

Higher preprojective algebras and stably Calabi–Yau properties

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In this paper, we give sufficient properties for a finite-dimensional graded algebra to be a higher preprojective algebra. These properties are of homological nature, they use Gorensteiness and bimodule isomorphisms in the stable category of Cohen–Macaulay modules. We prove that these properties are also necessary for 3-preprojective algebras using [18] and for preprojective algebras of higher representation finite algebras using [5].

1. Introduction

Preprojective algebras play an important role in many different parts of mathematics. Such an algebra is associated to a quiver Q without oriented cycles. It has been defined by Gelfand and Ponomarev in the 1970s to get a better understanding of the representation theory of the path algebra of the quiver Q . Recently, in the context of higher Auslander–Reiten theory, Iyama generalized the definition of preprojective algebras. If Λ is a finite-dimensional algebra of global dimension $d - 1$, its d -preprojective algebra $\Pi_d(\Lambda)$ is defined as the tensor algebra $T_\Lambda \text{Ext}_{\Lambda^e}^{d-1}(\Lambda, \Lambda^e)$ where Λ^e is the enveloping algebra $\Lambda \otimes_k \Lambda^{\text{op}}$. It is naturally a positively graded algebra. The bimodule $\text{Ext}_{\Lambda^e}^{d-1}(\Lambda, \Lambda^e)$ is the zeroth cohomology group of the inverse of the “canonical bundle” $\text{Hom}_k(\Lambda, k)[-d + 1]$ in the derived category $\mathcal{D}^b(\text{mod } \Lambda)$. In the case where Λ satisfies some geometrical properties (Fano, quasi-Fano, etc), its preprojective algebra has also been studied in the context of non-commutative algebraic geometry (see [12, 21, 22]).

In this paper, we are interested in the properties that characterize finite-dimensional preprojective algebras. For $d = 2$, the preprojective algebra $\Pi = \Pi_2(kQ)$ is finite-dimensional if and only if Q is a Dynkin quiver and, in

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that case by a classical result, the algebra Π is self-injective and there is a functorial isomorphism

$$\text{Ext}_{\Pi}^1(X, Y) \cong D \text{Ext}_{\Pi}^1(Y, X)$$

for any $X, Y \in \text{mod } \Pi$. So in other words, the triangulated category $\underline{\text{mod}} \Pi$ is 2-Calabi–Yau. The duality above comes from an isomorphism

$$\text{Hom}_{\Pi^e}(\Pi, \Pi^e) \cong \Omega_{\Pi^e}^3 \Pi(1)$$

in the stable category of graded bimodules $\underline{\text{gr}}\text{-}\Pi^e$ (where $\Pi(1)$ is the graded bimodule Π shifted by 1). This isomorphism can also be written as

$$\mathbf{R}\text{Hom}_{\Pi^e}(\Pi, \Pi^e)[3] \cong \Pi(1) \quad \text{in } D^b(\underline{\text{gr}}\text{-}\Pi^e)/\text{gr-perf } \Pi^e$$

using the triangle equivalence $\underline{\text{gr}}\text{-}\Pi^e \cong D^b(\underline{\text{gr}}\text{-}\Pi^e)/\text{gr-perf } \Pi^e$. The main result of this paper is the following.

Theorem 1.1 (Theorem 3.1). *Let $\Pi = \bigoplus_{i \geq 0} \Pi_i$ be a finite-dimensional graded algebra satisfying the following properties:*

- (a) $\text{pdim } D\Pi = \text{idim } \Pi \leq d - 2$, that is, Π is of Gorenstein dimension $\leq d - 2$;
- (b) $\mathbf{R}\text{Hom}_{\Pi^e}(\Pi, \Pi^e)[d + 1] \cong \Pi(1)$ in $D^b(\underline{\text{gr}}\text{-}\Pi^e)/\text{gr-perf } \Pi^e$; and
- (c) $\text{Ext}_{\underline{\text{gr}}\text{-}\Pi^e}^j(\Pi, \Pi^e(i)) = 0$ for all $i < 0$ and $j > 0$.

Then Π is isomorphic as a graded algebra to $\Pi_d(\Lambda)$ for some algebra Λ of global dimension at most $d - 1$.

Property (b) can be understood as an algebraic (and graded) enhancement of the property stably Calabi–Yau for the algebra Π . In particular, it implies that the stable category of maximal Cohen–Macaulay modules over Π is a d -Calabi–Yau triangulated category. Gorensteinness and stably Calabi–Yau property are homological properties that appear naturally in the study of preprojective algebras (see [20]). Therefore (a) and (b) are natural hypotheses to consider. Hypothesis (c) becomes also natural when the algebra Π has finite global dimension (see Observation 3.3).

In a second part of the paper, we show that in certain situations also the converse of the above theorem holds. We prove that properties (a)–(c) are satisfied by the finite-dimensional preprojective algebras for $d = 2$ and 3 (Theorems 5.1 and 5.7). Moreover, using the results of Dugas in [5], we

prove that properties (a)–(c) hold for self-injective d -preprojective algebras for any d . More precisely we prove the following.

Theorem 1.2 (Theorems 4.8 and 5.9). *The map $\Lambda \mapsto \Pi_d(\Lambda)$ gives a one-to-one correspondence between $(d - 1)$ -representation finite algebras Λ and finite-dimensional self-injective graded algebras Π satisfying $\mathbf{RHom}_{\Pi^e}(\Pi, \Pi^e)[d + 1] \cong \Pi(1)$ in $\mathbf{D}^b(\text{gr-}\Pi^e)/\text{gr-perf } \Pi^e$.*

This result is very similar to Theorem 4.35 of [12] that asserts that the preprojective construction gives a one-to-one correspondence between $(d - 1)$ representation-infinite algebras Λ and homologically smooth algebras Π satisfying $\mathbf{RHom}_{\Pi^e}(\Pi, \Pi^e)[d] \cong \Pi(1)$ in $\mathbf{D}^b(\text{gr-}\Pi^e)$.

Plan of the paper

The paper is organized as follows. We start in Section 2 with preliminaries on graded Gorenstein algebras, and Cohen–Macaulay modules, and define the notion of bimodule stably (1) -twisted d -Calabi–Yau algebras. The main result of the paper is proved in Section 3. Section 4 gives an interpretation of the Gorenstein dimension of a d -preprojective algebra $\Pi_d(\Lambda)$ in terms of some Hom-vanishing in the category $\mathbf{D}^b(\text{mod } \Lambda)$. In Section 5, we prove that the converse of Theorem 3.1 is true for $d = 2$ and 3, and for preprojective algebras of $(d - 1)$ -representation finite algebras.

Notation

All algebras in this paper are finite-dimensional algebras over a field k . For an algebra A , we denote by A^e the tensor algebra $A \otimes A^{\text{op}}$. The dual $\text{Hom}_k(A, k)$ of A is denoted by DA . When nothing else is stated explicitly, tensor products are over the field k .

2. Preliminaries

2.1. Graded algebras and graded modules

Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be a positively graded algebra. For a graded A -module $M = \bigoplus_{n \in \mathbb{Z}} M_n$, and for any $p \in \mathbb{Z}$, we denote by $M(p)$ the graded module $\bigoplus_{n \in \mathbb{Z}} M_{n+p}$, which is the degree n part of $M(p)$ is M_{p+n} . We denote by $\text{gr-}A$ the category of finitely generated graded A -modules. Morphisms in $\text{gr-}A$ are

graded morphisms homogeneous of degree 0. The category $\text{gr-}A$ is an abelian Krull–Schmidt category.

By an abuse of notation, for $M \in \text{gr-}A$ we will denote by $M \in \text{mod } A$ its image in $\text{mod } A$ under the forgetful functor $\text{gr-}A \rightarrow \text{mod } A$. Note that the A^{op} -module $\text{Hom}_A(M, A) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\text{gr-}A}(M, A(p))$ is naturally graded.

We write

$$\begin{aligned} \text{gr-proj}_{\leq 0} A &= \text{add}\{A(i) \mid i \geq 0\}, \\ \text{gr-proj}_{> 0} A &= \text{add}\{A(i) \mid i < 0\} \end{aligned}$$

for the subcategories of $\text{gr-proj } A$ of projectives generated in positive, respectively, in non-positive, degrees.

For an additive category \mathcal{A} , we denote by $\text{K}^{\text{b},-}(\mathcal{A})$ (resp. $\text{K}^{\text{b}}(\mathcal{A})$) the homotopy category of right bounded (resp. bounded) complexes of objects in \mathcal{A} .

Proposition 2.1. *Let A be a positively graded algebra, such that A_0 has finite global dimension. Then $\text{D}^{\text{b}}(\text{gr-}A) = \text{K}^{\text{b},-}(\text{gr-proj } A)$ has a semiorthogonal decomposition*

$$\text{K}^{\text{b},-}(\text{gr-proj } A) = \left\langle \text{K}^{\text{b}}(\text{gr-proj}_{\leq 0} A), \text{K}^{\text{b},-}(\text{gr-proj}_{> 0} A) \right\rangle.$$

That is, we have $\text{Hom}_{\text{K}^{\text{b},-}(\text{gr-proj } A)}(\text{K}^{\text{b}}(\text{gr-proj}_{\leq 0} A), \text{K}^{\text{b},-}(\text{gr-proj}_{> 0} A)) = 0$, and for $X \in \text{K}^{\text{b},-}(\text{gr-proj } A)$ there is a triangle

$$\underbrace{X_{\leq 0}}_{\in \text{K}^{\text{b}}(\text{gr-proj}_{\leq 0} A)} \longrightarrow X \longrightarrow \underbrace{X_{> 0}}_{\in \text{K}^{\text{b},-}(\text{gr-proj}_{> 0} A)} \longrightarrow .$$

Remark 2.2. It follows that the triangle in Proposition 2.1 is functorial in X . We will denote by $(-)>_0$ and $(-)\leq_0$ the functors in the triangle.

