Math. Res. Lett. **21** (2014), *no.* 03, 605–615

HILBERT–SAMUEL MULTIPLICITIES OF CERTAIN DEFORMATION RINGS

Fabian Sander

Abstract. We compute presentations of crystalline framed deformation rings of a two-dimensional representation $\bar{\rho}$ of the absolute Galois group of $\mathbb{Q}_p$, when $\bar{\rho}$ has scalar semi-simplification, the Hodge–Tate weights are small and $p > 2$. In the non-trivial cases, we show that the special fibre is geometrically irreducible, generically reduced and the Hilbert–Samuel multiplicity is either 1, 2 or 4 depending on $\bar{\rho}$. We show that in the last two cases the deformation ring is not Cohen–Macaulay.

1. Introduction

Let $p > 2$ be a prime. Let k be a finite field of characteristic p, E be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ with ring of integers $\mathcal{O}$ and uniformizer π. For a given continuous representation $\bar{\rho}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ we consider the universal framed deformation ring $R_{\bar{\rho}}^{\square}$ and the universal framed deformation $\rho^{\mathrm{univ}}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_{\bar{\rho}}^{\square})$. For all $\mathfrak{p} \in \text{m-Spec}(R_{\bar{\rho}}^{\square}[\frac{1}{p}])$, the set of maximal ideals of $R_{\bar{\rho}}^{\square}[\frac{1}{p}]$, we can specialize the universal representation at $\mathfrak{p}$ to obtain the representation

$$\rho_{\mathfrak{p}}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2\left(R_{\bar{\rho}}^{\square}\left[\frac{1}{p}\right]/\mathfrak{p}\right),$$

where $R_{\bar{\rho}}^{\square}[\frac{1}{p}]/\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$. Let $\tau\colon I_{G_{\mathbb{Q}_p}} \to \mathrm{GL}_2(E)$ be a representation with an open kernel, where $I_{G_{\mathbb{Q}_p}}$ is the inertia subgroup of $G_{\mathbb{Q}_p}$. We also fix integers a, b with $b \geq 0$ and a continuous character $\psi\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\overline{\psi\epsilon} = \det(\bar{\rho})$, where ϵ is the cyclotomic character. Kisin showed in [10] that there exist unique reduced $\mathcal{O}$-torsion free quotients $R_{\bar{\rho}}^{\square,\psi}(a,b,\tau)$ and $R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b,\tau)$ of $R_{\bar{\rho}}^{\square}$ with the property that $\rho_{\mathfrak{p}}$ factors through $R_{\bar{\rho}}^{\square,\psi}(a,b,\tau)$ resp. $R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b,\tau)$ if and only if $\rho_{\mathfrak{p}}$ is potentially semi-stable resp. potentially crystalline with Hodge–Tate weights $(a, a+b+1)$ and has determinant $\psi\epsilon$ and inertial type τ. If τ is trivial then $R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b) := R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b,\mathbb{1}\oplus\mathbb{1})$ parametrizes all the crystalline lifts of $\bar{\rho}$ with Hodge–Tate weights $(a, a+b+1)$ and determinant $\psi\epsilon$. The Breuil–Mézard conjecture, proved by Kisin for almost all $\bar{\rho}$, see also [2,3,7,8,14], says that the Hilbert–Samuel multiplicity of the ring $R_{\bar{\rho}}^{\square,\psi}(a,b,\tau)/\pi$ can be determined by computing certain automorphic multiplicities, which do not depend on $\bar{\rho}$, and the Hilbert–Samuel multiplicities of $R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b)$ in low weights for $0 \leq a \leq p-2,\ 0 \leq b \leq p-1$. For most $\bar{\rho}$, the Hilbert–Samuel multiplicities of $R_{\bar{\rho},\mathrm{cris}}^{\square,\psi}(a,b)$ have already been determined. Our goal in this paper is to compute the

Received by the editors December 20, 2012.

Hilbert–Samuel multiplicity of the ring $R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,b)$ with $0 \le a \le p-2,\ 0 \le b \le p-1$ when

$$\bar\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k), \quad g \mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}.$$

One may show that $R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,b)$ is zero if either $b \neq p-2$ or the restriction of χ to $I_{\mathbb{Q}_p}$ is not equal to ϵ^a modulo π.

Theorem 1. *Let a be an integer with $0 \le a \le p-2$ such that $\chi|_{I_{\mathbb{Q}_p}} \equiv \epsilon^a \pmod{\pi}$. Then $R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,p-2)/\pi$ is geometrically irreducible, generically reduced and*

$$e(R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,p-2)/\pi) = \begin{cases} 1 \text{ if } \bar\rho \otimes \chi^{-1} \text{ is ramified,} \\ 2 \text{ if } \bar\rho \otimes \chi^{-1} \text{ is unramified, indecomposable,} \\ 4 \text{ if } \bar\rho \otimes \chi^{-1} \text{ is split.} \end{cases}$$

In the last two cases, $R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,p-2)$ is not Cohen–Macaulay.

