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THE p-ADIC SHINTANI COCYCLE

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Abstract. The Shintani cocycle on $\mathrm{GL}_n(\mathbb{Q})$, as constructed by Hill, gives a cohomological interpretation of special values of zeta functions for totally real fields of degree n. We give an explicit criterion for a specialization of the Shintani cocycle to be p-adically interpolable. As a corollary, we recover the results of Deligne–Ribet, Cassou Noguès and Barsky on the construction of p-adic L-functions attached to totally real fields.

1. Introduction

The goal of this paper is to construct cocycles on arithmetic subgroups of GL_n, valued in spaces of p-adic pseudo-measures, which specialize to p-adic L-functions. Our starting point will be the *Shintani zeta functions*, which generalize the Hurwitz partial zeta functions

$$\zeta_{\mathrm{H}}([a+f\mathbb{Z}];s) := \sum_{\substack{n\in\mathbb{N}\\ n\equiv a\ (f)}} \frac{1}{n^s}$$

to several dimensions. Given a lattice $L \subset \mathbb{R}^n$ (more generally, a finite linear combination of characteristic functions of affine lattices, e.g., a test function) and a cone $C \in \mathbb{R}^n_+$, the zeta function is defined by

$$\zeta_{\mathrm{SH}}([L], C; s) := \sum_{v\in C\cap L} \frac{1}{N(v)^s},\ \Re(s) \gg 0,$$

where $N(v)$ is the product of the coordinates. Shintani [15] showed these enjoy meromorphic continuation to the entire complex plane and computed their special values in terms of generalized Bernoulli polynomials. Moreover, Shintani showed how to decompose the L-functions of totally real fields into Shintani zeta functions. The explicit formulas for these values implies the rationality results of Siegel and Klingen and provides the foundation for p-adic interpolation of these values by Cassou-Nogués [3] and Barsky [2]. Our contribution will be a simple criterion, in terms of the cone C and the "test function f", for the special values of $\zeta_{\mathrm{Sh}}([L], C; s)$ to be p-adically continuous.

Next, we turn to the *Shintani cocycle*, constructed by Hill in [13]. This $n-1$ cocycle on GL_n takes values in a module of cones. Pairing a cone C and a test function f gives rise to a Shintani zeta function $\zeta_{\mathrm{SH}}(f, C; s)$, and this pairing is obviously bilinear and GL_n-equivariant. Our approach will be to fix a test function f', away from p, and use this pairing to construct p-adic pseudo-measures corresponding to ζ_{SH}. After restricting to a subgroup $\Gamma \subset \mathrm{GL}_n$ stabilizing f, we get a cocycle $\Phi_{f'} : \Gamma^n \to \widetilde{\mathcal{M}}(\mathbb{Z}_p^n)$ valued in a space of pseudo-measures. We will describe which specializations (if any)

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yield p-adic measures in terms of our criterion. As an application, we will give a new construction of the p-adic L-functions of totally real fields.

Recently, Charollois and Dasgupta [4] have obtained similar results with Sczech's GL_n cocycle as part of a program to study the Gross–Stark units. In their work, they define an ℓ-smoothed Sczech cocycle and deduce integrality results from explicit formulas in terms of Dedekind sums. As a consequence of these integrality results, they construct the p-adic measures corresponding to the zeta values of totally real fields. They have announced similar results for a version of the Shintani cocycle, but their techniques are substantially different from ours. Concurrently, Spiess [17] has constructed p-adic measures from the Shintani cocycle, adapting the argument of Cassou-Noguès. Again, our techniques differ substantially and we believe our results are more general. Rather than constructing measures from integrality results, we find the p-adic pseudo-measures as specializations of the Shintani cocycle, then show that these specializations are in fact measures via our elementary arguments.

2. Notation and definitions

Let V be an n-dimensional $\mathbb{Q}$-vector space. We will always assume that we have picked a distinguished lattice $L \subset V$. In the case $V = \mathbb{Q}$, we let $L = \mathbb{Z}$. The $\mathbb{R}$-vector space $\mathbb{R}^n$ comes with the standard basis $e_1, \dots, e_n$.

For each prime p, we will write $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We write $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ and $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$. Given an embedding $V \hookrightarrow \mathbb{R}^n$ (equivalently, an isomorphism $V_{\mathbb{R}} \cong \mathbb{R}^n$), we will denote by $(V_{\mathbb{R}})_+$ the pre-image of the positive orthant with respect to this isomorphism.

The group of test functions on V, denoted $\mathcal{S}(V)$, is simply the $\mathbb{Z}$-module of Bruhat–Schwartz functions on the finite adeles $\mathbb{A}_V^{(\infty)}$. Denote by $\mathcal{S}(V_p)$ the space of step functions (locally constant, compact support) $f_p : V_p \to \mathbb{Z}$. For example, if $U \subset V_p$ is a compact open, we will write $[U]$ for the characteristic function of U. The group of test functions on V is defined by

$$\mathcal{S}(V) = \bigotimes_p{}' \mathcal{S}(V_p), \tag{2.1}$$

where the restricted product means $f_p = [L_p]$ almost everywhere.