This semiorthogonal decomposition has already been used in [23].

Proof of Proposition 2.1. Since A is positively graded the space $\text{Hom}_{\text{gr-}A}(A(i), A(j))$ vanishes whenever $j < i$. It follows that

$$\text{Hom}_{\text{gr-proj } A}(\text{gr-proj}_{\leq 0} A, \text{gr-proj}_{> 0} A) = 0,$$

and thus we have the Hom-vanishing of the proposition.

To obtain the triangle, we observe that any graded projective is the direct sum of a graded projective generated in non-positive degree and a

graded projective generated in positive degree. Choosing such decompositions for all terms of a right bounded complex X of graded projective A -modules gives rise to a short exact sequence $X_{\leq 0} \twoheadrightarrow X \twoheadrightarrow X_{> 0}$ and thus to a triangle in the homotopy category, where $X_{\leq 0} \in \mathbf{K}^{\mathbf{b},-}(\text{gr-proj}_{\leq 0} A)$ and $X_{> 0} \in \mathbf{K}^{\mathbf{b},-}(\text{gr-proj}_{> 0} A)$. Finally observe that, since A_0 has finite global dimension, we may assume that only finitely many terms of the complex are not generated in positive degrees, that is $X_{\leq 0}$ is also left bounded. \square

2.2. Graded Cohen–Macaulay modules

Definition 2.3. An algebra A is said to be *Gorenstein* if its injective dimension (denoted $\text{id } A$) and the projective dimension of its dual (denoted $\text{pd } DA$) are both finite. For such an algebra, the *Gorenstein dimension of A* is the integer $\text{id } A = \text{pd } A$. We define the category of (*maximal*) *Cohen–Macaulay* modules by

$$\text{CM}(A) := \{X \in \text{mod } A \mid \text{Ext}_A^i(X, A) = 0 \text{ for } i > 0\}.$$

If moreover A is a graded algebra, we denote by

$$\text{gr-CM}(A) := \{X \in \text{gr-}A \mid \text{Ext}_A^i(X, A) = 0 \text{ for } i > 0\}$$

the category of graded (maximal) Cohen–Macaulay A -modules.

The next result is the graded version of a famous triangle equivalence [3, Theorem 4.4.1] (see also [19, 24]).

Theorem 2.4. *Let A be a graded Gorenstein algebra, then $\text{gr-CM}(A)$ is a Frobenius category and there is a triangle equivalence*

$$\text{D}^{\mathbf{b}}(\text{gr-}A)/\text{gr-perf } A \xrightarrow{\sim} \underline{\text{gr-CM}}(A).$$

Because of this equivalence, we will use the notation $\underline{\text{gr-CM}}(A)$ for the category $\text{D}^{\mathbf{b}}(\text{gr-}A)/\text{gr-perf}(A)$. That is, we may write $M \cong N$ in $\underline{\text{gr-CM}}(A)$ even if M and N are not Cohen–Macaulay modules.

The next two lemmas are classical results on maximal Cohen–Macaulay modules that will be used in this paper.

Lemma 2.5. *Let A be a graded Gorenstein algebra of dimension g . For $X \in \text{gr-}A$, its g th syzygy $\Omega^g(X)$ is Cohen–Macaulay.*

Lemma 2.6. *Let A be a graded Gorenstein ring and M a graded A -module. Then there is a short exact sequence*

$$K \twoheadrightarrow {}_{\text{CM}}M \twoheadrightarrow M$$

with $\text{pd } K < \infty$ and ${}_{\text{CM}}M \in \text{gr-CM } A$.

In this situation ${}_{\text{CM}}M$ is called a Cohen Macaulay replacement of M .

Proof. This fact is well known, but we include a sketch of the proof for the convenience of the reader.

Sufficiently high syzygies of M are Cohen–Macaulay by Lemma 2.5, so the claim trivially holds for them. Assume we already found the upper sequence in the diagram below:

$$\begin{array}{ccccc}
 K_i & \twoheadrightarrow & {}_{\text{CM}}(\Omega^i M) & \twoheadrightarrow & \Omega^i M \\
 & & \downarrow \text{approx.} & & \downarrow \\
 & & \bar{P}_{i-1} & \dashrightarrow & P_{i-1} \\
 & & \downarrow & & \downarrow \\
 K_{i-1} & \dashrightarrow & {}_{\text{CM}}(\Omega^{i-1} M) & \dashrightarrow & \Omega^{i-1} M
 \end{array}$$

The right vertical sequence is the defining sequence of $\Omega^i M$, and the middle one is obtained by taking a left projective approximation of ${}_{\text{CM}}(\Omega^i M)$ and its cokernel. We obtain an induced map as indicated by the upper dashed arrow, which can be assumed to be split epi (by enlarging \bar{P}_{i-1} if necessary). Then we obtain the desired sequence in the lower row of the diagram. Indeed, K_{i-1} has finite projective dimension since both the kernel of $\bar{P}_{i-1} \twoheadrightarrow P_{i-1}$ and K_i do. □

Remark 2.7. The proof above also shows that, provided M is generated in degree 0, we may choose ${}_{\text{CM}}M$ to only have projective summands which are generated in degree 0.

Lemma 2.8. *In the setup of Lemma 2.6, the following are equivalent:*

- (1) $\forall j > 0 \forall i < 0: \text{Ext}_{\text{gr-}A}^j(M, A(i)) = 0$ and
- (2) K can be chosen such that it has a projective resolution in $\text{gr-proj}_{\leq 0} A$.

Proof. Since ${}_{\text{CM}}M$ is Cohen–Macaulay, applying the functor $\text{Hom}_{\text{gr-}A}(-, A(i))$ to the short exact sequence, we obtain isomorphisms

$$\text{Ext}_{\text{gr-}A}^j(M, A(i)) \cong \text{Ext}_{\text{gr-}A}^{j-1}(K, A(i)) \quad \text{for all } j > 1$$

and an exact sequence

$$\begin{aligned} \text{Hom}_{\text{gr-}A}(M, A(i)) &\twoheadrightarrow \text{Hom}_{\text{gr-}A}({}_{\text{CM}}M, A(i)) \longrightarrow \\ &\longrightarrow \text{Hom}_{\text{gr-}A}(K, A(i)) \longrightarrow \text{Ext}_{\text{gr-}A}^1(M, A(i)). \end{aligned}$$

(1) \implies (2): Pick a minimal projective resolution of K

$$0 \longrightarrow P_d \longrightarrow \dots \longrightarrow P_0 \longrightarrow K.$$

Let j be the leftmost position such that P_j is not contained in $\text{gr-proj}_{\leq 0} A$ (assuming such a term exists). It follows that $\text{Ext}_{\text{gr-}A}^j(K, A(i)) \neq 0$ for some $i < 0$. By (1) and the discussion above it follows that $j = 0$. Thus K has a non-zero direct summand in $\text{gr-proj}_{> 0} A$. Since $\text{Ext}_{\text{gr-}A}^1(M, \text{gr-proj}_{> 0} A) = 0$ such a summand can be split off of the short exact sequence.

(2) \implies (1): Since K has a projective resolution in $\text{gr-proj}_{\leq 0} A$, $\text{Ext}_{\text{gr-}A}^j(K, A(i))$ vanishes for all $i < 0$ and all j . The claim then immediately follows from the discussion above. □

2.3. Stably graded Calabi–Yau algebras

Definition 2.9. A Gorenstein graded algebra A is called *stably (p)-twisted d -Calabi–Yau* if there is a functorial isomorphism

$$D\underline{\text{Ext}}_{\text{gr-}A}^i(X, Y) \cong \underline{\text{Ext}}_{\text{gr-}A}^{d-i}(Y(p), X)$$

for any $X, Y \in \text{gr-CM } \Pi$ and for any $i \in \mathbb{Z}$.

Definition 2.10. A graded algebra A is called *bimodule stably (p)-twisted d -Calabi–Yau* if there is an isomorphism

$$\mathbf{R}\text{Hom}_{A^e}(A, A^e)[d + 1] \cong A(p) \text{ in } \underline{\text{gr-CM}}(A^e).$$

The integer p is called the Gorenstein parameter.

Remark 2.11 (see [10]). Recall that a bimodule $(d + 1)$ -Calabi–Yau algebra is a homologically smooth algebra satisfying

$$\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)[d + 1] \cong A \text{ in } \mathbf{D}^b(\mathrm{gr}\text{-}A^e).$$

So the choice of d for the Calabi–Yau dimension in our definition could seem strange. But the reason of our choice is motivated by the following result.

Theorem 2.12. *Let A be a finite-dimensional graded Gorenstein algebra which is bimodule stably (p) -twisted d -Calabi–Yau, then A is stably (p) -twisted d -Calabi–Yau.*

Proof. This can be shown using an Auslander–Reiten formula in $\mathrm{gr}\text{-CM } A$ as it is done in [27, 3.10; 14, Theorem 8.3] for local isolated singularities. Here we give a different argument using the description of the category $\underline{\mathrm{gr}\text{-CM } A}$ as the localization $\mathbf{D}^b(\mathrm{gr}\text{-}A)/\mathrm{gr}\text{-perf } A$.