The multiplicity 4 does not seem to have been anticipated in the literature, see for example [11, 1.1.6]. Our method is elementary in the sense that we do not use any integral p-adic Hodge theory. The only p-adic Hodge theoretic input is that if ρ is a crystalline lift of $\bar\rho$ with Hodge–Tate weights $(0,p-1)$, then we have an exact sequence

$$0 \longrightarrow \epsilon^{p-1}\chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0,$$

where $\chi_1, \chi_2\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters. This allows us to convert the problem into a linear algebra problem, which we solve in Lemma 2. This gives us an explicit presentation of the ring $R^{\square,\psi}_{\bar\rho,\mathrm{cris}}(a,p-2)$, using which we compute the multiplicities in Section 4. Our argument gives a proof of the existence of $R^{\square.\psi}_{\bar\rho,\mathrm{cris}}(a,p-2)$ independent of [10]. After writing this note we discovered that the idea to convert the problem into linear algebra already appears in [15].

2. The universal ring

After twisting we may assume that $\chi = 1$ and $a = 0$ so that

$$\bar\rho(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$

Since the image of $\bar\rho$ in $\mathrm{GL}_2(k)$ is a p-group, the universal representation factors through the maximal pro-p quotient of $G_{\mathbb{Q}_p}$, which we denote by G. We have the following commuting diagram:

$$\begin{array}{ccc} G_{\mathbb{Q}_p} & \longrightarrow & G \\ \downarrow & & \downarrow \\ G^{\mathrm{ab}}_{\mathbb{Q}_p} & \longrightarrow & G^{\mathrm{ab}}_{\mathbb{Q}_p}(p) \cong G^{\mathrm{ab}}, \end{array}$$

where $G^{\mathrm{ab}}_{\mathbb{Q}_p} := \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$ is the maximal abelian quotient of $G_{\mathbb{Q}_p}$ and can be described by the exact sequence

$$1 \longrightarrow \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p^{\mathrm{ur}}) \longrightarrow G^{\mathrm{ab}}_{\mathbb{Q}_p} \longrightarrow G_{\mathbb{F}_p} \longrightarrow 1$$

where $\mathbb{Q}_p^{\mathrm{ur}}$ is the maximal unramified extension of $\mathbb{Q}_p$ inside $\bar{\mathbb{Q}}_p$. Local class field theory implies that the natural map

$$G^{\mathrm{ab}}_{\mathbb{Q}_p} \to \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$$

is an isomorphism, where μ_{p^∞} is the group of p-power order roots of unity in $\bar{\mathbb{Q}}_p$. The cyclotomic character ϵ induces an isomorphism

$$\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \xrightarrow[\epsilon]{\cong} \mathbb{Z}_p^\times$$

and $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \hat{\mathbb{Z}}$, hence

$$G^{\mathrm{ab}} \cong (1 + p\mathbb{Z}_p) \times \mathbb{Z}_p,$$

where the map onto the first factor is given by ϵ^{p-1}. We choose a pair of generators $\bar{\gamma}, \bar{\delta}$ of G^{ab} such that $\bar{\gamma} \mapsto (1+p, 0)$ and $\bar{\delta} \mapsto (1, 1)$. With [1, Lemma 3.2] we obtain that G is a free pro-p group in two letters γ, δ which project to $\bar{\gamma}, \bar{\delta}$. The way we choose these generators will be of importance in the following.

Lemma 1. *Let $\eta\colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be a continuous character such that $\eta \equiv 1(p)$. Then $\eta = \epsilon^k \chi$ for an unramified character χ if and only if $\eta(\gamma) = \epsilon(\gamma)^k$ and $p-1|k$.*

Proof. " ⇒: " Since γ maps to identity in $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$, we clearly have $\chi(\gamma) = 1$ for every unramified character χ. Hence $\epsilon(\gamma)^k \equiv 1(p)$, which implies $p-1|k$.

" ⇐: " From $\eta\epsilon^{-k}(\gamma) = 1$ and the fact that δ maps to the image of identity in the maximal pro-p quotient of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, we see that $\eta\epsilon^{-k} = \chi$ for an unramified character χ. □

Since G is a free pro-p group generated by γ and δ, to give a framed deformation of $\bar{\rho}$ to $(A, \mathfrak{m}_A)$ is equivalent to give two matrices in $\mathrm{GL}_2(A)$ which reduce to $\bar{\rho}(\gamma)$ and $\bar{\rho}(\delta)$ modulo $\mathfrak{m}_A$. Thus

$$R^{\square}_{\bar{\rho}} = \mathcal{O}[[x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]]$$

and the universal framed deformation is given by

$$\rho^{\mathrm{univ}}\colon G \to \mathrm{GL}_2(R^{\square}_{\bar{\rho}}),$$
$$\gamma \mapsto \begin{pmatrix} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{pmatrix},$$
$$\delta \mapsto \begin{pmatrix} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{pmatrix},$$

where $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ where $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$.