Lemma 2.1. *Equipping V with the lattice topology, $\mathcal{S}(V)$ is the space of locally constant functions with bounded support $f : V \to \mathbb{Z}$. In other words, a test function $f \in \mathcal{S}(V)$ is a finite linear combination of characteristic functions of affine lattices.*

For each prime p, we will denote by $\mathcal{S}(V^{(p)})$ the space

$$\bigotimes_{\ell \neq p}{}' \mathcal{S}(V_\ell) \tag{2.2}$$

and refer to elements $f' \in \mathcal{S}(V^{(p)})$ as a test functions *away from p*.

Our convention will be to let $\mathrm{GL}(V)$ act on V on the left. By duality, this endows $\mathcal{S}(V)$ with a *right* $\mathrm{GL}(V)$ action, $(f|\gamma)(v) := f(\gamma v)$.

Let $v_1, \dots, v_r$, $r \leq n$ be linearly independent vectors in $V_{\mathbb{R}}$. The "open" cone $C^{\mathrm{o}}(v_1, \dots, v_r)$ is the positive span $C^{\mathrm{o}}(v_1, \dots, v_r) = \{\sum a_i v_i | a_i \in \mathbb{R}, a_i > 0\}$. Note that if $r < n$, an open cone is not an open subset of $V_{\mathbb{R}}$. The "closed" cone $C(v_1, \dots, v_r)$

is defined by $C(v_1,\dots,v_r)=\{\sum a_iv_i | a_i\in\mathbb{R}, a_i\geq 0\}$. Following [9], we will call finite disjoint union of open cones *Shintani sets*. We will say a Shintani set is rational if it is the union of open cones generated by vectors $v_1,\dots,v_r\in V\subset V_{\mathbb{R}}$. Finally, we say a Shintani set is pointed if it does not contain any lines.

3. Shintani's method

3.1. Shintani zeta functions. If C is a rational Shintani set in $(V_{\mathbb{R}})_+$ and $f\in\mathcal{S}(V)$ is a test function, the *Shintani zeta function* $\zeta_{\mathrm{Sh}}(f,C;s)$ is defined, for $\mathrm{Re}(s)\gg 0$, as the sum

$$\zeta_{\mathrm{Sh}}(f,C;s):=\sum_{v\in C\cap V}\frac{f(v)}{\mathrm{N}(v)^s},$$

where $\mathrm{N}(v)=e_1^*(v)\cdots e_n^*(v)$ is the product of the coordinates. One can show that the sum converges for $\mathrm{Re}(s)\gg 0$ (see, for example, [12]), and Shintani showed the these have meromorphic continuation to $s\in\mathbb{C}$. Moreover, the values $\zeta_{\mathrm{SH}}(f,C;-k)$ can be expressed in terms of Bernoulli polynomials.

Shintani used these results to study the special values of Hecke L-functions of totally real fields. Let us suppose that F is a totally real field of degree n. The Hecke L-functions of F decompose as sums of partial zeta functions, sometimes called *ray class zeta functions*. If $\mathfrak{f}$ is an integral ideal of F and $\mathfrak{a}$ is a fractional ideal relatively prime to $\mathfrak{f}$, then the ray class zeta function for $[\mathfrak{a}]_{\mathfrak{f}}$ is defined by

$$\zeta([\mathfrak{a}]_{\mathfrak{f}},s):=\sum_{\substack{\mathfrak{b}\subset\mathcal{O}_F\\ \mathfrak{b}\sim_{\mathfrak{f}}\mathfrak{a}}}\frac{1}{\mathrm{N}\,\mathfrak{b}^s}\ \text{for}\,\mathrm{Re}(s)\gg 0,$$

where the sum is over all integral ideals representing $[\mathfrak{a}]_{\mathfrak{f}}$ in the narrow ray class group. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent in the narrow ray class group if and only if there exists a totally positive $\alpha\equiv 1 \pmod{\mathfrak{f}}$, such that $(\alpha)=\mathfrak{b}\mathfrak{a}^{-1}$. Put

$$E(\mathfrak{f})=\{u\in\mathcal{O}_F^\times | u\gg 0 \text{ and } u\equiv 1 \pmod{\mathfrak{f}}\},$$

so that $(\alpha)\sim_{\mathfrak{f}}(\beta)$ if and only if $\alpha\beta^{-1}\in E(\mathfrak{f})$. We may rewrite the sum as