From [17, Lemma 4.1] we know that for any $X, Y \in \mathbf{D}^b(\mathrm{gr}\text{-}A)$ there is a functorial isomorphism

$$D\mathrm{Hom}_{\mathbf{D}^b(\mathrm{gr}\text{-}A)}(X, Y) \cong \mathrm{Hom}_{\mathbf{D}^b(\mathrm{gr}\text{-}A)}(\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e) \overset{\mathbf{L}}{\otimes}_A Y, X).$$

Moreover since the algebra A is Gorenstein, the functor $\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e) \overset{\mathbf{L}}{\otimes}_A -$ sends any perfect complex to a perfect complex. Therefore, we can apply [1, Proposition 4.3.1] to deduce that the category $\mathbf{D}^b(\mathrm{gr}\text{-}A)/\mathrm{gr}\text{-perf } A$ has a Serre functor whose inverse is given by $\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)[1] \overset{\mathbf{L}}{\otimes}_A -$.

Now for any objects $X, Y \in \mathrm{gr}\text{-CM } A$ and for any $i \in \mathbb{Z}$ we have functorial isomorphisms

$$\begin{aligned} D\underline{\mathrm{Ext}}^i_{\mathrm{gr}\text{-}A}(X, Y) &\cong D\underline{\mathrm{Hom}}_{\mathrm{gr}\text{-}A}(X, Y[i]) \\ &\cong \underline{\mathrm{Hom}}_{\mathrm{gr}\text{-}A}(\mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)[i + 1] \overset{\mathbf{L}}{\otimes}_A Y, X) \\ &\cong \underline{\mathrm{Hom}}_{\mathrm{gr}\text{-}A}(A[i - d](p) \overset{\mathbf{L}}{\otimes}_A Y, X) \\ &\cong \underline{\mathrm{Hom}}_{\mathrm{gr}\text{-}A}(Y(p), X[d - i]) \\ &\cong \underline{\mathrm{Ext}}^{d-i}_{\mathrm{gr}\text{-}A}(Y(p), X). \quad \square \end{aligned}$$

Note that if A is bimodule stably (p) -twisted d -Calabi–Yau, then it is bimodule stably d -Calabi–Yau as an ungraded algebra. Therefore, by the same argument the stable category $\underline{\mathrm{CM } A}$ is d -Calabi–Yau.

Remark 2.13. For a self-injective (or a Frobenius) algebra, the non-graded version of Definition 2.10 coincide with the definition of Frobenius d -Calabi–Yau algebra [8, Definition 2.3.6]. In this setup, Theorem 2.12 is [8, Theorem 2.3.21].

2.4. Higher preprojective algebras

Let Λ be a finite-dimensional algebra of global dimension $d - 1$. We denote by $\mathbb{S}_{d-1} = -\overset{\mathbf{L}}{\otimes}_{\Lambda} D\Lambda[-d + 1]$ the composition of the Serre functor with the $(d - 1)$ desuspension of the bounded derived category $D^b(\text{mod } \Lambda)$. It is an autoequivalence of $D^b(\text{mod } \Lambda)$. We denote by τ_{d-1} the composition

$$\text{mod } \Lambda \hookrightarrow D^b(\text{mod } \Lambda) \xrightarrow{\mathbb{S}_{d-1}} D^b(\text{mod } \Lambda) \xrightarrow{H^0} \text{mod } \Lambda.$$

The algebra Λ is called τ_{d-1} -finite if the functor τ_{d-1} is nilpotent.

Definition 2.14 ([16]). The d -preprojective algebra of Λ is defined to be the tensor algebra

$$\Pi_d(\Lambda) := T_{\Lambda} \text{Ext}_{\Lambda}^{d-1}(D\Lambda, \Lambda).$$

It is immediate to see that we have $\Pi_d\Lambda \cong \bigoplus_{p \geq 0} \tau_{d-1}^{-p}\Lambda$ as Λ -bimodules. Hence the algebra Λ is τ_{d-1} -finite if and only if $\Pi_d(\Lambda)$ is finite-dimensional.

3. Homological characterization of finite-dimensional preprojective algebras

In this section, we prove the main result of the paper, that gives a sufficient condition for a finite-dimensional graded algebra Π to be the d -preprojective algebra of its degree zero subalgebra.

3.1. Main result and strategy of the proof

Theorem 3.1. *Let Π be a finite-dimensional positively graded algebra, and an integer $d \geq 2$, such that*

- (1) Π is Gorenstein of dimension at most $d - 2$, that is $\text{id } \Pi = \text{pd } D\Pi = g$, for some $g \leq d - 2$;
- (2) Π is bimodule stably (1)-twisted d -Calabi–Yau; and

(3) $\text{Ext}_{\text{gr-}\Pi^e}^j(\Pi, \Pi^e(i)) = 0$ for all $i < 0$ and $j > 0$.

Then $\Lambda = \Pi_0$ is a τ_{d-1} -finite algebra of global dimension $\leq d - 1$, and Π is the d -preprojective algebra of Λ .

The main ingredient of the proof is the triangle

$$\Pi_{\leq 0} \longrightarrow \Pi \longrightarrow \Pi_{> 0} \longrightarrow \quad \text{in } D^b(\text{gr-}\Pi^e)$$

given by Proposition 2.1, that is a decomposition of the projective bimodule resolution of Π according to the degrees the projective bimodules are generated in. More precisely, the strategy of the proof consists of the computation of the cohomology groups of the triangle

$$\Pi_{\leq 0} \otimes_{\Pi}^{\mathbf{L}} \Lambda \longrightarrow \Pi \otimes_{\Pi}^{\mathbf{L}} \Lambda \longrightarrow \Pi_{> 0} \otimes_{\Pi}^{\mathbf{L}} \Lambda \longrightarrow \quad \text{in } D^b(\text{gr-}\Pi \otimes \Lambda^{\text{op}}).$$

Let us start by reformulating some of the conditions. By Lemmas 2.6 and 2.8 and Remark 2.7 there is a short exact sequence of Π^e -modules

$$K \twoheadrightarrow M \longrightarrow \Pi,$$

such that K has a finite projective resolution, with terms generated in non-positive degree, and M is a Cohen–Macaulay Π^e -module, all of whose projective summands are generated in degree zero.

Observation 3.2. We may reformulate the condition that Π is bimodule stably (1)-twisted d -Calabi–Yau (condition (2)) by adding the following isomorphisms (in $\underline{\text{gr-CM}}(\Pi^e)$) in front of and after the defining one:

$$\mathbf{R}\text{Hom}_{\Pi^e}(M, \Pi^e)[d + 1] \cong \mathbf{R}\text{Hom}_{\Pi^e}(\Pi, \Pi^e)[d + 1] \stackrel{\text{def}}{\cong} \Pi(1) \cong \Omega_{\Pi^e}^{d+1} \Pi[d + 1](1).$$

Note that since M is Cohen–Macaulay we may drop the \mathbf{R} in the left-most term. Moreover by condition (1) we have that also $\Omega_{\Pi^e}^{d+1} \Pi$ is Cohen–Macaulay. Thus, under condition (1), condition (2) is equivalent to

$$\text{Hom}_{\Pi^e}(M, \Pi^e)(-1) \cong \Omega_{\Pi^e}^{d+1} \Pi$$

up to projective summands generated in degree 1.

Observation 3.3. When we consider the special case where Π has finite global dimension, conditions (1)–(3) are very easy to check. Then we immediately have $\text{gl.dim } \Pi = g$, so condition (1) holds whenever $\text{gl.dim } \Pi \leq d - 2$.

Moreover condition (2) vacuously holds. Thus it remains to consider condition (3).

Lemma 3.4. *If Π is a positively graded algebra with finite global dimension. Then condition (3) is equivalent to Π being concentrated in degree 0.*

Proof. If Π is concentrated in degree 0, then clearly (3) holds.

Conversely, assume that Π is not concentrated in degree 0. Then there are two graded simples S and T such that $\text{Ext}_{\text{gr-}\Pi}^1(S, T) \neq 0$, where S is concentrated in degree 0 and T in some positive degree. Now, considering the injective resolution of S (note that this is concentrated in non-positive degrees) and the projective resolution of T (which, similarly, is concentrated in positive degrees), we find that $\text{Ext}^i(D\Pi, \Pi(j)) \neq 0$ for some positive i and negative j . □

On the other hand, we see that under condition (1) the degree 0 part of Π also has global dimension $\leq d - 2$, so the d -preprojective algebra of Π_0 is Π_0 . In other words, Π is the d -preprojective algebra of its degree 0 part if and only if it is concentrated in degree 0.

This indicates that, at least in the special case of Π having finite global dimension, condition (3) actually is the “correct” condition here.

3.2. The global dimension of Λ

Proposition 3.5. *In the setup of Theorem 3.1 we have $\text{gl.dim } \Lambda \leq d - 1$.*

Proof. It suffices to show that $\text{pd}_{\Lambda^e} \Lambda \leq d - 1$. We have the following isomorphisms of Λ^e -modules:

$$\begin{aligned} \Omega_{\Lambda^e}^d \Lambda &\cong (\Omega_{\Pi^e}^d \Pi)_0 \\ &\cong (\Omega_{\Pi^e}^{-1} \Omega_{\Pi^e}^{d+1} \Pi)_0 && \text{since } \Omega_{\Pi^e}^d \Pi \text{ is CM (Assumption (1))} \\ &\cong (\Omega_{\Pi^e}^{-1} \text{Hom}_{\Pi^e}(M, \Pi^e)(-1))_0 && \text{by Observation 3.2} \\ &\cong (\text{Hom}_{\Pi^e}(\Omega_{\Pi^e} M, \Pi^e)(-1))_0 \\ &= \text{Hom}_{\text{gr-}\Pi^e}(\Omega_{\Pi^e} M, \Pi^e(-1)). \end{aligned}$$

From the short exact sequence $K \twoheadrightarrow M \twoheadrightarrow \Pi$, it is easy to construct a short exact sequence $K' \twoheadrightarrow \Omega_{\Pi^e} M \twoheadrightarrow \Omega_{\Pi^e} \Pi$ with K' having a projective resolution

in $\text{gr-proj}_{\leq 0} \Pi^e$. Thus $\text{Hom}_{\text{gr-}\Pi^e}(K', \Pi^e(-1))$ vanishes and we have

$$\text{Hom}_{\text{gr-}\Pi^e}(\Omega_{\Pi^e} M, \Pi^e(-1)) \cong \text{Hom}_{\text{gr-}\Pi^e}(\Omega_{\Pi^e} \Pi, \Pi^e(-1)).$$

Therefore we have

$$\Omega_{\Lambda^e}^d(\Lambda) \cong \text{Ext}_{\text{gr-}\Pi^e}^1(\Pi, \Pi^e(-1)) = 0$$

by assumption (3). □

3.3. Proof of Theorem 3.1

We start with a technical lemma that will be useful.

