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Small energy scattering for the Klein–Gordon–Zakharov system with radial symmetry

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We prove small energy scattering for the three-dimensional Klein–Gordon–Zakharov system with radial symmetry. The idea of proof is the same as the Zakharov system studied in [6], namely to combine the normal form reduction and the radial-improved Strichartz estimates.

1. Introduction

In this paper, we consider the Cauchy problem for the three-dimensional (3D) Klein–Gordon–Zakharov system

$$\begin{cases} \ddot{u} - \Delta u + u = nu, \\ \ddot{n}/\alpha^2 - \Delta n = -\Delta u^2 \end{cases} \tag{1.1}$$

with the initial data

$$u(0,x) = u_0, \quad \dot{u}(0,x) = u_1, \quad n(0,x) = n_0, \quad \dot{n}(0,x) = n_1, \tag{1.2}$$

where $(u,n)(t,x) : \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{R}$, and $\alpha > 0, \alpha \neq 1$ denotes the ion sound speed. It preserves the energy

$$E = \int_{\mathbb{R}^3} |u|^2 + |\nabla u|^2 + |\dot{u}|^2 + \frac{|D^{-1}\dot{n}|^2/\alpha^2 + |n|^2}{2} - n|u|^2 dx, \tag{1.3}$$

where $D := \sqrt{-\Delta}$, as well as the radial symmetry

$$(u,n)(t,x) = (u,n)(t,|x|). \tag{1.4}$$

We consider those solutions with such symmetry and finite energy, hence

$$(u_0, u_1, n_0, n_1) \in H^1_r(\mathbb{R}^3) \times L^2_r(\mathbb{R}^3) \times L^2_r(\mathbb{R}^3) \times \dot{H}^{-1}_r(\mathbb{R}^3). \tag{1.5}$$

We are interested in the scattering for small data in the above function space.

This system describes the interaction between Langmuir waves and ion sound waves in a plasma (see [1, 3]). The local well-posedness (for arbitrary initial data) and global well-posedness (for small initial data) of (1.1) with $\alpha < 1$ in the energy space $H^1 \times L^2$ was proved by Ozawa, Tsutaya and Tsutsumi in [16]. We point out that (1.1) does not have null form structure as in Klainerman and Machedon [9] and this suggests that when $\alpha = 1$ the system (1.1) may be locally ill-posed in $H^1 \times L^2$ (the counter example of Lindblad [10] for similar equations). Hence, we suppose $\alpha \neq 1$ here. When the first equation of (1.1) is replaced by $c^{-2}\ddot{u} - \Delta u + c^2 u = -nu$ and $c, \alpha \to \infty$, Masmoudi and Nakanishi studied the limit system and the behavior of their solutions in a series of papers [11–13]. The instability of standing wave of Klein–Gordon–Zakharov system was studied in [4, 5, 14].

In this paper, inspired by Guo and Nakanishi [6], we combine the normal form technique, which was first used in a dispersive partial differential equation context by Shatah [17], and the improved radial Strichartz estimates to prove small energy scattering of (1.1) with radial symmetry. The normal form transform was also used in [15] for (1.1) and they got the scattering from initial data small in the Sobolev spaces with high regularity and in L^p with $p < 2$. Moreover, their scattering result is independent of radial symmetry. The main result of this paper is

Theorem 1.1. *If* (u_0, u_1, n_0, n_1) *are all radial and small enough in the norm of* (1.5)*, then the solution* (u, n) *scatters in this space as* $t \to \pm\infty$.

The main difficulties for the proof of scattering are derivative loss and slow dispersion of the wave equation together with the quadratic non- linearity. The loss of derivative can be overcome by the normal form transform (under the assumption $\alpha \neq 1$, so we have good non-linear structures mainly due to the different propagation speed). To handle the quadratic interaction, we have to assume radial symmetry so that we have wider class of Strichartz estimates.

2. Transformation of equations

This section is devoted to transform the equation by using the normal form. It is convenient first to change the system into first order as usual. Let

$$\mathcal{U} := u - i\langle D\rangle^{-1}\dot{u}, \quad \mathcal{N} := n - iD^{-1}\dot{n}/\alpha, \tag{2.1}$$

where $\langle x\rangle = (1+x^2)^{1/2}$, then $u = \mathrm{Re}\,\mathcal{U} = (\mathcal{U}+\bar{\mathcal{U}})/2$, $n = \mathrm{Re}\,\mathcal{N} = (\mathcal{N}+\bar{\mathcal{N}})/2$ and the equations for $(\mathcal{U},\mathcal{N})$ are

$$\begin{cases}(i\partial_t + \langle D\rangle)\mathcal{U} = \langle D\rangle^{-1}(\mathcal{N}\mathcal{U}/4 + \bar{\mathcal{N}}\mathcal{U}/4 + \mathcal{N}\bar{\mathcal{U}}/4 + \bar{\mathcal{N}}\bar{\mathcal{U}}/4),\\ (i\partial_t + \alpha D)\mathcal{N} = \alpha D(\mathcal{U}\bar{\mathcal{U}}/4 + \bar{\mathcal{U}}\mathcal{U}/4 + \mathcal{U}^2/4 + \bar{\mathcal{U}}^2/4).\end{cases} \tag{2.2}$$

Now we introduce some notations. We use $K(t), W_\alpha(t)$ to denote the Klein–Gordon and the wave propagators:

$$K(t)\phi = \mathcal{F}^{-1}e^{it\langle\xi\rangle}\hat{\phi}, \quad W_\alpha(t)\phi = \mathcal{F}^{-1}e^{i\alpha t|\xi|}\hat{\phi}, \quad \hat{\phi} = \mathcal{F}\phi.$$

Let $\eta_0 : \mathbb{R} \to [0,1]$ denote a radial smooth function supported in $\{|\xi| \le 2\}$ and equal to 1 in $\{|\xi| \le 1\}$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ and $\chi_{\le k}(\xi) = \eta_0(\xi/2^k)$. For $k \in \mathbb{Z}$ let $P_k, P_{\le k}$ denote the operators on $L^2(\mathbb{R}^3)$ defined by $\widehat{P_k u}(\xi) = \chi_k(|\xi|)\widehat{u}(\xi)$, $\widehat{P_{\le k}u}(\xi) = \chi_{\le k}(|\xi|)\widehat{u}(\xi)$.

For a quadratic term uv, we use $(uv)_{LH}$, $(uv)_{HH}$, $(uv)_{HL}$ to denote the three different interactions:

$$\begin{aligned}(uv)_{LH} &= \sum_{k\in\mathbb{Z}} P_{\le k-k_\alpha}uP_kv, \quad (uv)_{HL} = (vu)_{LH},\\ (uv)_{HH} &= \sum_{\substack{|k_1-k_2|<k_\alpha\\ k_1,k_2\in\mathbb{Z}}} P_{k_1}uP_{k_2}v,\end{aligned}$$

where k_α is a large number which is determined later, depending on α. It is obvious that we have

$$uv = (uv)_{HH} + (uv)_{LH} + (uv)_{HL} \tag{2.3}$$

and they are all radial if u, v are both radial. Moreover, for any such index $* = HH, HL, LH$, we denote the bilinear symbol (multiplier) by

$$\mathcal{F}(uv)_* = \int \mathcal{P}_*\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta \tag{2.4}$$

and finite sum of those bilinear operators are denoted by the sum of indices:

$$(uv)_{*_1+*_2+\cdots} = (uv)_{*_1} + (uv)_{*_2} + \cdots . \tag{2.5}$$

From Duhamel's formula and taking a Fourier transform, we get that the first equation of (2.2) is equivalent to

$$\begin{aligned}\hat{\mathcal{U}} = e^{it\langle\xi\rangle}\hat{\mathcal{U}}_0 - i\langle\xi\rangle^{-1}\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}(nu)_{HL}ds \\ - i\langle\xi\rangle^{-1}\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}(nu)_{HH+LH}ds\end{aligned}$$

