\mathfrak{g}^*\otimes V_\alpha\otimes V_\rho\right)^{\Phi(C)}\\ &\quad=\bigwedge\langle^{-w}dz_1\otimes v_{e^w},\, e^{-\bar w}d\bar z_1\otimes v_{e^{\bar w}},\, e^w dz_2\otimes v_{e^{-w}},\, e^{\bar w}d\bar z_2\otimes v_{e^{-\bar w}},\, dw,\, d\bar w\rangle\\ &\qquad\bigotimes\langle v_1\otimes v_{e^{-w}},\, v_2\otimes v_{e^{-\bar w}}, v_3\otimes v_{e^w},\, v_4\otimes v_{e^{\bar w}},\, v_5,\, v_6\rangle.\end{aligned}$$

For any lattice Γ we have $b_1(G/\Gamma) = b_1(\mathfrak{g}) = 2$. But we will see that

$$\dim H^1(G/\Gamma, E_{\mathrm{Ad}})$$

varies for a choice of Γ. If $b, d \in \pi\mathbb{Z}$, then we have

$$\begin{aligned} A^0_\Gamma &= \langle v_5, v_6\rangle, \\ A^1_\Gamma &= \langle e^{-w} dz_1 \otimes v_1,\ e^{-w} dz_1 \otimes v_{e^w} \otimes v_2 \otimes v_{e^{-\bar w}}, \\ &\quad e^{-\bar w} d\bar z_1 \otimes v_{e^{\bar w}} \otimes v_1 \otimes v_{e^w},\ e^{-\bar w} d\bar z_1 \otimes v_2, \\ &\quad e^{w} dz_2 \otimes v_3,\ e^{w} dz_2 \otimes v_{e^{-w}} \otimes v_4 \otimes v_{e^{\bar w}}, \\ &\quad e^{\bar w} d\bar z_2 \otimes v_{e^{-\bar w}} \otimes v_3 \otimes v_{e^{-w}},\ e^{\bar w} d\bar z_2 \otimes v_4, \\ &\quad dw \otimes v_5,\ dw \otimes v_6,\ d\bar w \otimes v_5,\ d\bar w \otimes v_6\rangle. \end{aligned}$$

Hence we have $\dim H^1(G/\Gamma, V_{\mathrm{Ad}}) = \dim H^1(A^*_\Gamma) = 6$.

On the other hand, if $b \notin \pi\mathbb{Z}$ or $d \notin \pi\mathbb{Z}$, then we have

$$\begin{aligned} A^0_\Gamma &= \langle v_5, v_6\rangle, \\ A^1_\Gamma &= \langle e^{-w} dz_1 \otimes v_1,\ e^{-\bar w} d\bar z_1 \otimes v_2, e^{w} dz_2 \otimes v_3, \\ &\quad e^{\bar w} d\bar z_2 \otimes v_4, dw \otimes v_5,\ dw \otimes v_6,\ d\bar w \otimes v_5,\ d\bar w \otimes v_6\rangle. \end{aligned}$$

Hence we have $\dim H^1(G/\Gamma, E_{\mathrm{Ad}}) = \dim H^1(A^*_\Gamma) = 2$.

7.2. Dolbeault cohomology $H^{*,*}_{\bar\partial}(G/\Gamma)$

For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes_\phi \mathbb{C}^2$, we consider the basis $(x_1, x_2, x_3) = (d\bar z_1, e^{-\bar z_1} d\bar z_2, e^{\bar z_1} d\bar z_3)$ of $\mathfrak{g}^*$. We consider $C = \mathbb{C} = \{(z_1)\}$ and $(\alpha_1, \alpha_2, \alpha_3) = (1, e^{z_1}, e^{-z_1})$ for C and $(\alpha_1, \alpha_2, \alpha_3)$ as in Section 5. If $b \notin \pi\mathbb{Z}$ or $c \notin \pi\mathbb{Z}$, then $(\star)$ holds and hence we have $H^{*,*}_{\bar\partial}(G/\Gamma) \cong \bigwedge \mathbb{C}^3 \otimes H^*(\mathfrak{g})$. If $b, d \in \pi\mathbb{Z}$, then the condition $(\square)$ holds and hence we have $H^{*,*}_{\bar\partial}(G/\Gamma) \cong \bigwedge \mathbb{C}^3 \otimes \bigwedge \mathbb{C}^3$. There exists a lattice Γ which satisfies the condition $(\star)$ or $(\square)$ (see [5]).

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