Lemma 3.6. *In the setup of Theorem 3.1 we have an isomorphism*

$$H^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) \cong H^{i+d}(\Pi(-1) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{R}\text{Hom}_{\Lambda}(D\Lambda, \Lambda)) \quad \text{in } \text{gr}(\Pi \otimes \Lambda^{\text{op}})$$

for all $i \geq -d + g + 1$, where g is the Gorenstein dimension of Π .
In particular

$$H^{-1}(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) \cong \Pi(-1) \otimes_{\Lambda} \text{Ext}_{\Lambda}^{d-1}(D\Lambda, \Lambda)$$

and

$$H^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) = 0 \quad \forall i \geq 0.$$

Proof. Since K has a projective resolution with terms in $\text{gr-proj}_{\leq 0} \Pi^e$ we have

$$\Pi_{>0} \cong M_{>0} \quad \text{in } D^b(\text{gr-}\Pi^e).$$

From Observation 3.2 we know that the graded Cohen–Macaulay Π^e -modules M and $\text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^{d+1} \Pi, \Pi^e)(-1)$ are isomorphic up to projective summands generated in degree 0. Thus

$$\Pi_{>0} \cong (\text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^{d+1} \Pi, \Pi^e)(-1))_{>0}.$$

Next we denote by P the complex formed by the first $d + 1$ terms of a graded projective resolution of Π as Π^e -module. Thus we have a triangle

$$\Omega_{\Pi^e}^{d+1} \Pi[d] \longrightarrow P \longrightarrow \Pi \longrightarrow \quad \text{in } D^b(\text{gr-}\Pi^e).$$

Applying the functor $(\mathbf{R}\mathrm{Hom}_{\Pi^e}(-[-d], \Pi^e)(-1))_{>0}$ we obtain

$$\begin{aligned} (\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1))_{>0} &\longrightarrow (\mathrm{Hom}_{\Pi^e}(P[-d], \Pi^e)(-1))_{>0} \\ &\longrightarrow \Pi_{>0} \longrightarrow . \end{aligned}$$

Observe that since $\Omega_{\Pi^e}^g \Pi$ is Cohen–Macaulay, the complex $\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Omega_{\Pi^e}^g \Pi, \Pi^e)$ is concentrated in homological degree 0, hence the complex $\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)$ is concentrated in homological degrees $0, \dots, g$. Therefore, the complex

$$\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1) = \mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)[d](-1)$$

is concentrated in homological degrees $-d, \dots, -d + g$. Thus the complex

$$\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1) \otimes_{\Pi}^{\mathbf{L}} \Lambda$$

is also concentrated in homological degrees $\leq -d + g$.

It follows that for $i \geq -d + g + 1$ we have

$$\begin{aligned} (3.1) \quad \mathrm{H}^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) &\cong \mathrm{H}^i(\mathrm{Hom}_{\Pi^e}(P[-d], \Pi^e)(-1))_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda \\ &\cong \mathrm{H}^{i+d}((\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda). \end{aligned}$$

Moreover, observe that

$$(\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \cong \mathrm{Hom}_{\Pi^e}(P, \Pi^e)_{>-1}(-1) \cong \mathrm{Hom}_{\Pi^e}(P_{\leq 0}, \Pi^e)(-1).$$

Since Π is a positively graded algebra, $P_{\leq 0}$ is generated in degree 0. Hence $P_{\leq 0} = \Pi \otimes Q \otimes \Pi$, where Q is the complex formed by the first $(d + 1)$ terms of a projective resolution of $\Pi_0 = \Lambda$ as Λ^e -module. Since $\mathrm{gl.dim} \Lambda \leq d - 1$, Q is a projective resolution of Λ as Λ^e -module, this leads to

$$(3.2) \quad (\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \cong \Pi \otimes_{\Lambda} \mathrm{Hom}_{\Lambda^e}(Q, \Lambda^e) \otimes_{\Lambda} \Pi(-1).$$

Combining (3.1) and (3.2) we obtain

$$\begin{aligned} \mathrm{H}^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) &\cong \mathrm{H}^{i+d}((\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) \\ &\cong \mathrm{H}^{i+d}(\Pi \otimes_{\Lambda} \mathrm{Hom}_{\Lambda^e}(Q, \Lambda^e) \otimes_{\Lambda} \Pi(-1) \otimes_{\Pi} \Lambda) \\ &\cong \mathrm{H}^{i+d}(\Pi(-1) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\Lambda^e}(\Lambda, \Lambda^e)) \\ &\cong \mathrm{H}^{i+d}(\Pi(-1) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\Lambda}(D\Lambda, \Lambda)) \end{aligned}$$

for $i \geq -d + g + 1$.

Now since $g \leq d - 2$, we have $g - d + 1 \leq -1$. Moreover the global dimension of Λ is at most $d - 1$, so the complex $\mathbf{R}\mathrm{Hom}_\Lambda(D\Lambda, \Lambda)$ is concentrated in degrees $\leq d - 1$. Hence we have

$$\begin{aligned} \mathrm{H}^{i+d}(\Pi(-1) \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_\Lambda(D\Lambda, \Lambda)) &\cong \Pi(-1) \otimes_\Lambda \mathrm{Ext}^{d-1}(D\Lambda, \Lambda) && \text{for } i = -1 \\ &= 0 && \text{for } i \geq 0. \end{aligned}$$

This finishes the proof. □

We now obtain the following short exact sequence, which is an essential ingredient to our proof.

Proposition 3.7. *In the setup of Theorem 3.1 we have a short exact sequence of graded $\Pi \otimes \Lambda^{\mathrm{op}}$ -modules*

$$\Pi(-1) \otimes_\Lambda \mathrm{Ext}_\Lambda^{d-1}(D\Lambda, \Lambda) \twoheadrightarrow \Pi \longrightarrow \Lambda,$$

where $\Pi \rightarrow \Lambda$ is the natural projection.

Proof. Consider Π as an object in $\mathrm{D}^b(\mathrm{gr}\text{-}\Pi^e)$, and the triangle

$$\Pi_{\leq 0} \longrightarrow \Pi \longrightarrow \Pi_{> 0} \longrightarrow$$

given by Proposition 2.1. Applying the functor $-\otimes_\Pi^{\mathbf{L}} \Lambda$ we obtain a triangle

$$\Pi_{\leq 0} \otimes_\Pi^{\mathbf{L}} \Lambda \longrightarrow \Pi \otimes_\Pi^{\mathbf{L}} \Lambda \longrightarrow \Pi_{> 0} \otimes_\Pi^{\mathbf{L}} \Lambda \longrightarrow \quad \text{in } \mathrm{D}^b(\mathrm{gr}\text{-}\Pi \otimes \Lambda^{\mathrm{op}}).$$

Recall that from Observation 3.2, $\Omega_{\Pi^e}^{d+1}\Pi$ and $\mathrm{Hom}_{\Pi^e}(M, \Pi^e)(-1)$ are isomorphic up to projective summands generated in degree 1. Hence we have the following isomorphisms in $\mathrm{D}^b(\mathrm{gr}\text{-}\Pi^e)$:

$$\begin{aligned} (\Omega_{\Pi^e}^{d+1}\Pi)_{\leq 0} &\cong (\mathrm{Hom}_{\Pi^e}(M, \Pi^e)(-1))_{\leq 0} \\ &\cong (\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{\leq 0} && \text{since the projective resolution} \\ & && \text{of } K \text{ is generated in non-positive} \\ & && \text{degrees} \\ &= 0 && \text{since } \Pi \text{ is a positively} \\ & && \text{graded algebra.} \end{aligned}$$

So we get $\Pi_{\leq 0} \cong P_{\leq 0} = \Pi \otimes_\Lambda Q \otimes \Pi$ where Q is a projective resolution of Λ as Λ^e -module. Therefore we obtain $\Pi_{\leq 0} \otimes_\Pi^{\mathbf{L}} \Lambda \cong \Pi$.

Since we clearly have $\Pi \otimes_{\Pi}^{\mathbb{L}} \Lambda = \Lambda$ we obtain an exact sequence

$$H^{-1}(\Pi_{>0} \otimes_{\Pi}^{\mathbb{L}} \Lambda) \rightarrow \Pi \rightarrow \Lambda \rightarrow H^0(\Pi_{>0} \otimes_{\Pi}^{\mathbb{L}} \Lambda).$$

Now the claim follows from the “in particular” part of Lemma 3.6 above. □

By Proposition 3.7 we have an isomorphism of graded $\Pi \otimes \Lambda^{\text{op}}$ -modules

$$\Pi(-1) \otimes_{\Lambda} \text{Ext}_{\Lambda}^{d-1}(D\Lambda, \Lambda) \cong \Pi_{>0}.$$

The next proposition shows that this is enough to identify Π as the tensor algebra $T_{\Lambda} \text{Ext}_{\Lambda}^{d-1}(D\Lambda, \Lambda)$ and thus finishes the proof of Theorem 3.1.