Especially, for the second term, we have

(2.6)

$$\begin{aligned}&- i\langle\xi\rangle^{-1}\int_0^t\int e^{i(t-s)\langle\xi\rangle}\mathcal{P}_{HL}\hat{n}(s,\xi-\eta)\hat{u}(s,\eta)d\eta ds\\ &= -\frac{i}{4}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{HL}e^{is\omega_1}[e^{-i\alpha s|\xi-\eta|}\hat{\mathcal{N}}(s,\xi-\eta)][e^{-is\langle\eta\rangle}\hat{\mathcal{U}}(s,\eta)]d\eta ds\\ &\quad - \frac{i}{4}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{HL}e^{is\omega_2}[e^{i\alpha s|\xi-\eta|}\hat{\bar{\mathcal{N}}}(s,\xi-\eta)][e^{-is\langle\eta\rangle}\hat{\mathcal{U}}(s,\eta)]d\eta ds\\ &\quad - \frac{i}{4}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{HL}e^{is\omega_3}[e^{-i\alpha s|\xi-\eta|}\hat{\mathcal{N}}(s,\xi-\eta)][e^{is\langle\eta\rangle}\hat{\bar{\mathcal{U}}}(s,\eta)]d\eta ds\\ &\quad - \frac{i}{4}\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{HL}e^{is\omega_4}[e^{i\alpha s|\xi-\eta|}\hat{\bar{\mathcal{N}}}(s,\xi-\eta)][e^{is\langle\eta\rangle}\hat{\bar{\mathcal{U}}}(s,\eta)]d\eta ds,\end{aligned}$$

where

$$\begin{aligned}\omega_1 &= -\langle\xi\rangle + \alpha|\xi-\eta| + \langle\eta\rangle, & \omega_2 &= -\langle\xi\rangle - \alpha|\xi-\eta| + \langle\eta\rangle,\\ \omega_3 &= -\langle\xi\rangle + \alpha|\xi-\eta| - \langle\eta\rangle, & \omega_4 &= -\langle\xi\rangle - \alpha|\xi-\eta| - \langle\eta\rangle.\end{aligned}$$

It is obvious that ω_2 and ω_4 will not vanish in the support of $\mathcal{P}_{HL}$: $|\xi| \sim |\xi-\eta| \gg |\eta|$. For example, if we choose $k_\alpha \ge 5$, then

$$|\omega_2|, |\omega_4| \sim_\alpha \langle\xi\rangle.$$

Therefore, there is no resonance in these cases.

In contrast, ω_1 and ω_3 are more problematic since they vanish when $|\eta| = 0$ and $|\xi| = c_\alpha := 2\alpha/|\alpha^2-1|$ in the support of $\mathcal{P}_{HL}$. Therefore, we need further to distinguish $(uv)_{HL}$ between resonant and non-resonant frequency parts as follows:

$$(2.7)\quad (uv)_{\alpha L} = \sum_{\substack{|2^k - c_\alpha|\le\delta_\alpha,\\ k\in\mathbb{Z}}} P_k u P_{\le k-k_\alpha} v, \quad (uv)_{XL} = \sum_{\substack{|2^k - c_\alpha|>\delta_\alpha,\\ k\in\mathbb{Z}}} P_k u P_{\le k-k_\alpha} v$$

and similarly denote $(uv)_{L\alpha}$, $(uv)_{LX}$. Then we use normal form only for non-resonant parts. We give the estimates of ω_1 and ω_3 precisely in the following lemma, similar to the estimates in [11].

Lemma 2.1. *Let $1 \neq \alpha > 0$, then there exist c_α, δ_α and k_α such that in the support of $\mathcal{P}_{XL}$,*

$$|\omega_1| \sim_\alpha |\xi|, \quad |\omega_3| \sim_\alpha \langle \xi \rangle.$$

Proof. We will use the simple fact $\langle \eta \rangle - 1 = \frac{|\eta|^2}{\langle \eta \rangle + 1} \leq |\eta|$.

(1) We consider the case $0 < \alpha < 1$.

For ω_1, by solving

$$\langle \xi \rangle - 1 = \alpha |\xi|,$$

we can get the resonant frequency

$$c_\alpha = \frac{2\alpha}{1 - \alpha^2}.$$

Now we estimate the function $f(r) := \alpha r - \langle r \rangle + 1$. Since $f'(r) = \alpha - r/\langle r \rangle$ and $f''(r) = -1/\langle r \rangle^3$, f is convex and has only maximum at

$$r_0 = \frac{\alpha}{\sqrt{1 - \alpha^2}} \in (0, c_\alpha).$$

There exists $\theta \in (0,1)$ such that $c_\alpha(1 - \theta) \in (r_0, c_\alpha)$. Let $\delta_\alpha = \theta c_\alpha$, then we can find a number $\rho = \rho(\alpha, \delta_\alpha)$ such that

$$|f(r)| \geq \rho r \quad \text{for} \quad r \in [0, c_\alpha - \delta_\alpha) \cup (c_\alpha + \delta_\alpha, \infty).$$

Choosing $k_\alpha \geq |\log_2 \rho| + 5$, we have

$$|\omega_1| \sim_\alpha |\xi|$$

in the support of $\mathcal{P}_{XL}$.

Now we consider ω_3. Choosing $k_\alpha \geq |\log_2(1 - \alpha)| + 5$, we have $(1 - \alpha)|\xi| \gg |\eta|$. Since

$$|\omega_3| \geq |\xi| - \alpha|\xi - \eta| + 1 \geq (1 - \alpha)|\xi| - |\eta| + 1 \geq c|\xi| + 1,$$

we have

$$|\omega_3| \sim_\alpha \langle \xi \rangle$$

in the support of $\mathcal{P}_{HL}$.

(2) We consider the case $\alpha > 1$.

For ω_1, by choosing $k_\alpha \geq |\log_2(\alpha-1)| + 5$, we have $|\xi| \gg |\eta|$ and $(\alpha-1)|\xi| \gg |\eta|$, and hence

$$|\omega_1| = |(-\langle\xi\rangle + 1) + \alpha|\xi - \eta| + (\langle\eta\rangle - 1)| \sim_\alpha |\xi|$$

in the support of $\mathcal{P}_{HL}$.

For ω_3, by solving

$$\langle\xi\rangle + 1 = \alpha|\xi|,$$

we can get the resonant frequency

$$c_\alpha = \frac{2\alpha}{\alpha^2 - 1}.$$

For the function $g(r) := \alpha r - \langle r\rangle - 1$, since $g'(r) = \alpha - r/\langle r\rangle > 0$, $g''(r) = -1/\langle r\rangle^3 < 0$ and the asymptotic line is $y(r) = (\alpha-1)r - 1$ when $r \to \infty$, $|g(r)|$ and the line $h(r) := (\alpha-1)r/2$ have two crossing points r_{c1} and r_{c2}. Let δ_α such that

$$\delta_\alpha = \max\{|c_\alpha - r_{c1}|,\ |c_\alpha - r_{c2}|\},$$

we have

$$|g(r)| \geq \frac{\alpha-1}{2} r \quad \text{for} \quad r \in [0, c_\alpha - \delta_\alpha) \cup (c_\alpha + \delta_\alpha, \infty).$$

Choosing $k_\alpha \geq |\log_2(\alpha-1)| + 5$, and noting that $|\omega_3| \sim 1$ for $1 \gg |\xi| \gg |\eta|$, we have

$$|\omega_3| \sim_\alpha \langle\xi\rangle$$

in the support of $\mathcal{P}_{XL}$.

□

By the lemma above, we gain $|\xi|^{-1}$ for high frequencies ($|\xi| > 1$) in all the cases, and lose $|\xi|^{-1}$ for low frequencies ($|\xi| < 1$) in the case ω_1. In general, the lower frequencies can be more problematic in the scattering problems, but it will turn out that we can absorb $|\xi|^{-1}$ by the Sobolev embedding.

By similar analysis, corresponding to the four non-linear terms of the second equation of (2.2), the resonance functions are

$$\tilde{\omega}_1 = -\alpha|\xi| + \langle \xi-\eta\rangle - \langle\eta\rangle, \qquad \tilde{\omega}_2 = -\alpha|\xi| - \langle \xi-\eta\rangle + \langle\eta\rangle,$$
$$\tilde{\omega}_3 = -\alpha|\xi| + \langle \xi-\eta\rangle + \langle\eta\rangle, \qquad \tilde{\omega}_4 = -\alpha|\xi| - \langle \xi-\eta\rangle - \langle\eta\rangle.$$

It is easy to check that $|\tilde{\omega}_j|$ behaves the same as $|\omega_j|$ for $j=1,2,3,4$. Indeed, ω_j and $\widetilde{\omega}_j$ are in the dual relation with the correspondence $\xi \mapsto \eta - \xi$.