Remark 1. We note that there are essentially three different cases:

(1) $\bar{\rho}$ is ramified $\Leftrightarrow \phi(\gamma) \neq 0 \Leftrightarrow x_{12} \in (R_{\bar{\rho}}^{\square})^{\times}$;
(2) $\bar{\rho}$ is unramified, non-split $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) \neq 0 \Leftrightarrow x_{12} \in \mathfrak{m}_{R_{\bar{\rho}}^{\square}}, y_{12} \in (R_{\bar{\rho}}^{\square})^{\times}$;
(3) $\bar{\rho}$ is split $\Leftrightarrow \phi(\gamma) = 0, \phi(\delta) = 0 \Leftrightarrow x_{12}, y_{12} \in \mathfrak{m}_{R_{\bar{\rho}}^{\square}}$.

Let $\psi\colon G_{\mathbb{Q}_p} \to \mathcal{O}^{\times}$ be a continuous character, such that $\det(\bar{\rho}) = \overline{\psi\epsilon}$, and let $R_{\bar{\rho}}^{\square,\psi}$ be the quotient of $R_{\bar{\rho}}^{\square}$ which parametrizes lifts of $\bar{\rho}$ with determinant $\psi\epsilon$. Since γ, δ generate G as a group, we obtain

$$\begin{aligned} R_{\bar{\rho}}^{\square,\psi} &\cong R_{\bar{\rho}}^{\square}/(\det(\rho^{\text{univ}}(\gamma) - \psi\epsilon(\gamma)), \det(\rho^{\text{univ}}(\delta) - \psi\epsilon(\delta))) \\ &\cong \mathcal{O}[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]], \end{aligned}$$

because we can eliminate the parameters t_γ, t_δ due to the relations $(1+t_\gamma)^2 = \psi\epsilon(\gamma) + x_{11}^2 + x_{12}x_{21}$, $t_\gamma \equiv 0(\mathfrak{m})$, $(1+t_\delta)^2 = \psi\epsilon(\delta) + y_{11}^2 + y_{12}y_{21}$, $t_\delta \equiv 0(\mathfrak{m})$. We let $v := \frac{1-\epsilon^{p-1}(\gamma)}{2}$ and define four polynomials

$$I_1 := (v + x_{11})(v - x_{11}) - x_{12}x_{21}, \tag{1}$$

$$I_2 := (v + x_{11})^2 y_{12} - 2(v + x_{11})x_{12}y_{11} - x_{12}^2 y_{21}, \tag{2}$$

$$I_3 := x_{21}^2 y_{12} - 2x_{21}(v - x_{11})y_{11} - (v - x_{11})^2 y_{21}, \tag{3}$$

$$I_4 := (v + x_{11})x_{21}y_{12} - 2x_{12}x_{21}y_{11} - x_{12}(v - x_{11})y_{21}. \tag{4}$$

Since for every representation with Hodge–Tate weights $(0, p-1)$ the determinant is a character of Hodge–Tate weight $p-1$ and $R_{\bar{\rho},\text{cris}}^{\square,\psi}(0, p-2)$ parametrizes all lifts $\rho_{\mathfrak{p}}$ with determinant $\psi\epsilon$, we let from now on ψ have Hodge–Tate weight $p-2$, as otherwise $R_{\bar{\rho},\text{cris}}^{\square,\psi}(0, p-2)$ would be trivial.

Definition 1. We set

$$R := R_{\bar{\rho}}^{\square,\psi}/(I_1, I_2, I_3, I_4).$$

Our goal is to show that $R_{\bar{\rho},\text{cris}}^{\square,\psi}(0, p-2)$ is isomorphic to R.

Lemma 2. *If* $\mathfrak{p} \in \text{m-Spec}(R_{\bar{\rho}}^{\square,\psi}[\frac{1}{p}])$, *then* $\mathfrak{p} \in \text{m-Spec}(R[\frac{1}{p}])$ *if and only if* $\rho_{\mathfrak{p}}$ *is reducible and* $\rho_{\mathfrak{p}}(\gamma)$ *acts on the* G*-invariant subspace with eigenvalue* $\epsilon^{p-1}(\gamma)$.

Proof. Let $\mathfrak{p} \in \text{m-Spec}(R_{\bar{\rho}}^{\square,\psi}[\frac{1}{p}])$, such that $\rho_{\mathfrak{p}}$ is reducible and $\rho_{\mathfrak{p}}(\gamma)$ acts on the G-invariant subspace with eigenvalue $\epsilon^{p-1}(\gamma)$. Since $\det(\rho_{\mathfrak{p}}(\gamma)) = \psi\epsilon(\gamma) = \epsilon(\gamma)^{p-1}$ and $\epsilon(\gamma)^{p-1}$ is an eigenvalue of $\rho_{\mathfrak{p}}(\gamma)$, the other eigenvalue must be 1. Therefore we can write $1 + t_\gamma = \frac{\epsilon(\gamma)^{p-1}+1}{2}$ and obtain

$$\begin{aligned} 0 &= \det\begin{pmatrix} 1 + t_\gamma + x_{11} - \epsilon(\gamma)^{p-1} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} - \epsilon(\gamma)^{p-1} \end{pmatrix} \\ &= (v + x_{11})(v - x_{11}) - x_{12}x_{21}. \end{aligned}$$