Proposition 3.8. *Let Π be a positively graded ring, $\Lambda = \Pi_0$, and X a $\Lambda \otimes \Lambda^{\text{op}}$ -module. Assume there is an isomorphism*

$$\Pi(-1) \otimes_{\Lambda} X \cong \Pi_{>0}$$

of graded $\Pi \otimes \Lambda^{\text{op}}$ -modules.

Then, as graded rings,

$$\Pi \cong T_{\Lambda} X.$$

Proof. Let $h: \Pi(-1) \otimes_{\Lambda} X \xrightarrow{\cong} \Pi_{>0}$ be as in the assumption.

We define an isomorphism of graded $\Lambda \otimes \Lambda^{\text{op}}$ -modules

$$\varphi: T_{\Lambda} X \rightarrow \Pi$$

iteratedly by $\varphi_0 = \text{id}_{\Lambda}$ and by letting φ_n be the composition

$$(T_{\Lambda} X)_n = X^{\otimes_{\Lambda} n} \xrightarrow{\varphi_{n-1} \otimes \text{id}_X} \Pi_{n-1} \otimes_{\Lambda} X \cong \Pi_n,$$

where the last isomorphism is the degree n -part of h .

It only remains to check that φ respects the ring-multiplication. It suffices to check that

$$\varphi_{n+m}(f \otimes g) = \varphi_n(f)\varphi_m(g)$$

for any $f \in X^{\otimes_{\Lambda} n}$ and $g \in X^{\otimes_{\Lambda} m}$. We show this by induction on m . For $m = 0$ this is just the Λ -linearity of φ_n . For $m = 1$ we have

$$\varphi_{n+1}(f \otimes g) \stackrel{\text{def}}{=} h(\varphi_n(f) \otimes g) = \varphi_n(f)h(1 \otimes g) \stackrel{\text{def}}{=} \varphi_n(f)\varphi_1(g).$$

For $m > 1$ we may assume that $g = x \otimes g'$ for some $x \in X$. (An arbitrary element is a sum of elementary tensors, but since all maps involved are linear it suffices to consider a single elementary tensor.) Now

$$\begin{aligned} \varphi_{n+m}(f \otimes g) &= \varphi_{n+m}(f \otimes x \otimes g') \\ &= \varphi_{n+1}(f \otimes x)\varphi_{m-1}(g') && \text{by inductive assumption} \\ &= \varphi_n(f)\varphi_1(x)\varphi_{m-1}(g') && \text{by the case } m = 1 \\ &= \varphi_n(f)\varphi_1(x \otimes g') && \text{by inductive assumption} \\ &= \varphi_n(f)\varphi_m(g). \end{aligned} \quad \square$$

Observation 3.9. Since Π is a finite-dimensional algebra, it follows that Λ is τ_{d-1} -finite, since this is equivalent to $T_\Lambda \text{Ext}_\Lambda^{d-1}(D\Lambda, \Lambda)$ being finite-dimensional.

4. The Gorenstein dimension

4.1. Gorenstein dimension and cluster tilting subcategories

In this section, we express the Gorenstein dimension of the algebra Π of Theorem 3.1 in terms of certain vanishing of extensions in the derived category $D^b(\text{mod } \Lambda)$. We start with the following result/definition.

Theorem 4.1 ([1, 5.4.2]; [13, 1.22]). *Let Λ be a τ_{d-1} -finite algebra. Then the category*

$$\mathcal{U} = \text{add}\{\mathbb{S}_{d-1}^i \Lambda \mid i \in \mathbb{Z}\} \subseteq D^b(\text{mod } \Lambda)$$

is a $(d - 1)$ -cluster tilting subcategory, that is,

$$\begin{aligned} \mathcal{U} &= \{X \in D^b(\text{mod } \Lambda) \mid \text{Ext}_\Lambda^i(\mathcal{U}, X) = 0 \ \forall i = 1, \dots, d - 2\} \\ &= \{X \in D^b(\text{mod } \Lambda) \mid \text{Ext}_\Lambda^i(X, \mathcal{U}) = 0 \ \forall i = 1, \dots, d - 2\}. \end{aligned}$$

Note that if Π is an algebra as in Theorem 3.1, its degree zero subalgebra Λ is always τ_{d-1} -finite, so the result above applies. The aim here is to express the Gorenstein dimension g of Π using the subcategory \mathcal{U} . More precisely, the main result of this section is the following.

Theorem 4.2. *In the setup of Theorem 3.1 the Gorenstein-dimension g of Π is given by*

$$g = d - 1 + \max\{i < 0 \mid \text{Hom}_{D^b(\text{mod } \Lambda)}(\mathcal{U}, \mathcal{U}[i]) \neq 0\}.$$

The proof of this theorem consists of two main steps: first, in Lemma 4.3 and Proposition 4.4, we calculate g in terms of the non-vanishing of homologies of the complex $\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)$. Second we show that this description coincides with the right-hand side term given in the theorem.

Lemma 4.3. *In the setup of Theorem 3.1 we have*

$$H^i(\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)) = 0 \quad \forall i \in \{g + 1, \dots, d - 2\}.$$

Proof. By Lemma 3.6 we have

$$H^i(\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)) = H^{i-d}(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) \quad \forall i \geq g + 1.$$

Looking at the proof of Proposition 3.7 we see that

$$H^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) = 0 \quad \forall i \neq -1.$$

The claim follows from these two statements. □

We now prove a converse of Lemma 4.3.

Proposition 4.4. *In the setup of Theorem 3.1 we have*

$$g = \max\{i \leq d - 2 \mid H^i(\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)) \neq 0\}.$$

Proof. We have the inequality “ \geq ” by Lemma 4.3. It remains to show that $H^g(\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{RHom}_{\Lambda}(D\Lambda, \Lambda)) \neq 0$. To do so, we analyse what happens in the proof of Lemma 3.6 for $i = -d + g$. As in the proof there, we obtain the triangle

$$\begin{aligned} (\mathbf{RHom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1))_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda &\rightarrow (\mathrm{Hom}_{\Pi^e}(P[-d], \Pi^e)(-1))_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda \\ &\rightarrow \Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda \rightarrow \end{aligned}$$

where P is the complex formed by the first $(d+1)$ terms of a projective resolution of Π as a graded Π -bimodule. Since $H^i(\Pi_{>0} \otimes_{\Pi}^{\mathbf{L}} \Lambda) = 0$ for $i \neq -1$

it follows that

$$\begin{aligned} & \mathbb{H}^i((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \\ & \cong \mathbb{H}^i((\mathrm{Hom}_{\Pi^e}(P[-d], \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \end{aligned}$$

whenever $i < -1$. In particular

$$\begin{aligned} & \mathbb{H}^g((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \\ & = \mathbb{H}^{g-d}((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi[-d], \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \\ & \cong \mathbb{H}^{g-d}((\mathrm{Hom}_{\Pi^e}(P[-d], \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \\ & \quad (\text{since } g - d \leq -2) \\ & = \mathbb{H}^g((\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda). \end{aligned}$$

As in the proof of Lemma 3.6 we observe that

$$\begin{aligned} (\mathrm{Hom}_{\Pi^e}(P, \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda & \cong (\mathrm{Hom}_{\Pi^e}(P_{\leq 0}, \Pi^e)(-1)) \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda \\ & \cong \Pi(-1) \otimes_{\Lambda}^{\mathbf{L}} \mathrm{Hom}_{\Lambda^e}(Q, \Lambda^e) \\ & \cong \Pi(-1) \otimes_{\Lambda}^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\Lambda}(D\Lambda, \Lambda), \end{aligned}$$

where Q is a projective resolution of Λ as a Λ -bimodule. Hence it suffices to show that

$$\mathbb{H}^g((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) \neq 0.$$

Since the complex $\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)$ is concentrated in homological degrees at most g we have

$$\mathbb{H}^g((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0} \otimes_{\mathbb{H}}^{\mathbf{L}} \Lambda) = \mathbb{H}^g((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0}) \otimes_{\mathbb{H}} \Lambda.$$

Tensoring with Λ cannot kill a finitely generated Π -module, so it suffices to show that

$$\mathbb{H}^g((\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0}) \neq 0.$$

By assumption (3) the homology of the complex $\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1)$ of graded Π^e -modules is concentrated in positive degrees, and hence we have

$$(\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))_{>0} = \mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1).$$

Thus it suffices that

$$\underbrace{\mathbb{H}^g(\mathbf{R}\mathrm{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1))}_{=\mathrm{Ext}_{\Pi^e}^g(\Pi, \Pi^e)(-1)} \neq 0,$$

which holds by definition of g . □

Therefore to prove Theorem 4.2 it is sufficient to prove

$$\begin{aligned} & \max\{i \leq d - 2 \mid H^i(\Pi \otimes_{\Lambda}^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_{\Lambda}(D\Lambda, \Lambda)) \neq 0\} \\ & = d - 1 + \max\{i < 0 \mid \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda)}(\mathcal{U}, \mathcal{U}[i]) \neq 0\}. \end{aligned}$$

For the proof, we prepare the following two lemmas.