In order to simplify the presentation, we assume $\alpha < 1$, which is the physical case in plasma; and we also suppose the non-linear terms in the first and second equation of (2.2) are $\mathcal{N}\mathcal{U}$ and $\mathcal{U}\bar{\mathcal{U}}$, respectively. For other cases, the proof is almost the same. Then we get that the first equation of (2.2) is equivalent to

$$\begin{aligned}\hat{\mathcal{U}} = e^{it\langle\xi\rangle}\hat{\mathcal{U}}_0 - i\langle\xi\rangle^{-1}\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}(\mathcal{N}\mathcal{U})_{XL}ds - i\langle\xi\rangle^{-1} \\ \times\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}(\mathcal{N}\mathcal{U})_{HH+LH+\alpha L}ds := I + II + III.\end{aligned}$$

Using equation (2.2) again, we get that

$$\partial_t(e^{-it\langle\xi\rangle}\hat{\mathcal{U}}) = -\, ie^{-it\langle\xi\rangle}\langle\xi\rangle^{-1}(\hat{\mathcal{N}} * \hat{\mathcal{U}})(\xi), \tag{2.8}$$
$$\partial_t(e^{-i\alpha t|\xi|}\hat{\mathcal{N}}) = -\, ie^{-i\alpha t|\xi|}\alpha|\xi|(\hat{\mathcal{U}} * \hat{\bar{\mathcal{U}}})(\xi). \tag{2.9}$$

Thus we have

$$\begin{aligned}II &= -i\langle\xi\rangle^{-1}\int_0^t\int e^{i(t-s)\langle\xi\rangle}\mathcal{P}_{XL}\hat{\mathcal{N}}(s,\xi-\eta)\hat{\mathcal{U}}(s,\eta)d\eta ds \\ &= -i\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{XL}e^{is\omega}[e^{-i\alpha s|\xi-\eta|}\hat{\mathcal{N}}(s,\xi-\eta)][e^{-is\langle\eta\rangle}\hat{\mathcal{U}}(s,\eta)]d\eta ds,\end{aligned}$$

where the resonance function is given by

$$\omega := -\langle\xi\rangle + \alpha|\xi-\eta| + \langle\eta\rangle.$$

In fact, this is just the most problematic term ω_1 in (2.6).

From integration by parts, we get

$$\begin{aligned}
II &= -\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\int_0^t\int \mathcal{P}_{XL}\omega^{-1}\partial_s(e^{is\omega})e^{-i\alpha s|\xi-\eta|}\\
&\quad\times\hat{\mathcal{N}}(s,\xi-\eta)e^{-is\langle\eta\rangle}\hat{\mathcal{U}}(s,\eta)d\eta ds\\
&= -\langle\xi\rangle^{-1}\int \mathcal{P}_{XL}\omega^{-1}[\hat{\mathcal{N}}(t,\xi-\eta)\hat{\mathcal{U}}(t,\eta)-e^{it\langle\xi\rangle}\hat{\mathcal{N}}(0,\xi-\eta)\hat{\mathcal{U}}(0,\eta)]d\eta\\
&\quad - i\alpha\langle\xi\rangle^{-1}\int_0^t\int \mathcal{P}_{XL}\omega^{-1}e^{i(t-s)\langle\xi\rangle}|\xi-\eta|\widehat{|\mathcal{U}|^2}(s,\xi-\eta)\hat{\mathcal{U}}(s,\eta)d\eta ds\\
&\quad - i\langle\xi\rangle^{-1}\int_0^t\int \mathcal{P}_{XL}\omega^{-1}e^{i(t-s)\langle\xi\rangle}\hat{\mathcal{N}}(s,\xi-\eta)\langle\eta\rangle^{-1}(\hat{\mathcal{N}}*\hat{\mathcal{U}})(s,\eta)d\eta ds.
\end{aligned}$$

We introduce a bilinear Fourier multiplier in the form

$$\Omega(f,g)=\mathcal{F}^{-1}\int \mathcal{P}_{XL}\omega^{-1}\hat{f}(\xi-\eta)\hat{g}(\eta)d\eta. \tag{2.10}$$

Then we have

$$\begin{aligned}
II &= -\langle\xi\rangle^{-1}\mathcal{F}\Omega(\mathcal{N},\mathcal{U})(t)+\langle\xi\rangle^{-1}e^{it\langle\xi\rangle}\mathcal{F}\Omega(\mathcal{N},\mathcal{U})(0)\\
&\quad - i\alpha\langle\xi\rangle^{-1}\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}\Omega(D|\mathcal{U}|^2,\mathcal{U})(s)ds\\
&\quad - i\langle\xi\rangle^{-1}\int_0^t e^{i(t-s)\langle\xi\rangle}\mathcal{F}\Omega(\mathcal{N},\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}))(s)ds.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathcal{U} &= K(t)\mathcal{U}_0+K(t)\langle D\rangle^{-1}\Omega(\mathcal{N},\mathcal{U})(0)-\langle D\rangle^{-1}\Omega(\mathcal{N},\mathcal{U})(t)\\
&\quad - i\alpha\langle D\rangle^{-1}\int_0^t K(t-s)\Omega(D|\mathcal{U}|^2,\mathcal{U})(s)ds\\
&\quad - i\langle D\rangle^{-1}\int_0^t K(t-s)\Omega(\mathcal{N},\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}))(s)ds\\
&\quad - i\langle D\rangle^{-1}\int_0^t K(t-s)(\mathcal{N}\mathcal{U})_{HH+LH+\alpha L}ds.
\end{aligned} \tag{2.11}$$

For the second equation in (2.2), similarly, we can apply the normal form reduction for the high–low and low–high interactions, and then get that it is equivalent to

(2.12)

$$\begin{aligned}\mathcal{N} = & W_\alpha(t)N_0 + \alpha W_\alpha(t)D\tilde{\Omega}(\mathcal{U},\mathcal{U})(0) - \alpha D\tilde{\Omega}(\mathcal{U},\mathcal{U})(t) \\ & - i\alpha\int_0^t W_\alpha(t-s)D(\mathcal{U}\bar{\mathcal{U}})_{HH+\alpha L+L\alpha}ds \\ & - i\alpha\int_0^t W_\alpha(t-s)(D\tilde{\Omega}(\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}),\mathcal{U}) + D\tilde{\Omega}(\mathcal{U},\langle D\rangle^{-1}(\mathcal{N}\mathcal{U})))(s)ds,\end{aligned}$$

where $\tilde{\Omega}$ is a bilinear Fourier multiplier in the form

$$\tilde{\Omega}(f,g) = \mathcal{F}^{-1}\int \mathcal{P}_{XL+LX}\frac{\hat{f}(\xi-\eta)\hat{\bar{g}}(\eta)}{\langle\xi-\eta\rangle - \langle\eta\rangle - \alpha|\xi|}d\eta. \tag{2.13}$$

3. Strichartz estimates and non-linear estimates

In this section, we introduce the Strichartz norm we need. Because of the quadratic term, our space relies heavily on the radial symmetry. For $\mathcal{U}$ and $\mathcal{N}$, we use the radial-improved Strichartz norms

$$\mathcal{U} \in X|Y, \quad \mathcal{N} \in L_t^\infty L_x^2 \cap L_t^2 \dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon),2}, \tag{3.1}$$

for fixed $0 < \varepsilon \ll 1$, where $\|\mathcal{U}\|_{X|Y} := \|P_{<0}\mathcal{U}\|_X + \|P_{\geq 0}\mathcal{U}\|_Y$ and

$$X = L_t^\infty L_x^2 \cap L_t^2 \dot{B}^{1/4+\varepsilon}_{q(\varepsilon),2}, \quad Y = L_t^\infty H_x^1 \cap L_t^2 B^{2/3}_{q(\varepsilon),2}, \quad \frac{1}{q(\varepsilon)} = \frac{1}{4} + \frac{\varepsilon}{3}.$$