If we now take $\mathfrak{p}$ as above but with $I_1 := (v+x_{11})(v-x_{11}) - x_{12}x_{21} \in \mathfrak{p}$, it is easy to see that the vectors $v_1 = \begin{pmatrix} -x_{12} \\ v+x_{11} \end{pmatrix}$ and $v_2 = \begin{pmatrix} v-x_{11} \\ -x_{21} \end{pmatrix}$ are eigenvectors for $\rho_{\mathfrak{p}}(\gamma)$ with eigenvalue $\epsilon(\gamma)^{p-1}$ if they are non-zero. But at least one of them is non-zero because otherwise we obtain $v=0$ and thus $\epsilon(\gamma)^{p-1} = 1$, which is a contradiction to the definition of γ. So $\rho_{\mathfrak{p}}$ is reducible with an invariant subspace on which $\rho_{\mathfrak{p}}(\gamma)$ acts by $\epsilon(\gamma)^{p-1}$ if and only if the vectors $v_1, v_2, \rho^{\mathrm{univ}}(\delta)v_1, \rho^{\mathrm{univ}}(\delta)v_2$ are pairwise linear dependent. It is easy to check that this is equivalent to the satisfaction of the equations $I_1 = I_2 = I_3 = I_4 = 0$. □

Lemma 3.

$$\text{m-Spec}\left(R\left[\frac{1}{p}\right]\right) = \text{m-Spec}\left(R_{\bar\rho}^{\square,\psi}(0,p-2)\left[\frac{1}{p}\right]\right).$$

Proof. From Khare and Wintenberger [9, Proposition 3.5(i)] we know that every crystalline lift $\rho_{\mathfrak{p}}$ of a reducible two-dimensional representation $\bar\rho$, such that $\rho_{\mathfrak{p}}$ has Hodge–Tate-weights $(0, p-1)$, is reducible itself. Moreover, Brinon and Conrad [4, Theorem 8.3.5] say that if ρ is a reducible two-dimensional crystalline representation, then we have an exact sequence

$$0 \longrightarrow \epsilon^{p-1}\chi_1 \longrightarrow \rho \longrightarrow \chi_2 \longrightarrow 0.$$

Thus $\rho_{\mathfrak{p}}(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and hence from Lemma 2 it is clear that

$$\text{m-Spec}\left(R\left[\frac{1}{p}\right]\right) \supset \text{m-Spec}\left(R_{\bar\rho}^{\square,\psi}(0,p-2)\left[\frac{1}{p}\right]\right).$$

For the other inclusion we note that it is also clear from Lemma 2 that any maximal ideal $\mathfrak{p} \in \text{m-Spec}(R[\frac{1}{p}])$ gives rise to a reducible representation $\rho_{\mathfrak{p}}$ such that $\rho_{\mathfrak{p}}(\gamma)$ acts on the invariant subspace as $\epsilon(\gamma)^{p-1}$ and that the other eigenvalue of $\rho_{\mathfrak{p}}(\gamma)$ is 1. So we obtain with Lemma 1 that $\rho_{\mathfrak{p}}$ is an extension of two crystalline characters

$$0 \to \eta_1 \to * \to \eta_2 \to 0,$$

where the Hodge–Tate weight of η_1 is equal to $p-1$ and the weight of η_2 is equal to 0. Then we can conclude from [13, Proposition 128] that it is semi-stable and from [4, Theorem 8.3.5, Proposition 8.38] that it is crystalline and hence $\mathfrak{p} \in$ m-Spec $(R_{\bar\rho}^{\square,\psi}(0,p-2)[\frac{1}{p}])$. □

Remark 2. We have the following identities mod I_1:

$$x_{21}I_2 = (v+x_{11})I_4, \tag{5}$$

$$(v-x_{11})I_2 = x_{12}I_4, \tag{6}$$

$$x_{21}I_4 = (v+x_{11})I_3, \tag{7}$$

$$(v-x_{11})I_4 = x_{12}I_3. \tag{8}$$

3. Reducedness

In order to show that $R_{\bar\rho}^{\square,\psi}(0,p-2)$ is equal to R, it is enough to show that R is reduced and $\mathcal{O}$-torsion free, since then the assertion follows from Lemma 3, as $R[\frac{1}{p}]$ is Jacobson because R is a quotient of a formal power series ring over a complete discrete valuation ring.

Lemma 4. *If $\mathcal{O} = W(k)$, then R is an $W(k)$-torsion-free integral domain.*

Proof. We distinguish two cases.

If $\bar\rho$ is ramified, i.e., x_{12} is invertible, we consider the fact that for every complete local ring A with $a \in \mathfrak{m}_A, u \in A^\times$, there is a canonical isomorphism $A[[z]]/(uz-a) \cong A$. Using this we see from (1),(2),(6) and (8) that

$$\begin{aligned} R &= \mathcal{O}[[x_{11},\hat{x}_{12},x_{21},y_{11},\hat{y}_{12},y_{21}]]/(I_1,I_2) \\ &\cong \mathcal{O}[[x_{11},\hat{x}_{12},y_{11},\hat{y}_{12}]], \end{aligned}$$

which shows the claim.