Lemma 4.5. *Let Λ be an algebra of global dimension at most $d - 1$. Assume for some $j \geq 0$ and some $p \leq -1$ we have*

$$H^i(\mathbb{S}_{d-1}^{-j}(\Lambda)) = 0 \quad \forall i \in \{p, \dots, -1\}.$$

Then

$$H^i(\mathbb{S}_{d-1}^{-(j+1)}(\Lambda)) \cong H^i(\mathbb{S}_{d-1}^{-1}(\tau_{d-1}^{-j}\Lambda)) \quad \forall i \geq p.$$

Proof. Since Λ has global dimension $\leq d - 1$ an easy induction shows that the functor \mathbb{S}_{d-1}^{-j} preserves the left aisle $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda)^{\leq 0}$ of the canonical t -structure of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda)$. Therefore $\mathbb{S}_{d-1}^{-j}\Lambda$ is in negative degrees and we can consider the triangle

$$\mathrm{trunc}^{<0}(\mathbb{S}_{d-1}^{-j}(\Lambda)) \longrightarrow \mathbb{S}_{d-1}^{-j}(\Lambda) \longrightarrow \underbrace{H^0(\mathbb{S}_{d-1}^{-j}(\Lambda))}_{=\tau_{d-1}^{-j}\Lambda} \longrightarrow,$$

where $\mathrm{trunc}^{<0}X$ is the usual truncation of the complex X . Applying \mathbb{S}_{d-1}^{-1} to it we obtain the triangle

$$\mathbb{S}_{d-1}^{-1}(\mathrm{trunc}^{<0}(\mathbb{S}_{d-1}^{-j}(\Lambda))) \longrightarrow \mathbb{S}_{d-1}^{-(j+1)}(\Lambda) \longrightarrow \mathbb{S}_{d-1}^{-1}(\tau_{d-1}^{-j}\Lambda) \longrightarrow.$$

By assumption we have that $\mathrm{trunc}^{<0}(\mathbb{S}_{d-1}^{-j}(\Lambda))$ is concentrated in degrees $\leq p - 1$. Hence $\mathbb{S}_{d-1}^{-1}(\mathrm{trunc}^{<0}(\mathbb{S}_{d-1}^{-j}(\Lambda)))$ is also concentrated in degrees $\leq p - 1$. Thus the triangle gives the desired isomorphism of homologies. □

Lemma 4.6. *Let Λ be an algebra of global dimension at most $d - 1$. The following are equivalent:*

- (1) $\forall i \in \{p, \dots, -1\} \forall j \geq 0: H^i(\mathbb{S}_{d-1}^{-j}(\Lambda)) = 0$; and
- (2) $\forall i \in \{p, \dots, -1\} \forall j \geq 0: H^i(\mathbb{S}_{d-1}^{-1}(\tau_{d-1}^{-j}\Lambda)) = 0$.

Proof. Condition (1) \implies (2) follows immediately from Lemma 4.5.

Condition (2) \implies (1) follows from Lemma 4.5 by induction on j . □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We first note that $\mathbf{RHom}_\Lambda(D\Lambda, \Lambda) \otimes_\Lambda^{\mathbf{L}} -$ is the inverse Serre functor on $\mathbf{D}^b(\text{mod } \Lambda)$, and that Π is isomorphic to $\bigoplus_{j \geq 0} \tau_{d-1}^{-j} \Lambda$ as a Λ -module. Hence we get the following equivalences for $\ell \leq d - 2$:

$$\begin{aligned} H^i(\mathbf{RHom}_\Lambda(D\Lambda, \Lambda) \otimes_\Lambda^{\mathbf{L}} \Pi) &= 0 \quad \forall i \in \{\ell, \dots, d - 2\} \\ \iff H^i(\mathbb{S}^{-1}(\Pi)) &= 0 \quad \forall i \in \{\ell, \dots, d - 2\} \\ \iff H^i(\mathbb{S}^{-1}(\tau_{d-1}^{-j} \Lambda)) &= 0 \quad \forall i \in \{\ell, \dots, d - 2\} \forall j \geq 0 \\ \iff H^i(\mathbb{S}_{d-1}^{-1}(\tau_{d-1}^{-j} \Lambda)) &= 0 \quad \forall i \in \{-d + 1 + \ell, \dots, -1\} \forall j \geq 0 \\ \iff H^i(\mathbb{S}_{d-1}^{-(j+1)}(\Lambda)) &= 0 \quad \forall i \in \{-d + 1 + \ell, \dots, -1\} \forall j \geq 0, \end{aligned}$$

where the last equivalence is Lemma 4.6 for $p = -d + 1 + \ell$.

We may drop the restriction to non-negative j , since $\mathbb{S}_{d-1}^{-j}(\Lambda)$ is concentrated in positive degrees for negative j . So we have

$$\begin{aligned} H^i(\mathbf{RHom}_\Lambda(D\Lambda, \Lambda) \otimes_\Lambda^{\mathbf{L}} \Pi) &= 0 \quad \forall i \in \{\ell, \dots, d - 2\} \\ \iff H^i(\mathbb{S}_{d-1}^{-j}(\Lambda)) &= 0 \quad \forall i \in \{-d + 1 + \ell, \dots, -1\} \forall j \\ \iff \text{Hom}_{\mathbf{D}^b(\text{mod } \Lambda)}(\Lambda, \mathcal{U}[i]) &= 0 \quad \forall i \in \{-d + 1 + \ell, \dots, -1\}. \end{aligned}$$

Therefore we get the equality

$$\begin{aligned} \max\{i \leq d - 2 \mid H^i(\Pi \otimes_\Lambda^{\mathbf{L}} \mathbf{RHom}_\Lambda(D\Lambda, \Lambda)) \neq 0\} \\ = d - 1 + \max\{i < 0 \mid \text{Hom}_{\mathbf{D}^b(\text{mod } \Lambda)}(\mathcal{U}, \mathcal{U}[i]) \neq 0\}, \end{aligned}$$

which finishes the proof of Theorem 4.2. □

4.2. The self-injective case

The situation of Theorem 3.1 is especially nice when the Gorenstein dimension g of the algebra Π is 0, that is when Π is self-injective. In that case, we prove that the algebra Π is the preprojective algebra of a $(d - 1)$ -representation finite algebra.

Definition 4.7 ([15, Definition 2.2; 16, Theorem 3.1]). A τ_{d-1} -algebra Λ is said to be $(d - 1)$ -representation finite if the subcategory $\mathcal{U} \subset \mathbf{D}^b(\text{mod } \Lambda)$ is stable under the Serre functor, that is $\mathbb{S}\mathcal{U} = \mathcal{U}$.

Theorem 4.8. *Let Π be a finite-dimensional self-injective positively graded algebra which is bimodule stably (1)-twisted d -Calabi–Yau. Then $\Lambda = \Pi_0$ is a $(d - 1)$ -representation finite algebra and there is an isomorphism of graded algebras $\Pi \cong \Pi_d(\Lambda)$.*

In order to prove this result, we introduce a technical definition.

Definition 4.9 ([16]). Let Λ be a τ_{d-1} -finite algebra. We say that Λ has the vanishing of small negative extensions property (**vosnex** for short) if

$$\text{Hom}_{\text{D}^b(\text{mod } \Lambda)}(\mathcal{U}, \mathcal{U}[i]) = 0 \quad \forall i \in \{-(d - 3), \dots, -1\}.$$

From Theorem 4.2, we immediately deduce the following result, giving an equivalent but more transparent characterization of what it means for an algebra to satisfy the vosnex property.

Corollary 4.10. *In the setup of Theorem 3.1 the following are equivalent:*

- (a) *the graded algebra Π has Gorenstein dimension ≤ 1 ,*
- (b) *the algebra $\Lambda = \Pi_0$ has the vosnex property.*

Using this corollary together with Theorem 3.1 and some results in [16], we achieve the proof of Theorem 4.8.

Proof of Theorem 4.8. First note that an algebra is self-injective if and only if it is Gorenstein of dimension 0. Moreover if Π is self-injective, then it clearly satisfies hypothesis (3) of Theorem 3.1. Hence Π is the d -preprojective algebra of its degree zero part Λ . Moreover, by Iyama and Oppermann [16, Corollary 3.7], the vosnex property implies that Λ is a $(d - 1)$ -representation finite algebra. □

5. Bimodule Calabi–Yau properties of preprojective algebras

Throughout this section k is assumed to be an algebraically closed field.

By a classical result due to Ringel [25], if Q is an acyclic quiver, the 2-preprojective algebra Π of the hereditary algebra kQ is the usual preprojective algebra of Q . If Q is Dynkin, then it is well known that Π is self-injective, finite-dimensional and that $\underline{\text{mod}} \Pi$ is 2-Calabi–Yau.

On the other hand, if Λ is a τ_2 -finite algebra of global dimension 2, its preprojective algebra $\Pi = \Pi_3(\Lambda)$ is the endomorphism algebra of a cluster-tilting object in a 2-Calabi–Yau category [2, Theorem 4.10]. Hence by Keller

and Reiten [20, Theorem 3.3], the algebra Π is Gorenstein and the stable category $\underline{\text{CM}}\Pi$ is 3-Calabi–Yau.

More generally, if Λ is a τ_{d-1} -finite algebra of global dimension $d - 1$, its preprojective algebra $\Pi = \Pi_d(\Lambda)$ is the endomorphism algebra of a $(d - 1)$ -cluster-tilting object in a $(d - 1)$ -Calabi–Yau category by Guo [9, Theorem 4.9]. If moreover Λ is $(d - 1)$ -representation finite, the stable category $\underline{\text{mod}}\Pi$ is d -Calabi–Yau by Iyama and Oppermann [16, Corollary 4.6] (see also [5, Proposition 3.3]).

In this section, we prove that these Calabi–Yau properties can be deduced from bimodule properties of the preprojective algebra. More precisely, we prove that in the above cases, the preprojective algebra satisfies the properties (1)–(3) of Theorem 3.1.