By the Sobolev embedding,

$$\begin{aligned}\dot{H}_x^1 = \dot{B}^1_{q(3/4),2} \subset \dot{B}^{1/4+\varepsilon}_{q(\varepsilon),2} \subset \dot{B}^{1/4-\varepsilon}_{q(-\varepsilon),2} \subset L_x^6, \\ H_x^{\frac{17}{12}-\varepsilon} \subset B^{\frac{2}{3}}_{q(\varepsilon),2} \subset B^{\frac{2}{3}-2\varepsilon}_{q(-\varepsilon),2} \subset B^{\frac{5}{12}-\varepsilon}_{6,2} \subset L_x^6.\end{aligned}$$

From now on, the third exponent of the Besov space will be fixed to 2 and so omitted. The condition $0 < \varepsilon \ll 1$ ensures that

$$\frac{10}{3} < q(\varepsilon) < 4 < q(-\varepsilon) < \infty, \tag{3.2}$$

such that the norms in (3.1) are Strichartz-admissible for radial solutions. The Strichartz estimates that we will use are given in the following lemma.

Lemma 3.1. *Assume that $\phi(x)$, $f(t,x)$ are spatially radially symmetric in $\mathbb{R}^3$. Then*

(a) Assume $(q,r),(\tilde q,\tilde r)\in[2,\infty]^2$ both satisfy the condition:

$$\frac{2}{q}+\frac{5}{r}<\frac{5}{2} \text{ or } (q,r)=(\infty,2) \tag{3.3}$$

and $\tilde q>2$. Let

$$\beta(q,r)=\begin{cases}\dfrac{3}{2}-\dfrac{3}{r}-\dfrac{1}{q}, & \dfrac{1}{q}+\dfrac{2}{r}<1 \text{ or } (q,r)=(\infty,2),\\[2mm] \dfrac{1}{r}+\dfrac{1}{q}-\dfrac{1}{2}, & \dfrac{1}{q}+\dfrac{2}{r}>1 \text{ and } \dfrac{2}{q}+\dfrac{5}{r}<\dfrac{5}{2},\\[2mm] \left(\dfrac{1}{2}-\dfrac{1}{r}\right)+, & \dfrac{1}{q}+\dfrac{2}{r}=1,\end{cases} \tag{3.4}$$

where we used the notation $a+$ to denote $a+\varepsilon$ for arbitrary fixed $\varepsilon>0$. Then

$$\|K(t)P_{\geq 0}\phi\|_{L_t^q\dot B_{r,2}^{-\beta(q,r)}}\lesssim\|\phi\|_{L_x^2}, \tag{3.5}$$

$$\left\|\int_0^t K(t-s)P_{\geq 0}f(s)ds\right\|_{L_t^q\dot B_{r,2}^{-\beta(q,r)}}\lesssim\|P_{\geq 0}f\|_{L_t^{\tilde q'}\dot B_{\tilde r',2}^{\beta(\tilde q,\tilde r)}}, \tag{3.6}$$

$$\|K(t)P_{<0}\phi\|_{L_t^q\dot B_{r,2}^{\frac{2}{q}+\frac{3}{r}-\frac{3}{2}}}\lesssim\|\phi\|_{L_x^2}, \tag{3.7}$$

$$\left\|\int_0^t K(t-s)P_{<0}f(s)ds\right\|_{L_t^q\dot B_{r,2}^{\frac{2}{q}+\frac{3}{r}-\frac{3}{2}}}\lesssim\|P_{<0}f\|_{L_t^{\tilde q'}\dot B_{\tilde r',2}^{\frac{3}{2}-\frac{3}{\tilde r}-\frac{2}{\tilde q}}}. \tag{3.8}$$

(b) If $(q,r),(\tilde q,\tilde r)\in[2,\infty]^2$ both satisfy the condition:

$$\frac{1}{q}+\frac{2}{r}<1 \text{ or } (q,r)=(\infty,2) \tag{3.9}$$

and $\tilde q>2$, then

$$\|W_\alpha(t)\phi\|_{L_t^q\dot B_{r,2}^{\frac{1}{q}+\frac{3}{r}-\frac{3}{2}}}\lesssim\|\phi\|_{L_x^2}, \tag{3.10}$$

$$\left\|\int_0^t W_\alpha(t-s)f(s)ds\right\|_{L_t^q\dot B_{r,2}^{\frac{1}{q}+\frac{3}{r}-\frac{3}{2}}}\lesssim\|f\|_{L_t^{\tilde q'}\dot B_{\tilde r',2}^{\frac{3}{2}-\frac{3}{\tilde r}-\frac{1}{\tilde q}}}. \tag{3.11}$$

Proof. The proof of (b) can be found in [7], and the previous references therein. Using their idea, we give a rough proof for both (a) and (b). By

Riesz–Thorin interpolation and the classical Strichartz estimates, it suffices to prove the lemma for $(q,r)=(2,r)$. Consider a free solution on $\mathbb{R}^3$ with $|\xi|\sim 2^k$ frequency in the form

$$u_k(t,x)=e^{it\omega(D)}P_k\phi(x), \tag{3.12}$$

where $\phi\in L_x^2$ is radial, and $\omega(|\xi|)$ is the dispersion function. Computing it in polar coordinate, we have

$$u_k(t,x)=\frac{4\pi}{|x|}\int_{\rho\sim 1}e^{it\omega(2^k\rho)}\chi_0(\rho)2^{2k}\rho\hat{\phi}(2^k\rho)\sin(2^k|x|\rho)d\rho. \tag{3.13}$$

Hence if for some j, we have an estimate of the form

$$\|\chi_j(|x|)\int_{\mathbb{R}}e^{it\omega(2^k\rho)+i2^kx\rho}\chi_0(\rho)f(\rho)d\rho\|_{L_t^2L_x^r(\mathbb{R}^2)}\lesssim 2^{\alpha j+\beta k}\|f\|_{L_x^2(\mathbb{R})} \tag{3.14}$$

with some α,β and $r\ge 2$, then we get

$$\|\chi_j(|x|)u_k(t,x)\|_{L_t^2L_x^r(\mathbb{R}^{1+3})}\lesssim 2^{(\alpha-1+2/r)j+(1/2+\beta)k}\|\phi\|_{L_x^2(\mathbb{R}^3)}. \tag{3.15}$$

Let Tf be the inside of the norm on the left of (3.14). Then we have

$$\begin{aligned} T^*F&=\iint e^{-is\omega(2^k\xi)-i2^ky\xi}\chi_0(\xi)\chi_j(|y|)F(s,y)dyds,\\ TT^*F&=\iint e^{i(t-s)\omega(2^k\xi)+i2^k(x-y)\xi}\chi_0^2(\xi)d\xi\cdot\chi_j(|x|)\chi_j(|y|)F(s,y)dyds,\end{aligned} \tag{3.16}$$

and so

$$|TT^*F|\le|(e^{it\omega(2^kD)}\mathcal{F}^{-1}\chi_0^2)(2^kx)\chi_{\le j+1}(|x|)|*|F|. \tag{3.17}$$

Let $K(t,x)=(e^{it\omega(2^kD)}\mathcal{F}^{-1}\chi_0^2)(2^kx)$. Then (3.14) will follow from

$$\|K(t,x)\|_{L^1_{t\in\mathbb{R}}L^{r/2}_{|x|<2^j}}\lesssim 2^{2\alpha j+2\beta k}. \tag{3.18}$$

(a) In the Klein–Gordon case $\omega(\rho)=\langle\rho\rangle$,

$$K(t,x)=\int e^{it\langle 2^k\rho\rangle+i2^kx\rho}\chi_0^2(\rho)d\rho. \tag{3.19}$$

Simple computation shows that $\omega'(\rho)=\rho\langle\rho\rangle^{-1}$, $\omega''(\rho)=\langle\rho\rangle^{-3}$. For $r=2$, we use the local smoothing estimates. Indeed, using the Plancherel's identity in

t and Cauchy–Schwartz inequality in x, we get

$$\|Tf\|_{L_t^2 L_x^2} \lesssim 2^{j/2} 2^{-k/2} \langle 2^k \rangle^{1/2} \|f\|_2,$$

and hence

$$\|e^{it\langle D\rangle} P_k \phi\|_{L_t^2 L^2_{|x|\sim 2^j}} \lesssim 2^{j/2} 2^{-(k\wedge 0)/2} \|\phi\|_{L_x^2} \tag{3.20}$$

where we used the notation $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$.