In the second case, where $\bar\rho$ is unramified, i.e., $x_{12} \notin R^\times$, we consider the ideal $I := (\pi, x_{11}, x_{12}, x_{21})$ and have

$$\mathrm{gr}_I R_{\bar\rho}^{\square,\psi} \cong k[[y_{11},\hat{y}_{12},y_{21}]][\bar\pi,\bar{x}_{11},\bar{x}_{12},\bar{x}_{21}].$$

Since $\mathcal{O} = W(k)$ we have $v \in I \setminus I^2$ and hence the elements I_1, I_2, I_3, I_4 are homogeneous of degree 2, so that

$$\mathrm{gr}_I R \cong k[[y_{11},\hat{y}_{12},y_{21}]][\bar\pi,\bar{x}_{11},\bar{x}_{12},\bar{x}_{21}]/(I_1,I_2,I_3,I_4),$$

see [6, Example 5.3]. Because R is noetherian it follows from [6, Corollary 5.5] that it is enough to show that $\mathrm{gr}_I R$ is an integral domain.

We define

$$A := k[[y_{11},\hat{y}_{12},y_{21}]][\bar{x}_{11},\bar{x}_{12},\bar{x}_{21},\bar\pi]/(\bar{I}_1)$$

and look at the map

$$\phi\colon A \to A[\bar{x}_{12}^{-1}]/(\bar{I}_2).$$

The latter ring is isomorphic to $(k[[y_{11},\hat{y}_{12},y_{21}]][\bar{x}_{11},\bar{x}_{12},\bar{x}_{11}^{-1},\bar\pi]/(I_2))$ and since I_2 is irreducible it is an integral domain. So we would be done by showing that $\ker(\phi) = (\bar{I}_2,\bar{I}_3,\bar{I}_4)$. The inclusion $(I_2,I_3,I_4) \subset \ker(\phi)$ is clear from (6) and (8). For the other one we consider the fact that

$$\ker(\phi) = \{a \in A :\ \exists n \in \mathbb{N} \cup \{0\}, b,c,d \in A : \bar{x}_{12}^n a = b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4\}.$$

To show that $\ker(\phi) \subset (I_2,I_3,I_4)$, we let $a \in A$ and n be minimal with the property that there exist $b,c,d \in A$ such that

$$\bar{x}_{12}^n a = b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4. \tag{9}$$

If $n = 0$ there is nothing to show. Now we assume that $n > 0$ and consider the prime ideal $\mathfrak{p} := (\bar{x}_{12}, \bar{v} - \bar{x}_{11}) \subset A$ and see that

$$A/\mathfrak{p} \cong k[[y_{11}, y_{12}, y_{21}]][\bar{x}_{11}, \bar{x}_{12}]$$

is a unique factorization domain. We also observe that

$$I_2 \equiv y_{12}(\bar{v} + \bar{x}_{11})^2 \bmod \mathfrak{p}, \tag{10}$$

$$I_3 \equiv y_{12}\bar{x}_{21}^2 \bmod \mathfrak{p}, \tag{11}$$

$$I_4 \equiv y_{12}(\bar{v} + \bar{x}_{11})\bar{x}_{21} \bmod \mathfrak{p}. \tag{12}$$

Modulo $\mathfrak{p}$ (9) becomes

$$0 \equiv y_{12}b(\bar{v} + \bar{x}_{11})^2 + y_{12}c\bar{x}_{21}^2 + y_{12}d(\bar{v} + \bar{x}_{11})\bar{x}_{21}. \tag{13}$$

Since $A/\mathfrak{p}$ is a unique factorization domain there are $b_1, c_1 \in A$ such that

$$y_{12}b \equiv b_1\bar{x}_{21} \bmod \mathfrak{p}, \tag{14}$$

$$y_{12}c \equiv c_1(\bar{v} + \bar{x}_{11}) \bmod \mathfrak{p} \tag{15}$$

and we see that

$$d \equiv -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} \bmod \mathfrak{p}. \tag{16}$$

Hence we can find $b_2, b_3, c_2, c_3, d_1, d_2 \in A$ such that

$$\begin{aligned} b &= b_1\bar{x}_{21} + b_2\bar{x}_{12} + b_3(\bar{v} - \bar{x}_{11}), \\ c &= c_1(\bar{v} + \bar{x}_{11}) + c_2\bar{x}_{12} + c_3(\bar{v} - \bar{x}_{11}), \\ d &= -\frac{b_1\bar{x}_{21} + c_1(\bar{v} + \bar{x}_{11})}{2} + d_1\bar{x}_{12} + d_2(\bar{v} - \bar{x}_{11}). \end{aligned}$$

Substituting this in (9) we get

$$\begin{aligned} \bar{x}_{12}^n a &= b\bar{I}_2 + c\bar{I}_3 + d\bar{I}_4 && (17) \\ &= \bar{x}_{12}(b_2I_2 + b_3I_4 + c_2I_3 + d_1I_4 + d_2I_3) \\ &\quad + \frac{1}{2}(b_1(\bar{v} + \bar{x}_{11}) + c_1\bar{x}_{21})I_4 + (\bar{v} - \bar{x}_{11})c_3I_3. && (18) \end{aligned}$$