5.1. Classical preprojective algebras

Let Q be an acyclic quiver. Then the 2-preprojective algebra $\Pi_2(kQ) = \text{Tr}_{kQ} \text{Ext}_{kQ}^1(DkQ, kQ)$ is given by the double quiver \bar{Q} , obtained from Q by adding for any $a: i \rightarrow j$ an arrow $\bar{a}: j \rightarrow i$, with the preprojective relations: $\sum_{a \in Q_1} a\bar{a} - \bar{a}a$. The functor τ_1 is isomorphic to the Auslander–Reiten translation of $\text{D}^b(\text{mod } kQ)$. Thus kQ is τ_1 -finite if and only if the quiver Q is of Dynkin type.

Using this description, we prove the converse of Theorem 3.1 for the case $d = 2$.

Theorem 5.1. *Let Λ be a basic τ_1 -finite algebra of global dimension ≤ 1 . Then the 2-preprojective algebra $\Pi := \Pi_2(\Lambda)$ satisfies the following properties:*

- (1) Π is self-injective (=Gorenstein of dimension 0); and
- (2) Π is bimodule stably (1)-twisted 2-Calabi–Yau.

In particular, $\underline{\text{mod}}\Pi$ is a 2-Calabi–Yau category.

Note that the self-injectivity of Π immediately implies that $\text{Ext}_{\text{gr-}\Pi^e}^j(\Pi, \Pi^e(i))$ vanishes for all i and all $j > 0$, so condition (3) of Theorem 3.1 is automatically satisfied.

Proof. Condition (1) is well known (see [7]).

Denote by P_\bullet the first three terms of the minimal projective resolution of Π as a graded Π -bimodule. Then P_\bullet is of the following form:

$$\begin{aligned} \bigoplus_{i \in Q_0} \Pi e_i \otimes e_i \Pi(-1) &\xrightarrow{d_2} \bigoplus_{a \in Q_1} (\Pi e_{\mathbf{t}(\bar{a})} \otimes e_{\mathbf{s}(\bar{a})} \Pi(-1) \oplus \Pi e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} \Pi) \\ &\xrightarrow{d_1} \bigoplus_{i \in Q_0} \Pi e_i \otimes e_i \Pi, \end{aligned}$$

where the maps d_1 and d_2 are given on components by

$$\begin{aligned} d_1^{a \rightarrow i} : \Pi e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} \Pi &\longrightarrow \Pi e_i \otimes e_i \Pi \\ e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} &\longmapsto a e_i \otimes e_i - e_i \otimes e_i a \end{aligned}$$

and

$$\begin{aligned} d_2^{i \rightarrow a} : \Pi e_i \otimes e_i \Pi &\longrightarrow \Pi e_{\mathbf{t}(\bar{a})} \otimes e_{\mathbf{s}(\bar{a})} \Pi \oplus \Pi e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} \Pi \\ e_i \otimes e_i &\longmapsto \sum_{a, \mathbf{s}(a)=i} e_i \otimes e_{\mathbf{s}(\bar{a})} a + \bar{a} e_{\mathbf{t}(a)} \otimes e_i \end{aligned}$$

([6, 26] for a non-graded version).

Then one easily checks that $\text{Hom}_{\Pi^e}(d_2, \Pi^e) \cong d_1(1)$, that is $\text{Hom}_{\Pi^e}(P_\bullet, \Pi^e)[2] \cong P_\bullet(1)$ in $\text{K}^b(\text{gr-proj } \Pi^e)$. Hence taking H^{-2} , one obtains $\text{Hom}_{\Pi^e}(\Pi, \Pi^e) \cong \Omega_{\Pi^e}^3(\Pi)(1)$ in $\text{gr-}\Pi^e$, which implies (2) by Observation 3.2 (note that Π is Cohen–Macaulay by (1)). □

5.2. The case $d = 3$

For the case $d = 3$, we use a result due to Keller which shows that any 3-preprojective algebra is given by a quiver with potential. So we start by recalling some definitions due to Ginzburg [10] and Derksen *et al.* [4].

Definition 5.2. Let Q be a quiver, and W a potential, that is a (possibly infinite) linear combination of cycles in Q . Then the associated *Jacobian algebra* is

$$J = \text{Jac}(Q, W) = \widehat{kQ} / (\partial_a W \mid \varphi \in Q_1),$$

where \widehat{kQ} is the completion of path algebra kQ , and ∂_a is the unique linear map such that $\partial_a p = \sum_{p=uaav} vu$ for a path p .

Observation 5.3. Let (Q, W) be a quiver with potential. We have

$$kQ_0 = J / \text{Rad } J, \text{ and}$$

$$kQ_1 = \text{the } kQ_0 \otimes kQ_0^{\text{op}}\text{-module generated by the arrows of } Q.$$

The complex

$$(5.1) \quad \bigoplus_{a \in Q_1} J e_{\mathbf{s}(a)} \otimes e_{\mathbf{t}(a)} J \xrightarrow{d_2} \bigoplus_{a \in Q_1} J e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} J \xrightarrow{d_1} \bigoplus_{i \in Q_0} J e_i \otimes e_i J$$

is the beginning of a projective resolution of J as J^e -module.

Here the maps are given on components by

$$d_1^{a \rightarrow i} : J e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} J \longrightarrow J e_i \otimes e_i J$$

$$e_{\mathbf{t}(a)} \otimes e_{\mathbf{s}(a)} \longmapsto a e_i \otimes e_i - e_i \otimes e_i a$$

and $d_2^{a \rightarrow b} = \partial_{a,b} W$, where for a cyclic path p we define

$$\partial_{a,b}(p) : J e_{\mathbf{s}(a)} \otimes e_{\mathbf{t}(a)} J \longrightarrow J e_{\mathbf{t}(b)} \otimes e_{\mathbf{s}(b)} J$$

$$e_{\mathbf{s}(a)} \otimes e_{\mathbf{t}(a)} \longmapsto \sum_{p=u_1 a u_2 b u_3} e_{\mathbf{s}(a)} u_2 e_{\mathbf{t}(b)} \otimes e_{\mathbf{s}(b)} u_3 u_1 e_{\mathbf{t}(a)}$$

$$+ \sum_{p=u_1 b u_2 a u_3} e_{\mathbf{s}(a)} u_3 u_1 e_{\mathbf{t}(b)} \otimes e_{\mathbf{s}(b)} u_2 e_{\mathbf{t}(a)}$$

Observation 5.4. Let Π be a finite-dimensional algebra, and let $\{e_i \mid i \in \{1, \dots, n\}\}$ be a complete set of idempotents. Then

$$\Pi e_i \otimes e_j \Pi \longrightarrow \text{Hom}_{\Pi^e}(\Pi e_j \otimes e_i \Pi, \Pi^e),$$

$$a_1 e_i \otimes e_j a_2 \longmapsto \left[b_1 e_j \otimes e_i b_2 \longmapsto b_1 e_j a_2 \otimes a_1 e_i b_2 \right]$$

is an isomorphism for any $i, j \in \{1, \dots, n\}$.

Lemma 5.5. Let (Q, W) be a quiver with potential. Let a and b be two arrows of Q .

Then, with the identification of Observation 5.4,

$$\mathrm{Hom}_{J^e}(\partial_{a,b}W, J^e) = \partial_{b,a}W.$$

Proof. It is enough to check this for a cyclic path p . More precisely, we have to check that

$$i \circ \partial_{b,a}(p) = \mathrm{Hom}_{J^e}(\partial_{a,b}(p), J^e) \circ i,$$

where i is the isomorphism of Observation 5.4. This can be verified by a straightforward calculation. □

There is a link between 3-preprojective algebras and Jacobian algebras given by the following result.

Theorem 5.6 (18, Theorem 6.12(a)). *Let Λ be a basic finite-dimensional algebra of global dimension ≤ 2 . Let Q be the quiver of Λ , and let R be a minimal set of relations, such that $\Lambda \cong kQ/(R)$ and such that R is the disjoint union of sets representing a basis of the Ext_{Λ}^2 -space between any two simple Λ -modules.*

Then there is an isomorphism $T_{\Lambda} \mathrm{Ext}_{\Lambda}^2(D\Lambda, \Lambda) \cong \mathrm{Jac}(\bar{Q}, W)$, where \bar{Q} is obtained by adding to Q an arrow $a_r : t(r) \rightarrow s(r)$ for each $r \in R$, and $W = \sum_{r \in R} a_r r$. The grading on $T_{\Lambda} \mathrm{Ext}_{\Lambda}^2(D\Lambda, \Lambda)$ is given by the arrows of Q having degree 0, and the arrows corresponding to the relations having degree 1.

Using Keller’s description of 2-preprojective algebras as Jacobian algebras, we can prove the converse of Theorem 3.1 for the case $d = 3$.

Theorem 5.7. *Let Λ be a τ_2 -finite algebra of global dimension ≤ 2 . Let $\Pi = T_{\Lambda} \mathrm{Ext}_{\Lambda}^2(D\Lambda, \Lambda)$ be the associated 3-preprojective algebra. Then*

- (1) Π is Gorenstein of dimension ≤ 1 ;
- (2) Π is bimodule stably (1)-twisted 3-Calabi–Yau; and
- (3) $\mathrm{Ext}_{\Pi^e}^j(\Pi, \Pi^e(i)) = 0$ for all $i < 0$ and all $j > 0$.

In particular, the category $\underline{\mathrm{CM}} \Pi$ is 3-Calabi–Yau.

We start by the following lemma which gives an equivalent condition for (2) in terms of the second syzygy of Π .