Let $\psi(\rho) = t\langle 2^k \rho\rangle + 2^k \rho x$. Then $\psi'(\rho) = t 2^{2k} \rho \langle 2^k \rho\rangle^{-1} + 2^k x$. Thus if $|t| 2^{k\wedge 0} \gg 2^j$, then $|\psi'(\rho)| \gtrsim |t| 2^{2k} \langle 2^k \rangle^{-1}$, using integration by parts twice we get

$$|K(t,x)| \le \int \big|\partial_\rho [\psi'(\rho)^{-1} \partial_\rho (\chi_0^2(\rho) \psi'(\rho)^{-1})]\big| d\rho \lesssim |t|^{-2} 2^{-4k} \langle 2^k \rangle^2.$$

Combining with the trivial bound $|K| \lesssim 1$, we get that for $k \ge 0$ and $j \ge -k$,

$$\|K\|_{L_t^1 L^\infty_{|x|<2^j}} \lesssim \int_{|t|<2^{j+2}} dt + \int_{|t|>2^{j+2}} 2^{-2k} t^{-2} dt \lesssim 2^j. \tag{3.21}$$

Hence, for $k \ge 0$ and $j \ge -k$,

$$\|e^{it\langle D\rangle} P_k \phi\|_{L_t^2 L^\infty_{|x|\sim 2^j}} \lesssim 2^{(k-j)/2} \|\phi\|_{L_x^2}. \tag{3.22}$$

Interpolating (3.20) with (3.22) and classical Strichartz estimate

$$\|e^{it\langle D\rangle} P_k \phi\|_{L_t^2 L^\infty_{x\in\mathbb{R}^3}} \lesssim 2^{(k\wedge 0)/2} 2^{k\vee 0} \|\phi\|_{L_x^2}, \tag{3.23}$$

we can get the homogeneous estimates in part (a) for $1/q + 2/r < 1$ or $(q,r) = (\infty, 2)$.

Now we use the stationary phase method to get an improvement due to the non-vanishing second derivative. Indeed

$$|\psi''(\rho)| = \Big|\frac{2^{2k} t}{\langle 2^k \rho\rangle^3}\Big| \gtrsim |t| 2^{-(k\vee 0)} 2^{2(k\wedge 0)} \tag{3.24}$$

in the support of χ_0. Hence by the stationary phase method

$$|K(t,x)| \lesssim |t|^{-1/2} 2^{(k\vee 0)/2} 2^{-(k\wedge 0)}. \tag{3.25}$$

Thus eventually we have

$$|K(t,x)\chi_j(x)| \lesssim |t|^{-1/2}2^{(k\vee 0)/2}2^{-(k\wedge 0)}1_{\{|t|2^{k\wedge 0}\lesssim 2^j\}} \\ + |t|^{-2}2^{-4k}\langle 2^k\rangle^2 1_{\{|t|2^{k\wedge 0}\gg 2^j\}}.$$

Therefore,

$$\|K\|_{L_t^1 L^\infty_{|x|\lesssim 2^j}} \lesssim 2^{j/2}2^{(k\vee 0)/2}2^{-3(k\wedge 0)/2} + 2^{-j}2^{k\wedge 0}2^{-4k}\langle 2^{2k}\rangle,$$

and then for $j \geq -\frac{5}{3}(k\vee 0) - (k\wedge 0)$ we have

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2 L^\infty_{|x|\sim 2^j}} \lesssim 2^{-3j/4+3(k\vee 0)/4-(k\wedge 0)/4}\|\phi\|_{L_x^2}. \tag{3.26}$$

In particular, by interpolation between (3.26) and (3.22), we get for $k \geq 0, j \geq -k$,

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2 L^\infty_{|x|\sim 2^j}} \lesssim 2^{\theta(k-j)}\|\phi\|_{L_x^2},\ \frac{1}{2} \leq \theta \leq \frac{3}{4}. \tag{3.27}$$

Interpolating (3.20) with (3.27) and classical Strichartz estimates, we get that for $k < 0$, if $r > 10/3$, then

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2 L_x^r} \lesssim \left(\sum_{j\leq -k} 2^{\frac{j-k}{r}+\frac{k}{2}(1-\frac{2}{r})} + \sum_{j>-k} 2^{\frac{j-k}{r}-\frac{3j+k}{4}(1-\frac{2}{r})}\right) \\ \times \|\phi\|_{L_x^2} \lesssim 2^{k(\frac{1}{2}-\frac{3}{r})}\|\phi\|_{L_x^2};$$

for $k \geq 0$, if $\frac{10}{3} < r < 4$, then

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2 L_x^r} \lesssim \left(\sum_{j\leq -k} 2^{\frac{j}{r}+k(1-\frac{2}{r})} + \sum_{j\leq k} 2^{\frac{j}{r}+\frac{1}{2}(k-j)(1-\frac{2}{r})}\right. \\ \left. + \sum_{j\geq k} 2^{\frac{j}{r}+\frac{3}{4}(k-j)(1-\frac{2}{r})}\right)\|\phi\|_{L_x^2} \lesssim 2^{\frac{k}{r}}\|\phi\|_{L_x^2};$$

for $k \geq 0$, if $r = 4$, then

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2L_x^r} \lesssim \left(\sum_{j\leq -k} 2^{\frac{j}{r}+k(1-\frac{2}{r})} + \sum_{-k\leq j\leq k} 2^{k/4} + \sum_{j>k} 2^{\frac{j}{r}+\frac{3}{4}(k-j)(1-\frac{2}{r})}\right)\|\phi\|_{L_x^2} \lesssim \langle k\rangle 2^{\frac{k}{4}}\|\phi\|_{L_x^2};$$

for $k \geq 0$, if $r > 4$, then

$$\|e^{it\langle D\rangle}P_k\phi\|_{L_t^2L_x^r} \lesssim \left(\sum_{j\leq -k} 2^{\frac{j}{r}+k(1-\frac{2}{r})} + \sum_{j>-k} 2^{\frac{j}{r}+\frac{1}{2}(k-j)(1-\frac{2}{r})}\right) \times \|\phi\|_{L_x^2} \lesssim 2^{k(1-\frac{3}{r})}\|\phi\|_{L_x^2}.$$

Therefore, by interpolation the homogeneous estimates for $2/q + 5/r < 5/2$ or $1/q + 2/r = 1$ in part (a) were proved.

The inhomogeneous linear estimates follow from the duality argument and the Christ–Kiselev lemma, similar to [7].

(b) In the wave case $\omega(\rho) = |\rho|$, we have

$$\|\mathcal{F}^{-1}(\chi_0^2)(x \pm t)\|_{L_t^1 L^{r/2}_{|x|<2^j}} \sim 2^j, \tag{3.28}$$

hence $\alpha = 1/2$ (independent of r). Thus we obtain

$$\||x|^{1/2-2/r}e^{it|D|}\chi_0(D)\phi\|_{L_t^2L^r_{|x|\sim 2^j}} \lesssim \|\phi\|_{L_x^2}. \tag{3.29}$$

In particular, we have

$$\|e^{it|D|}\chi_0(D)\phi\|_{L_t^2L_x^r} \lesssim \|\phi\|_{L_x^2}\ (\forall r > 4). \tag{3.30}$$

By scaling,

$$\|e^{it|D|}P_k\phi\|_{L_t^2L_x^r} \lesssim (2^k)^{\frac{3}{2}-\frac{1}{2}-\frac{3}{r}}\|\phi\|_{L_x^2}\ (\forall r > 4). \tag{3.31}$$

This yields the radial improvement of the wave Strichartz in 3D. $\square$

Remark 1. The generalized Strichartz estimates for Klein–Gordon equation was also studied by Cho and Lee [2] which also addresses the non-radial versions. Our proof is different from theirs, and the idea is from [7]. Our results give better bound on the regularity, but the range of (q, r) is the same except some endpoints. More precisely, they prove that the borderline case $2/q + 5/r = 5/2$ is also admissible except for the endpoint $(q, r) = (2, 10/3)$. The borderline case for the Schrödinger equation was partially proved in [7], which was extended except for the endpoint by Ke [8]. The borderline case for the wave equation is prohibited except for the trivial energy norm.