Modulo $\mathfrak{p}$ we have $b_1(\bar{v}+\bar{x}_{11})+c_1\bar{x}_{21} \equiv 0$ and hence there are $b_4, b_5, b_6, c_4, c_5, c_6$ with

$$b_1 = \bar{x}_{21}b_4 + \bar{x}_{12}b_5 + (\bar{v} - \bar{x}_{11})b_6, \tag{19}$$

$$c_1 = (\bar{v} + \bar{x}_{11})c_4 + \bar{x}_{12}c_5 + (\bar{v} - \bar{x}_{11})c_6. \tag{20}$$

Hence we can rewrite (18) to

$$\bar{x}_{12}^n a = \bar{x}_{12}z + \frac{1}{2}(b_4 + c_4)(\bar{v} + \bar{x}_{11})^2I_3 + (\bar{v} - \bar{x}_{11})c_3I_3 \tag{21}$$

for a certain $z \in (I_2, I_3, I_4)$. So with (21) we see that $b_4 + c_4 \equiv 0$ modulo $\mathfrak{p}$ and $c_3 \equiv 0$ modulo the prime ideal $\mathfrak{p}' := (\bar{x}_{12}, \bar{v} + \bar{x}_{11})$. Therefore we can find some $c_7, c_8, e_1, e_2 \in A$ with

$$c_3 = c_7\bar{x}_{12} + c_8(\bar{v} + \bar{x}_{11}),$$
$$b_4 + c_4 = e_1\bar{x}_{12} + e_2(\bar{v} - \bar{x}_{11}).$$

But since we have $(v + x_{11})(v - x_{11}) = x_{12}x_{21}$ in A we can finally transform (21) to

$$\bar{x}_{12}^n a = \bar{x}_{12} z'$$

for some $z' \in (I_2, I_3, I_4)$ which shows that $\bar{x}_{12}^{n-1} a \in (I_2, I_3, I_4)$, since A is an integral domain. But this is a contradiction to the minimality of n. □

Proposition 1. *R is reduced and $\mathcal{O}$-torsion free for any choice of $\mathcal{O}$.*

Proof. Since $\mathcal{O}$ is flat over $W(k)$ and we have seen in Lemma 3 that

$$S := W(k)[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(I_1, I_2, I_3, I_4)$$

is an integral domain, we get an injection

$$\mathcal{O} \otimes_{W(k)} S \to \mathcal{O} \otimes_{W(k)} \operatorname{Quot}(S).$$

As S is $W(k)$-torsion free by Lemma 3, we obtain an isomorphism

$$\mathcal{O} \otimes_{W(k)} \operatorname{Quot}(S) \xrightarrow{\cong} \mathcal{O}\left[\frac{1}{p}\right] \otimes_{W(k)[\frac{1}{p}]} \operatorname{Quot}(S).$$

Since $\mathcal{O}[\frac{1}{p}]$ is a separable field extension of $W(k)[\frac{1}{p}]$, we deduce that $\mathcal{O}[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} \operatorname{Quot}(S)$ is reduced and $\mathcal{O}$-torsion free. □

4. The multiplicity

We want to compute the Hilbert–Samuel multiplicity of the ring R/π for the given representation

$$\bar{\rho}\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k), \quad g \mapsto \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}.$$

We denote the maximal ideal of R/π by $\mathfrak{m}$.

Theorem 2.

$$e(R/\pi) = \begin{cases} 1 \text{ if } \bar{\rho} \text{ is ramified,} \\ 2 \text{ if } \bar{\rho} \text{ is unramified, indecomposable,} \\ 4 \text{ if } \bar{\rho} \text{ is split.} \end{cases}$$

Proof. If we set $J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21}$ we obtain modulo π the relations

(22) $$I_2 \equiv -x_{12}J,$$

(23) $$I_3 \equiv x_{21}J,$$

(24) $$I_4 \equiv x_{11}J.$$

We split the proof into three cases as in Remark 1. If $\bar{\rho}$ is ramified, i.e., x_{12} is invertible, we see as in the proof of Lemma 4 that

$$\begin{aligned} R/\pi &\cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J) \\ &\cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]]. \end{aligned}$$

Hence it is a regular local ring and therefore $e(R/\pi) = 1$.

Let us assume in the following that $\bar{\rho}$ is unramified, i.e., $x_{12} = \hat{x}_{12} \in \mathfrak{m}_R$, and we can consider the exact sequence

$$0 \to (R/\pi)/\mathrm{Ann}_{R/\pi}(J) \to R/\pi \to R/(\pi, J) \to 0. \tag{25}$$