$$\zeta([\mathfrak{a}]_{\mathfrak{f}},s)=\sum_{\substack{\beta\in(a+\mathfrak{a}^{-1}\mathfrak{f})/E(\mathfrak{f})\\ \beta\gg 0}}\frac{1}{\mathrm{N}(\mathfrak{a}\beta)^s}=\mathrm{N}\,\mathfrak{a}^{-s}\sum_{\substack{\beta\in(a+\mathfrak{a}^{-1}\mathfrak{f})/E(\mathfrak{f})\\ \beta\gg 0}}\frac{1}{\mathrm{N}\,\beta^s},\tag{3.1}$$

where $a\in\mathfrak{a}^{-1}$ is any fixed element congruent to 1 (mod $\mathfrak{f}$). If $\mathfrak{a}$ is integral, then it suffices to take $a=1$.

In order to interpret these ray class zeta functions as Shintani zeta functions, we embed F in $\mathbb{R}^n$. Write $\tau_1,\dots,\tau_n$ for the n embeddings of F into $\mathbb{R}$ and $F\hookrightarrow\mathbb{R}^n$ by $\alpha\mapsto(\tau_1(\alpha),\dots,\tau_n(\alpha))$. The norm $N=e_1^*\cdots e_n^*$ on $\mathbb{R}^n$ extends the usual norm on F to $\mathbb{R}^n$. Shintani's insight was to construct a fundamental domain for the action of $E(\mathfrak{f})$ by decomposing $\mathbb{R}^n_+$ (where the totally positive elements live) into disjoint polyhedral cones. For example, if F is a real quadratic field and ε is a totally positive unit generating $E(\mathfrak{f})$, then the Shintani set $C^{\mathrm{o}}(1,\varepsilon)\cup C^{\mathrm{o}}(1)$ forms a fundamental domain for the action of $\mathcal{O}_F^\times$ (extended continuously to $\mathbb{R}^n$). More generally, Shintani proved

Proposition 3.1 (Proposition 4 of [15]). *Let $E \subset (\mathcal{O}_F^\times)_+$ be a finite index subgroup of totally positive units. Then there exists a rational Shintani set $C \subset \mathbb{R}_+^n$, such that $\varepsilon C \cap C = \emptyset$ for all $\varepsilon \in E$ and*

$$\mathbb{R}_+^n = \coprod_{\varepsilon \in E} \varepsilon C.$$

Such a collection of cones will be called a *Shintani domain* for E. A Shintani domain lets us decompose the ray class zeta functions as

$$\zeta([\mathfrak{a}]_{\mathfrak{f}}, s) = (\mathrm{N}\,\mathfrak{a})^{-s} \sum_{\alpha \in (a + \mathfrak{a}^{-1}\mathfrak{f} \cap C)} \frac{1}{\mathrm{N}\,\alpha^s} = (\mathrm{N}\,\mathfrak{a})^{-s} \zeta_{\mathrm{Sh}}([a + \mathfrak{a}^{-1}\mathfrak{f}], C; s) \tag{3.2}$$

reducing the study of special values of Hecke L-functions to the study of Shintani zeta functions.

3.2. Special values. Let us now suppose that C is the rational "open" cone $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$, with $v_1, \ldots, v_r \in (V_{\mathbb{R}})_+ \cap V$. Since a cone is finite disjoint union of open cones, it suffices to treat only this case. Shintani's meromorphic continuation of

$$\zeta_{\mathrm{SH}}(f, C; s) = \sum_{v \in C} \frac{f(v)}{N(v)^s}$$

generalizes Riemann's arguments for the meromorphic continuation of $\zeta(s)$. First, one expresses $\zeta_{\mathrm{SH}}(f, C; s)$ as Mellin-transform by

$$\begin{aligned}\sum_{v \in C} \frac{f(v)}{N(v)^s} &= \sum_{v \in C} f(v) \frac{1}{\Gamma(s)^n} \int_{(0,\ldots,0)}^{(\infty,\ldots,\infty)} e^{-(e_1^*(v)x_1 + \cdots + e_n^*(v)x_n)} (x_1 \cdots x_n)^{s-1} dx_1 \cdots dx_n \\ &= \frac{1}{\Gamma(s)^n} \int_{(0,\ldots,0)}^{(\infty,\ldots,\infty)} \sum_{v \in C} f(v) e^{-(e_1^*(v)x_1 + \cdots + e_n^*(v)x_n)} (x_1 \cdots x_n)^{s-1} dx_1 \cdots dx_n.\end{aligned}$$