The regularity in our estimates is optimal for all (q, r) in the admissible range. Indeed, there exists radial L^2 function $\phi \neq 0$ such that

$$\|e^{it\langle D\rangle} P_k \phi\|_{L_t^q L_x^r} \gtrsim C(q, r, k)\|\phi\|_{L_x^2}, \tag{3.32}$$

where $C(q, r, k) = \langle k\rangle^{1/q} 2^{(1/2-1/r)k}$ for (q, r) satisfying $1/q + 2/r = 1$, and $C(q, r, k) = 2^{\beta(q,r)k}$ for all other (q, r) in the admissible range. By (3.13), (3.32) is equivalent to the existence of f such that

(3.33)
$$\|s^{\frac{2}{r}-1} \int_{\rho\sim 1} e^{it\langle 2^k\rho\rangle} \chi_0(\rho) f(\rho) \sin(2^k s\rho) d\rho\|_{L_t^q L_{s>0}^r} \gtrsim C(q, r, k) 2^{-k/2} \|f\|_{L^2}.$$

Take $f = 1_{[0,10]}(\rho)$, then we have

$$\begin{aligned} I &:= \int_{\rho\sim 1} e^{it\langle 2^k\rho\rangle} \chi_0(\rho) f(\rho) \sin(2^k s\rho) d\rho \\ &= \frac{1}{2i} \int_{\rho\sim 1} e^{it\langle 2^k\rho\rangle} \chi_0(\rho)(e^{i2^k s\rho} - e^{-i2^k s\rho}) d\rho := I_1 - I_2. \end{aligned}$$

In the region $E = \{2^{-k} \ll |t| \ll 2^k,\ |t - s| \ll 2^{-k}\}$, using integration by parts we get $|I_1| \ll 1$; on the other hand,

$$|I_2| \sim |\int_{\rho\sim 1} e^{\frac{it}{\langle 2^k\rho\rangle + 2^k\rho}} e^{i2^k(t-s)\rho} \chi_0(\rho) d\rho| \sim 1.$$

Thus $|I| \sim 1$ on E. Hence,

$$\text{L.H.S. of (3.33)} \gtrsim 2^{-k/r} \left(\int_{2^{-k}\ll|t|\ll 2^k} |t|^{\frac{2q}{r}-q} dt \right)^{1/q} \gtrsim C(q, r, k) 2^{-k/2} \|f\|_2,$$

and (3.32) is proved.

We will apply this lemma to the integral equations. Then in order to close the argument, we need to do some non-linear estimates.

3.1. Bilinear terms

The above Strichartz norms neatly fit in the bilinear terms on the right, which are partially resonant. Indeed we have

Lemma 3.2. *(1) For any $\mathcal{N}$ and $\mathcal{U}$, the following estimates hold*

$$\|\langle D\rangle^{-1}(\mathcal{N}\mathcal{U})_{LH}\|_{L_t^1 H_x^1} \lesssim \|\mathcal{N}\|_{L_t^2 \dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \|\mathcal{U}\|_{L_t^2 \dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}},$$
$$\|\langle D\rangle^{-1}(\mathcal{N}\mathcal{U})_{HH}\|_{L_t^1 H_x^1} \lesssim \|\mathcal{N}\|_{L_t^2 \dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \|\mathcal{U}\|_{L_t^2 \dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}}.$$

If $0 \le \theta \le 1$, $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{\theta}{2}$, $\frac{1}{\tilde{r}} = \frac{1}{4} + \frac{\theta}{3} + \frac{\varepsilon}{3}$, *then*

$$\|\langle D\rangle^{-1}(\mathcal{N}\mathcal{U})_{\alpha L}\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|\mathcal{N}\|_{L_t^2 \dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \|\mathcal{U}\|_{X|Y}. \tag{3.34}$$

(2) For any $\mathcal{U}$, the following estimate holds:

$$\|D(\mathcal{U}\bar{\mathcal{U}})_{HH}\|_{L_t^1 L_x^2} \lesssim \|\mathcal{U}\|^2_{L_t^2(\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)} | B^{2/3}_{q(\varepsilon)})}. \tag{3.35}$$

If $0 \le \theta \le 1$, $\frac{1}{\tilde{q}} = \frac{1}{2} - \frac{\theta}{2}$, $\frac{1}{\tilde{r}} = \frac{1}{4} + \frac{\theta}{3} - \frac{\varepsilon}{3}$, *then*

$$\|D(\mathcal{U}\bar{\mathcal{U}})_{\alpha L + L\alpha}\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \lesssim \|\mathcal{U}\|^2_{X|Y}. \tag{3.36}$$

Proof. (1) For the first inequality, it suffices to prove

$$\|(\mathcal{N}\mathcal{U})_{LH}\|_{L_x^2} \lesssim \|\mathcal{N}\|_{\dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \|\mathcal{U}\|_{\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}}.$$

By dyadic decomposition, we have $(\mathcal{N}\mathcal{U})_{LH} = \sum_{k_1 \le k_2 - k_\alpha} P_{k_1}\mathcal{N} P_{k_2}\mathcal{U}$. Then by Hölder inequality, we get

$$\begin{aligned}
\|(\mathcal{N}\mathcal{U})_{LH}\|_{L_x^2} &\le \left(\sum_{k_2} \left| \sum_{k_1 \le k_2 - k_\alpha} \|P_{k_1}\mathcal{N}\|_{L_x^{q(-\varepsilon)}} \|P_{k_2}\mathcal{U}\|_{L_x^{q(\varepsilon)}} \right|^2 \right)^{1/2} \\
&\lesssim \left(\sum_{k_2} \left| \sum_{k_1 \le k_2 - k_\alpha} 2^{k_1(\frac{1}{4}+\varepsilon)} 2^{k_1(-\frac{1}{4}-\varepsilon)} \|P_{k_1}\mathcal{N}\|_{L_x^{q(-\varepsilon)}} \|P_{k_2}\mathcal{U}\|_{L_x^{q(\varepsilon)}} \right|^2 \right)^{1/2} \\
&\lesssim \|N\|_{\dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \|u\|_{\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}}.
\end{aligned}$$

Similarly, we can get the second one. For the third inequality, by Hölder inequality and Sobolev embedding,

$$\|(\mathcal{N}\mathcal{U})_{\alpha L}\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}} \lesssim \|\mathcal{N}_\alpha\|_{L_t^2 L_x^{q(-\varepsilon)}}\|\mathcal{U}\|_{L_t^{\frac{2}{\theta}} L_x^{\frac{6}{3-2\theta}}} \lesssim \|\mathcal{N}_\alpha\|_{L_t^2 L_x^{q(-\varepsilon)}}\|\mathcal{U}\|_X.$$

(2) For the first inequality, we have

$$\begin{aligned}\|D(\mathcal{U}\bar{\mathcal{U}})_{HH}\|_{L_x^2} &\le \sum_{|k_1-k_2|<k_\alpha} 2^{k_2}\|P_{k_1}\mathcal{U}\|_{L_x^{q(-\varepsilon)}}\|P_{k_2}\bar{\mathcal{U}}\|_{L_x^{q(\varepsilon)}} \\ &\lesssim \|\mathcal{U}\|_{\dot{B}^{1/2-\varepsilon}_{q(-\varepsilon)}}\|\mathcal{U}\|_{\dot{B}^{1/2+\varepsilon}_{q(\varepsilon)}} \lesssim \|\mathcal{U}\|^2_{\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}|B^{2/3}_{q(\varepsilon)}}.\end{aligned}$$

The proof of the second one is similar with the third one in (1). □

3.2. Boundary terms

Next, we estimate the boundary terms.