Since $x_{11}, x_{12}, x_{21} \in \mathrm{Ann}_{R/\pi}(J)$, see (22)–(24), we have $\dim((R/\pi)/\mathrm{Ann}_{R/\pi}(J)) \leq 3$. But $\dim R/\pi = 4$ so that (25) gives us $e(R/\pi) = e(R/(\pi, J))$, see [12, Theorem 14.6]. We obtain that

$$\begin{aligned} R/(\pi, J) &\cong k[[x_{11}, x_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J) \\ &\cong (k[[x_{11}, x_{12}, x_{21}]]/(x_{11}^2 + x_{12}x_{21}))[[y_{11}, \hat{y}_{12}, y_{21}]]/(J) \end{aligned}$$

is a complete intersection of dimension 4. So if $\mathfrak{q} \subset R/(\pi, J)$ is an ideal generated by four elements, such that $R/(\pi, J, \mathfrak{q})$ has finite length as a $R/(\pi, J)$-module, then these elements form a regular sequence in $R/(\pi, J)$ and $e_{\mathfrak{q}}(R/(\pi, J)) = l(R/(\pi, J, \mathfrak{q}))$, see [12, Theorem 17.11]. Besides, if there exists an integer n such that $\mathfrak{q}\mathfrak{m}^n = \mathfrak{m}^{n+1}$, then $e(R/(\pi, J)) = e_{\mathfrak{q}}(R/(\pi, J))$, see [12, Theorem 14.13]. So to finish the proof it would suffice to find such an ideal $\mathfrak{q}$.

If $\bar{\rho}$ is indecomposable, i.e., $\phi(\delta)$ is non-zero and therefore y_{12} is a unit in R, we can write the equation $J = 0$ as

$$x_{21} = -y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})$$

and $I_1 = 0$ as

$$x_{11}^2 = x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})$$

so that

$$R/(\pi, J) \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12})).$$

Hence it is clear that $x_{12}, x_{21}, y_{11}, \hat{y}_{12}$ is a system of parameters for $R/(\pi, J)$ that generates an ideal $\mathfrak{q}$ with $\mathfrak{q}\mathfrak{m} = \mathfrak{m}^2$. So we obtain

$$e_{\mathfrak{q}}(R/(\pi, J)) = l(R/(\pi, J, \mathfrak{q})) = l(k[[x_{11}]]/(x_{11}^2)) = 2$$

and hence $e(R/\pi) = 2$.

If $\bar{\rho}$ is split, which is equivalent to $x_{12}, y_{12} \notin R^\times$, we take $\mathfrak{q} := (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11})$ and claim that $\mathfrak{q}\mathfrak{m}^2 = \mathfrak{m}^3$. If we write $\mathfrak{m} = (x_{12} - x_{21}, x_{12} - y_{12}, x_{12} - y_{21}, y_{11}, x_{11}, x_{12})$ we just have to check that $x_{11}^3, x_{11}^2 x_{12}, x_{11}x_{12}^2, x_{12}^3 \in \mathfrak{q}\mathfrak{m}^2$. Therefore it is enough to see that

$$\begin{aligned} x_{11}^2 &= x_{11}y_{11} - \frac{1}{2}(x_{12} - y_{12})x_{21} - \frac{1}{2}(x_{21} - y_{21})x_{12} \ \in \mathfrak{m}\mathfrak{q}, \\ x_{12}^2 &= -x_{11}^2 + x_{12}(x_{12} - x_{21}) \ \in \mathfrak{m}\mathfrak{q}. \end{aligned}$$

Hence

$$e(R/\pi) = l(R/(\pi, J, \mathfrak{q})) = l(k[[x_{11}, x_{12}]]/(x_{11}^2, x_{12}^2)) = 4.$$

□

Corollary 1. *If $\bar{\rho}$ is unramified, then the ring R is not Cohen–Macaulay.*

Proof. Since R is $\mathcal{O}$-torsion free, π is R-regular and hence R is CM if and only if R/π is CM. In (25) we have constructed a non-zero submodule of R/π of dimension strictly less than the dimension of R/π. It follows from [5, Theorem 2.1.2(a)] that R/π cannot be CM. □

Proposition 2. $\mathrm{Spec}(R/\pi)$ *is geometrically irreducible and generically reduced.*

To prove the proposition we need the following lemma. As in the proof of Theorem 2 we define $J := y_{12}x_{21} + 2x_{11}y_{11} + x_{12}y_{21}$.

Lemma 5. *$R/(\pi, J)$ is an integral domain.*

Proof. We again distinguish between three cases as in Remark 1. If $\bar{\rho}$ is ramified, i.e., x_{12} is invertible, we have already seen in the proof of Theorem 2 that

$$\begin{aligned} R/(\pi, J) &\cong k[[x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 + x_{12}x_{21}, J) \\ &\cong k[[x_{11}, \hat{x}_{12}, y_{11}, \hat{y}_{12}]]. \end{aligned}$$

If $\bar{\rho}$ is unramified and indecomposable, i.e., $x_{12} = \hat{x}_{12} \in \mathfrak{m}_R, y_{12} \in R^\times$ we saw that

$$R/(\pi, J) \cong k[[x_{11}, x_{12}, y_{11}, \hat{y}_{12}, y_{21}]]/(x_{11}^2 - x_{12}y_{12}^{-1}(2x_{11}y_{11} + y_{21}x_{12}))$$

which is easily checked to be an integral domain. If $\bar{\rho}$ is unramified and split, i.e., $x_{12}, y_{12} \in \mathfrak{m}_R$, let $\mathfrak{n}$ denote the maximal ideal of $R/(\pi, J)$. It is enough to show that the graded ring $\mathrm{gr}_{\mathfrak{n}} R/(\pi, J)$ is a domain. Since J is homogeneous we have

$$\mathrm{gr}_{\mathfrak{n}} R/(\pi, J) \cong k[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]/(x_{11}^2 + x_{12}x_{21}, J).$$