Lemma 5.8. *A finite-dimensional graded algebra of Gorenstein dimension ≤ 1 is bimodule stably (1)-twisted 3-Calabi–Yau if and only if there is an isomorphism $\text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^2 \Pi, \Pi^e) \cong \Omega_{\Pi^e}^2 \Pi(1)$ in $\text{gr-CM } \Pi^e$.*

Proof. First note that since the Gorenstein dimension of Π is ≤ 1 , $\Omega_{\Pi^e} \Pi$ is Cohen–Macaulay by Lemma 2.5. Therefore $\Omega_{\Pi^e}^2(\Pi)$ is also Cohen–Macaulay and does not have any projective direct summands. Then we observe that

$$\begin{aligned} \mathbf{R}\text{Hom}_{\Pi^e}(\Pi, \Pi^e)[4] &\cong \Pi(1) \text{ in } \underline{\text{gr-CM}}(\Pi^e) \\ \iff \mathbf{R}\text{Hom}_{\Pi^e}(\Pi[-2], \Pi^e) &\cong \Pi[-2](1) \text{ in } \underline{\text{gr-CM}}(\Pi^e) \\ \iff \text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^2 \Pi, \Pi^e) &\cong \Omega_{\Pi^e}^2 \Pi(1) \text{ in } \underline{\text{gr-CM}}(\Pi^e) \\ \iff \text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^2 \Pi, \Pi^e) &\cong \Omega_{\Pi^e}^2 \Pi(1) \text{ in } \text{gr-CM}(\Pi^e). \end{aligned}$$

The last equivalence holds since both $\text{Hom}_{\Pi^e}(\Omega_{\Pi^e}^2 \Pi, \Pi^e)$ and $\Omega_{\Pi^e}^2 \Pi(1)$ are Cohen–Macaulay without projective summands. □

Proof of Theorem 5.7. Condition (1) holds by Keller and Reiten [20, Proposition 2.1].

By Theorem 5.6, there exists a quiver with potential (\bar{Q}, W) with an isomorphism $\Pi \cong \text{Jac}(\bar{Q}, W)$. We consider the graded version of the exact sequence in (5.1), and its terms by

$$\begin{aligned} P_2 &= \bigoplus_{a \in \bar{Q}_1} \Pi e_{\mathfrak{s}(a)} \otimes e_{\mathfrak{t}(a)} \Pi(-1 + \deg a), \\ P_1 &= \bigoplus_{a \in \bar{Q}_1} \Pi e_{\mathfrak{t}(a)} \otimes e_{\mathfrak{s}(a)} \Pi(-\deg a), \text{ and} \\ P_0 &= \bigoplus_{i \in \bar{Q}_0} \Pi e_i \otimes e_i \Pi(0). \end{aligned}$$

By Lemma 5.5, keeping track of the external grading, we have

$$(5.2) \quad \text{Hom}_{\Pi^e}(d_2, \Pi^e) = d_2(1).$$

We denote $(-)^{\vee} = \text{Hom}_{\Pi^e}(-, \Pi^e)$. Since the Gorenstein dimension of Π is ≤ 1 , the cokernel of d_2 is Cohen–Macaulay, and we have an isomorphism

$$(\text{Coker } d_2)^{\vee} \cong \text{Ker}(d_2^{\vee}).$$

Using (5.2) we get the following isomorphisms:

$$\begin{aligned}
 (\Omega_{\Pi^e}^2 \Pi)^\vee &\cong (\text{Im } d_2)^\vee \cong \text{Coker}((\text{Coker } d_2)^\vee \twoheadrightarrow P_1^\vee) \\
 &\cong \text{Coker}(\text{Ker}(d_2^\vee) \twoheadrightarrow P_1^\vee) \\
 &\cong \text{Coker}(\text{Ker}(d_2(1)) \twoheadrightarrow P_2(1)) \\
 &\cong \text{Im } d_2(1) \cong \Omega_{\Pi^e}^2 \Pi(1).
 \end{aligned}$$

Hence we get (2) by Lemma 5.8.

By (1), $\text{Ext}_{\Pi^e}^j(\Pi, \Pi^e)$ vanishes for $j \geq 2$.

Denote by N the maximal summand of $\Omega_{\Pi^e} \Pi$ without projective summands. The module N is Cohen–Macaulay and we have $\Omega_{\Pi^e} \Pi \cong N \oplus P$. The projective module P is a summand of P_1 and the induced map $P_2 \rightarrow P$ vanishes. Since the arrows of degree 1 correspond to minimal relations of Λ , and hence also to certain minimal relations of $T_\Lambda \text{Ext}_\Lambda^2(D\Lambda, \Lambda)$, P is generated in degree 0, that is $P \in \text{add } \Pi^e(0)$.

Now since Π is bimodule stably (1)-twisted 3-Calabi–Yau, we have isomorphisms

$$\text{Hom}_{\Pi^e}(N, \Pi^e) \cong \Omega_{\Pi^e}^2 N(1) \cong \Omega_{\Pi^e}^3 \Pi(1) \text{ in } \text{gr-CM } \Pi^e$$

The right isomorphism holds since $\Omega_{\Pi^e}^3 \Pi$ does not have any projective summands.

Now we have

$$\begin{aligned}
 \text{Ext}_{\text{gr-}\Pi^e}^1(\Pi, \Pi^e(i)) &\cong \text{Hom}_{\text{gr-}\Pi^e}(\Omega_{\Pi^e} \Pi, \Pi^e(i)) \\
 &\cong (\text{Hom}_{\Pi^e}(\Omega_{\Pi^e} \Pi, \Pi^e))_i \\
 &\cong (\text{Hom}_{\Pi^e}(N, \Pi^e))_i \oplus (\text{Hom}_{\Pi^e}(P, \Pi^e))_i \\
 &\cong (\Omega_{\Pi^e}^3 \Pi)_{i+1} \oplus (\text{Hom}_{\Pi^e}(P, \Pi^e))_i.
 \end{aligned}$$

Now if $i < 0$, $(\text{Hom}_{\Pi^e}(P, \Pi^e))_i$ clearly vanishes since P , hence $\text{Hom}_{\Pi^e}(P, \Pi^e)$ are generated in degree 0. Moreover since $\text{gl.dim } \Lambda \leq 2$ the minimal projective resolution of Π is generated in strictly positive degrees from position 3 on, that is $(\Omega_{\Pi^e}^3 \Pi)_{\leq 0}$ vanishes and thus claim (3) here holds. □

5.3. The $(d - 1)$ -representation finite case

Theorem 5.9. *Let Λ be a $(d - 1)$ -representation finite algebra. Then its d-preprojective algebra Π is finite-dimensional self-injective and stably bimodule (1)-twisted d-Calabi–Yau. In particular, mod Π is d-Calabi–Yau.*

Proof. Finite-dimensionality and self-injectivity are proved in [16, Corollary 3.4].

Let Λ be a $(d - 1)$ -representation-finite algebra. Then the category \mathcal{U} is a $(d - 1)$ -cluster tilting subcategory of $D^b(\text{mod } \Lambda)$, which satisfies $\mathcal{U} = \mathcal{U}[d - 1]$. Moreover, the category $D^b(\text{mod } \Lambda)$ is an algebraic triangulated category, that is it is equivalent to the stable category of some Frobenius category. Hence we can apply Theorem 3.2 of [5] and we obtain an isomorphism

$$\Omega_{\mathcal{U} \otimes \mathcal{U}^{\text{op}}}^{d+1}(\text{Hom}_{\mathcal{U}}(-, -)) \cong \text{Hom}_{\mathcal{U}}(-, -[-d + 1])$$

in $\text{mod}(\mathcal{U} \otimes \mathcal{U}^{\text{op}})$ up to projective summands.

So we have the following isomorphisms in $\underline{\text{gr}}\text{-}\Pi^e$:

$$\begin{aligned} \Omega_{\Pi^e}^{d+1}(\Pi) &\cong \bigoplus_{i \geq 0} \text{Hom}_{D^b(\text{mod } \Lambda)}(\Lambda, \mathbb{S}_{d-1}^{-i} \Lambda[-d + 1]) \\ &\cong \bigoplus_{i \geq 0} \text{Hom}_{D^b(\text{mod } \Lambda)}(\mathbb{S} \Lambda, \mathbb{S}_{d-1}^{-i+1} \Lambda). \end{aligned}$$

On the other hand

$$\begin{aligned} \text{Hom}_{\Pi^e}(\Pi, \Pi^e) &\cong \text{Hom}_{\Pi}(D\Pi, \Pi) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\text{mod } \Lambda)}(D\Lambda, \mathbb{S}_{d-1}^{-i} \Lambda). \end{aligned}$$

Combining these two isomorphisms, and observing that $\mathbb{S}\Lambda = D\Lambda$, we obtain $\Omega_{\Pi^e}^{d+1}(\Pi) \cong \text{Hom}_{\Pi^e}(\Pi, \Pi^e)(-1)$ in $\underline{\text{gr}}\text{-}\Pi^e$. Since Π is self-injective, Π is Cohen–Macaulay, so Π is bimodule stably (1)-twisted d -Calabi–Yau by Observation 3.2. □

Remark 5.10. Let Λ be an algebra which is τ_{d-1} -finite and satisfying the vosnex property. In [16], the authors prove that the algebra $\Pi = \Pi_d(\Lambda)$ is Gorenstein of dimension ≤ 1 and that the stable category $\underline{\text{CM}}\Pi$ is d -Calabi–Yau (Theorem 1.2(1)). It follows from the proof (see [16, Theorem 5.11]) that moreover the category of graded Cohen–Macaulay modules $\underline{\text{gr}}\text{-}\underline{\text{CM}}\Pi$ is (1)-twisted d -Calabi–Yau.

Unfortunately, it is not clear from the proof that this Calabi–Yau property comes from a bimodule property. So we cannot use the results in [16] to prove a result similar to Theorem 5.9 in the case where the algebra Π has Gorenstein dimension 1.

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