Lemma 3.3. *For any $\mathcal{N}_0$ and $\mathcal{U}_0$, we have*

$$\|\langle D\rangle^{-1}\Omega(\mathcal{N}_0,\mathcal{U}_0)\|_{H_x^1} \lesssim \|\mathcal{N}_0\|_{L_x^2}\|\mathcal{U}_0\|_{H_x^1},\ \|D\tilde{\Omega}(\mathcal{U}_0,\mathcal{U}_0)\|_{L_x^2} \lesssim \|\mathcal{U}_0\|^2_{H_x^1}.$$

As a consequence, for any $\mathcal{N}$ and $\mathcal{U}$

$$\|\langle D\rangle^{-1}\Omega(\mathcal{N},\mathcal{U})\|_{L_t^\infty H_x^1} \lesssim \|\mathcal{N}\|_{L_t^\infty L_x^2}\|\mathcal{U}\|_{L_t^\infty H_x^1},\ \|D\tilde{\Omega}(\mathcal{U},\mathcal{U})\|_{L_t^\infty L_x^2} \lesssim \|\mathcal{U}\|^2_{L_t^\infty H_x^1}.$$

Proof. We only prove $\|\Omega(\mathcal{N}_0,\mathcal{U}_0)\|_{L_x^2} \lesssim \|\mathcal{N}_0\|_{L_x^2}\|\mathcal{U}_0\|_{H_x^1}$, since the others are similar. From the Plancherel equality we have

$$\|\Omega(\mathcal{N}_0,\mathcal{U}_0)\|_{L_x^2} \lesssim \left\|\int_{|\xi-\eta|\gg|\eta|}|\eta|^{-1}|\hat{N}_0(\xi-\eta)|\cdot|\hat{\mathcal{U}}_0(\eta)|d\eta\right\|_{L^2} \lesssim \|\mathcal{N}_0\|_{L_x^2}\|\mathcal{U}_0\|_{H_x^1},$$

where we used the Sobolev embedding $\|\mathcal{F}^{-1}|\xi|^{-1}|\hat{u_0}(\xi)|\|_{L^\infty} \lesssim \|u_0\|_{H^1}$. □

To handle the other component, we will need a Coifman–Meyer-type bilinear multiplier estimates (see Lemma 3.5 in [6]).

Lemma 3.4. *For any $\mathcal{N}$ and $\mathcal{U}$ we have*

$$\begin{aligned}\|\langle D\rangle^{-1}\Omega(\mathcal{N},\mathcal{U})\|_{L_t^2(\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}|B^{2/3}_{q(\varepsilon)})} &\lesssim \|\mathcal{N}\|_{L_t^\infty L_x^2}\|\mathcal{U}\|_{L_t^2 L_x^6}, \\ \|D\tilde{\Omega}(\mathcal{U},\mathcal{U})\|_{L_t^2\dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} &\lesssim \|\mathcal{U}\|_{L_t^\infty H_x^1}\|\mathcal{U}\|_{L_t^2 L_x^6}.\end{aligned}$$

Proof. For the first inequality, it suffices to prove

$$\|\Omega(\mathcal{N},\mathcal{U})\|_{\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}|B^{-1/3}_{q(\varepsilon)}} \lesssim \|\mathcal{N}\|_{L^2_x}\|\mathcal{U}\|_{L^6_x}.$$

By Sobolev embedding, we get

$$\|\Omega(\mathcal{N},\mathcal{U})\|_{\dot{B}^{1/4+\varepsilon}_{q(\varepsilon)}} \lesssim \|D\Omega(\mathcal{N},\mathcal{U})\|_{L^2_x}.$$

It is easy to see that $D\Omega(\mathcal{N},\mathcal{U})$ is a bilinear multiplier with the symbol

$$m(\xi,\eta) = \frac{|\xi+\eta|\sum \chi_{\le k-5}(\eta)\chi_k(\xi)}{-\langle\xi+\eta\rangle + \alpha|\xi| + \langle\eta\rangle},$$

and m satisfies the condition in Lemma 3.5 in [6]. Thus applying dyadic decomposition and Bernstein inequality, we get

$$\begin{aligned}
\|P_{<0}D\Omega(\mathcal{N},\mathcal{U})\|_{L^2_x} &\lesssim \left(\sum_{k_2<2}\left\|\sum_{k_1\le k_2-k_\alpha} D\Omega(P_{k_2}\mathcal{N}, P_{k_1}\mathcal{U})\right\|^2_{L^2_x}\right)^{1/2} \\
&\lesssim \left(\sum_{k_2<2}\left(\sum_{k_1\le k_2-k_\alpha} \|P_{k_2}\mathcal{N}\|_{L^2_x}\|P_{k_1}\mathcal{U}\|_{L^\infty_x}\right)^2\right)^{1/2} \\
&\lesssim \left(\sum_{k_2<2}\left(\sum_{k_1\le k_2-k_\alpha} (2^{k_1})^{\frac{1}{2}}\|P_{k_2}\mathcal{N}\|_{L^2_x}\|P_{k_1}\mathcal{U}\|_{L^6_x}\right)^2\right)^{1/2} \\
&\lesssim \|\mathcal{N}\|_{L^2_x}\|\mathcal{U}\|_{L^6_x}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|P_{\ge 0}\Omega(\mathcal{N},\mathcal{U})\|_{B^{-1/3}_{q(\varepsilon)}} \\
&\quad\lesssim \|P_{\ge 0}\langle D\rangle^{\frac{5}{12}-\varepsilon}\Omega(\mathcal{N},\mathcal{U})\|_{L^2_x} \\
&\quad\lesssim \left(\sum_{k_2\ge -2}\left\|\sum_{k_1\le k_2-k_\alpha} \langle D\rangle^{\frac{5}{12}-\varepsilon}\Omega(P_{k_2}\mathcal{N}, P_{k_1}\mathcal{U})\right\|^2_{L^2_x}\right)^{1/2} \\
&\quad\lesssim \left(\sum_{k_2\ge -2}\left(\sum_{k_1\le k_2-k_\alpha} \langle 2^{k_2}\rangle^{-\frac{7}{12}-\varepsilon}\|P_{k_2}\mathcal{N}\|_{L^2_x}\|P_{k_1}\mathcal{U}\|_{L^\infty_x}\right)^2\right)^{1/2}
\end{aligned}$$

$$\lesssim \left(\sum_{k_2 \geq -2} \left(\sum_{k_1 \leq k_2 - k_\alpha} (2^{k_1})^{\frac{1}{2}} \langle 2^{k_2} \rangle^{-\frac{7}{12}-\varepsilon} \|P_{k_2}\mathcal{N}\|_{L_x^2} \|P_{k_1}\mathcal{U}\|_{L_x^6} \right)^2 \right)^{1/2}$$
$$\lesssim \|\mathcal{N}\|_{L_x^2} \|\mathcal{U}\|_{L_x^6}.$$

We proved the desired result.

Similarly, for the second inequality, by Sobolev embedding we get

$$\|D\tilde{\Omega}(\mathcal{U},\mathcal{U})\|_{\dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon)}} \lesssim \|D^{3/2}\tilde{\Omega}(\mathcal{U},\mathcal{U})\|_{L_x^2}$$

and $D\tilde{\Omega}$ behaves similarly to $D\Omega$. Then applying dyadic decomposition and Bernstein inequality, we get

$$\begin{aligned}
\|D^{3/2}\tilde{\Omega}(\mathcal{U},\mathcal{U})\|_{L_x^2} &\lesssim \left(\sum_{k_2} \left\| \sum_{k_1 \leq k_2 - k_\alpha} D^{3/2}\tilde{\Omega}(P_{k_2}\mathcal{U}, P_{k_1}\mathcal{U}) \right\|_{L_x^2}^2 \right)^{1/2} \\
&\lesssim \left(\sum_{k_2} \left(\sum_{k_1 \leq k_2 - k_\alpha} 2^{k_2/2} \|P_{k_2}\mathcal{U}\|_{L_x^2} \|P_{k_1}\mathcal{U}\|_{L_x^\infty} \right)^2 \right)^{1/2} \\
&\lesssim \left(\sum_{k_2} \left(\sum_{k_1 \leq k_2 - k_\alpha} 2^{(k_1+k_2)/2} \|P_{k_2}\mathcal{U}\|_{L_x^2} \|P_{k_1}\mathcal{U}\|_{L_x^6} \right)^2 \right)^{1/2} \\
&\lesssim \|\mathcal{U}\|_{H_x^1} \|\mathcal{U}\|_{L_x^6}.
\end{aligned}$$

Thus we finish the proof of the lemma. □

3.3. Cubic terms

Finally, we deal with the cubic terms.