We set $A := k[x_{11}, x_{12}, x_{21}, y_{11}, y_{12}, y_{21}]/(x_{11}^2 + x_{12}x_{21})$ and have to prove that $(J) \subset A$ is a prime ideal. We look at the localization map $A \xrightarrow{\iota} A[y_{21}^{-1}]$, which is an inclusion because y_{21} is regular in A. This gives us a map $A \xrightarrow{\bar{\iota}} A[y_{21}^{-1}]/(J)$. Since

$$A[y_{21}^{-1}]/(J) \cong k[x_{11}, x_{21}, y_{11}, y_{12}, y_{21}, y_{21}^{-1}]/(x_{11}^2 - x_{21}y_{21}^{-1}(2x_{11}y_{11} + x_{21}y_{12}))$$

is a domain, we would be done by showing that $\ker(\bar{\iota}) = (J)$. We have

$$\ker(\bar{\iota}) = \{a \in A : y_{21}^i a = bJ \text{ for some } i \in \mathbb{Z}_{\geq 0}, b \in A : y_{21} \nmid b\}.$$

But since $(y_{21}) \subset A$ is a prime ideal and y_{21} does not divide J, we see that $i = 0$ in all these equations and hence $\ker(\bar{\iota}) = (J)$. □

Proof of Proposition 2. Let $\mathfrak{p}$ be a minimal prime ideal of $S := R/\pi$. It follows from (22)–(24) that $J^2 = 0$ and thus $J \in \mathrm{rad}(S) = \bigcap_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$. So Lemma 5 gives us that JS is the only minimal prime ideal of S, hence $\mathrm{Spec}(S)$ is irreducible. If we replace the field k by an extension k', we obtain the irreducibility of $\mathrm{Spec}(S \otimes_k k')$ analogously, thus $\mathrm{Spec}(S)$ is geometrically irreducible.

$\mathrm{Spec}(S)$ is called generically reduced if $S_{\mathfrak{p}}$ is reduced for any minimal prime ideal $\mathfrak{p}$. We have already seen that there is only one minimal prime ideal $\mathfrak{p} = JS$. By localizing (25) we obtain $S_{\mathfrak{p}} \cong R/(\pi, J)$. Lemma 5 implies that $S_{\mathfrak{p}}$ is reduced. □

Acknowledgment

This paper was written as a part of my PhD thesis. I want to thank my advisor Prof. Dr. Vytautas Paškūnas for his great support.

References

[1] G. Böckle, *Demuškin groups with group actions and applications to deformations of Galois representations*, Compos. Math. **121**(2) (2000), 109–154.

[2] C. Breuil and A. Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115**(2) (2002), 205–310 (with an appendix by Guy Henniart).

[3] ———, *Multiplicités modulaires raffinées*, Bull. Soc. Math. France **142**(1) (2014), 127–175.

[4] O. Brinon and B. Conrad, *CMI summer school notes on p-adic Hodge theory* (2009), `http://math.stanford.edu/~conrad/papers/notes.pdf`.

[5] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Math., **39**, Cambridge University Press, Cambridge, 1993, ISBN 0-521-41068-1.

[6] D. Eisenbud, *Commutative algebra*, Graduate Texts in Math., **150**, Springer-Verlag, New York, 1995, ISBN 0-387-94268-8; 0-387-94269-6 (with a view toward algebraic geometry).

[7] M. Emerton and T. Gee, *A geometric perspective on the Breuil–Mézard conjecture*, J. Inst. Math. Jussieu **13**(1) (2014), 183–223.

[8] Y. Hu and F. Tan, *The Breuil–Mézard conjecture for non-scalar split residual representations*, 2013, `arXiv:1309.1658`.

[9] C. Khare and J.-P. Wintenberger, *Serre's modularity conjecture II*, Invent. Math. **178**(3) (2009), 505–586.

[10] M. Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21**(2) (2008), 513–546.

[11] ———, *The Fontaine–Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22**(3) (2009), 641–690.

[12] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Math., **8**, Cambridge University Press, Cambridge, 1989, ISBN 0-521-36764-6 (translated from the Japanese by M. Reid).

[13] J. Nekovář, *On p-adic height pairings*, in 'Séminaire de Théorie des Nombres (Paris, 1990–91)', 127–202, Progr. Math., **108**, Birkhäuser Boston, Boston, MA, 1993.

[14] V. Paškūnas, *On the Breuil–Mézard conjecture*, 2012, `arXiv:1209.5205`.

[15] A. Snowden, *Singularities of ordinary deformation rings*, 2011, `arXiv:1111.3654`.

Department of Mathematics, University of Duisburg-Essen, Thea-Leymann-Strasse 9, 45127 Essen, Germany

E-mail address: `fabian.sander@uni-due.de`