Lemma 3.5. *For any $\mathcal{N}$ and $\mathcal{U}$ we have*

$$\begin{aligned}
\|\langle D\rangle^{-1}\Omega(D|\mathcal{U}|^2,\mathcal{U})\|_{L_t^1 H_x^1} &\lesssim \|\mathcal{U}\|_{L_t^2 L_x^6}^2 \|\mathcal{U}\|_{L_t^\infty H_x^1}, \\
\|\langle D\rangle^{-1}\Omega(\mathcal{N},\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}))\|_{L_t^2(L_x^{6/5}|B_{6/5}^{1+5/6})} &\lesssim \|\mathcal{N}\|_{L_t^\infty L_x^2}^2 \|\mathcal{U}\|_{L_t^2 L_x^6}, \\
\|D\tilde{\Omega}(\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}),\mathcal{U})\|_{L_t^1 L_x^2} &\lesssim \|\mathcal{N}\|_{L_t^\infty L_x^2} \|\mathcal{U}\|_{L_t^2 L_x^6}^2.
\end{aligned}$$

Proof. As in the proof of the previous lemma, applying dyadic decomposition, we get

$$\begin{aligned}
\|\Omega(D|\mathcal{U}|^2,\mathcal{U})\|_{L^2_x} &\lesssim \left(\sum_{k_2}\left\|\sum_{k_1\le k_2-k_\alpha}\Omega(P_{k_2}D|\mathcal{U}|^2,P_{k_1}\mathcal{U})\right\|_{L^2_x}^2\right)^{1/2}\\
&\lesssim \left(\sum_{k_2}\left(\sum_{k_1\le k_2-k_\alpha}\|P_{k_2}|\mathcal{U}|^2\|_{L^2_x}\|P_{k_1}\mathcal{U}\|_{L^\infty_x}\right)^2\right)^{1/2}\\
&\lesssim \||\mathcal{U}|^2\|_{H^{1/2}_x}\|\mathcal{U}\|_{L^6_x} \lesssim \|\mathcal{U}\|_{H^1_x}\|\mathcal{U}\|_{L^6_x}^2.
\end{aligned}$$

Similarly, for the second inequality, we have

$$\begin{aligned}
&\|\langle D\rangle^{5/6}\Omega(\mathcal{N},\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}))\|_{L^{6/5}_x}\\
&\lesssim \left(\sum_{k_2}\left\|\sum_{k_1\le k_2-k_\alpha}\langle D\rangle^{5/6}\Omega(P_{k_2}\mathcal{N},P_{k_1}\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}))\right\|_{L^{6/5}_x}^2\right)^{1/2}\\
&\lesssim \left(\sum_{k_2}\left(\sum_{k_1\le k_2-k_\alpha}2^{-k_2}\langle 2^{k_2}\rangle^{5/6}\langle 2^{k_1}\rangle^{-1}\|P_{k_2}\mathcal{N}\|_{L^2_x}\|P_{k_1}(\mathcal{N}\mathcal{U})\|_{L^3_x}\right)^2\right)^{1/2}\\
&\lesssim \left(\sum_{k_2}\left(\sum_{k_1\le k_2-k_\alpha}2^{k_1-k_2}\langle 2^{k_2}\rangle^{5/6}\langle 2^{k_1}\rangle^{-1}\|P_{k_2}\mathcal{N}\|_{L^2_x}\|P_{k_1}(\mathcal{N}\mathcal{U})\|_{L^{3/2}_x}\right)^2\right)^{1/2}\\
&\lesssim \|\mathcal{N}\|_{L^2_x}^2\|\mathcal{U}\|_{L^6_x},
\end{aligned}$$

and for the last inequality we have

$$\begin{aligned}
&\|D\tilde{\Omega}(\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}),\mathcal{U})\|_{L^2_x}\\
&\quad\lesssim \left(\sum_{k_2}\left\|\sum_{k_1\le k_2-k_\alpha}D\tilde{\Omega}(\langle D\rangle^{-1}(\mathcal{N}\mathcal{U}),\mathcal{U})\right\|_{L^2_x}^2\right)^{1/2}\\
&\quad\lesssim \left(\sum_{k_2}\left(\sum_{k_1\le k_2-k_\alpha}2^{k_1/2}\langle 2^{k_2}\rangle^{-1}\|P_{k_2}(\mathcal{N}\mathcal{U})\|_{L^2_x}\|P_{k_1}\mathcal{U}\|_{L^6_x}\right)^2\right)^{1/2}
\end{aligned}$$

$$\lesssim \left(\sum_{k_2} \left(\sum_{k_1 \le k_2 - k_\alpha} 2^{(k_1+k_2)/2} \langle 2^{k_2} \rangle^{-1} \| P_{k_2}(\mathcal{N}\mathcal{U}) \|_{L_x^{3/2}} \| P_{k_1} \mathcal{U} \|_{L_x^6} \right)^2 \right)^{1/2}$$
$$\lesssim \| \mathcal{N}\mathcal{U} \|_{L_x^{3/2}} \| \mathcal{U} \|_{L_x^6} \| \mathcal{N} \|_{L_x^2} \| \mathcal{U} \|_{L_x^6}^2 .$$

□

4. Proof of Theorem 1.1

Now we are ready to use the estimates obtained in the previous section to prove Theorem 1.1. For any $(u_0, u_1, n_0, n_1) \in H_r^1(\mathbb{R}^3) \times L_r^2(\mathbb{R}^3) \times L_r^2(\mathbb{R}^3) \times \dot{H}_r^{-1}(\mathbb{R}^3)$, we define an operator $\Phi_{u_0,u_1,n_0,n_1}(\mathcal{U}, \mathcal{N})$ by the right-hand side of (2.11)–(2.12). Our resolution space is

$$S_\eta = \{ (\mathcal{U}, \mathcal{N}) : \| (\mathcal{U}, \mathcal{N}) \|_S = \| \mathcal{U} \|_{X|Y} + \| \mathcal{N} \|_{L_t^\infty L_x^2 \cap L_t^2 \dot{B}^{-1/4-\varepsilon}_{q(-\varepsilon),2}} \le \eta \}$$

endowed with the norm metric $\|\cdot\|_S$.

We will show that $\Phi_{u_0,u_1,n_0,n_1} : S_\eta \to S_\eta$ is a contraction mapping, provided that $\eta \ll 1$ and (u_0, u_1, n_0, n_1) are sufficiently small. By the estimates in the previous section, we have for any $(\mathcal{U}, \mathcal{N}) \in S_\eta$

$$\begin{aligned} \| \Phi_{u_0,u_1,n_0,n_1}(\mathcal{U}, \mathcal{N}) \|_S \lesssim & \| \mathcal{U}_0 \|_{H_x^1} + \| \mathcal{N}_0 \|_{L_x^2} + (\| \mathcal{U}_0 \|_{H_x^1} + \| \mathcal{N}_0 \|_{L_x^2})^2 \\ & + \| (\mathcal{U}, \mathcal{N}) \|_S^2 + \| (\mathcal{U}, \mathcal{N}) \|_S^3 \le \eta \end{aligned}$$

if $\varepsilon_0 = \| \mathcal{U}_0 \|_{H_x^1} + \| \mathcal{N}_0 \|_{L_x^2} = \| u_0 \|_{H_x^1} + \| u_1 \|_{L_x^2} + \| n_0 \|_{L_x^2} + \| n_1 \|_{\dot{H}_x^{-1}} \ll 1$, and we set $\eta = C\varepsilon_0$. Similarly, we can prove $\Phi_{u_0,u_1,n_0,n_1} : S_\eta \to S_\eta$ is a contraction mapping. Our estimates are time global, therefore Theorem 1.1 follows immediately.

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