To simplify notation, write $v \cdot x$ for $e_1^*(v)x_1 + \cdots + e_n^*(v)x_n$. Since C is rational, there exist $a_1, \ldots, a_r \in \mathbb{Q}$, such that f is periodic with respect to the lattice $a_1 v_1 \mathbb{Z} + \cdots + a_r v_r \mathbb{Z}$. After rescaling $v_1, \ldots, v_r$, we may assume that f is periodic with respect to the lattice $v_1\mathbb{Z} + \cdots + v_r\mathbb{Z}$, then we can rewrite $\sum_{v \in C} f(v) e^{-v \cdot x}$ as the "rational function"

$$\sum_{v \in C} f(v) e^{-v \cdot x} = \frac{1}{1 - e^{-v_1 \cdot x}} \cdots \frac{1}{1 - e^{-v_r \cdot x}} \sum_{v \in \mathcal{P}} f(v) e^{-v \cdot x},$$

where $\mathcal{P} \subset C$ is the fundamental domain for translation by $v_1\mathbb{Z}_{\geq 0} + \cdots + v_n\mathbb{Z}_{\geq 0}$. Switching signs, write

$$G(x_1, \ldots, x_n) = \frac{1}{1 - e^{v_1 \cdot x}} \cdots \frac{1}{1 - e^{v_r \cdot x}} \sum_{v \in \mathcal{P}} f(v) e^{v \cdot x}.$$

The function $\frac{1}{e^z - 1}$ has a simple pole at $z = 0$ with residue 1, so $G(x_1, \ldots, x_n)$ potentially has simple poles along the hyperplanes $v_1 \cdot x = 0, \ldots, v_r \cdot x = 0$.

Next, one would like to find the Mellin-transform of $G(-x)$ as a term in the complex contour integral

$$\int_C G(-z) e^{(s-1)\log(z_1) + \cdots + (s-1)\log(z_n)} dz_1 \cdots dz_n,$$

where C is a product of keyhole contours $+\infty \to +\infty$ around 0. However, when $n > 1$, the poles of G will intersect any sphere about the origin, hence our contour C, and we come to an impasse. Shintani managed to circumvent these problems by cleverly decomposing the domain of the Mellin transform. For details, we refer the reader to Shintani's original paper [15] or the notes of Dasgupta and Greenberg [11] for very readable accounts.

The following theorem is a reformulation of Proposition 1 of [15].

Theorem 3.2 (Shintani)**.** *The function $\zeta_{\mathrm{SH}}(f, C; s)$ has meromorphic continuation to the whole complex plane with at most a simple pole at $s = 1$. Moreover, the special values are given by*

$$\zeta_{\mathrm{SH}}(f, C; -k) = \frac{1}{nk!^n}\left(\sum_{i=1}^{n} \mathrm{Coeff}(G(ux_1, ux_2, \ldots, ux_n)\,|_{x_i=1}; u^{nk}x_2^k \cdots x_n^k)\right), \tag{3.3}$$

where $\mathrm{Coeff}(F(x_1, \ldots, x_n), x_1^{k_1} \cdots x_n^{k_1})$ *denotes the coefficient of* $x_1^{k_1} \cdots x_n^{k_1}$ *in the Laurent series of F about the origin.*

Note that $G(x_1, \ldots, x_n)$ is not necessarily a sum of monomials $x_1^{k_1} \cdots x_n^{k_n}$ near $(0, \ldots, 0)$ (consider $\frac{e^{x+y}}{e^{x+y}-1} = \frac{1}{x+y}\sum_{n,m\geq 0} B_{n+m}\frac{x^n y^m}{n!m!}$). However, it is not hard to see that $G(ux_1, \ldots, ux_n)|_{x_1=1}$ has a well-defined Laurent series in powers of $u, x_2, \ldots, x_n$. If G happens to be holomorphic at the origin, then equation (3.3) simplifies to

$$\begin{aligned}\zeta_{\mathrm{SH}}(f, C; -k) &= \frac{1}{k!^n}\,\mathrm{Coeff}(G(x_1, \ldots, x_n), x_1^k \cdots x_n^k)\\ &= \mathrm{Coeff}\left(\frac{\partial^{nk}}{\partial^k x_1 \cdots \partial^k x_n} G(x_1, \ldots, x_n); x_1^0 \cdots x_n^0\right)\\ &= \frac{\partial^{nk}}{\partial^k x_1 \cdots \partial^k x_n} G(x_1, \ldots, x_n)\,|_{x=0}\,.\end{aligned}$$

4. Pseudo-measures and zeta values

4.1. Pseudo-measures. Fix p a prime. Let $U \subset V_p$ be a compact open. We will focus primarily on the case $U = L_p$, the p-adic completion of the lattice $L \subset V$.

Definition 4.1. $\mathcal{C}(U)$ is the $\mathbb{Q}_p$ vector space of continuous functions $f : U \to \mathbb{Q}_p$. This is a $\mathbb{Q}_p$-Banach space under the sup-norm, $|f| = \sup_{v\in U} |f(v)|_p$.

Definition 4.2. A p-adic measure μ on U is a continuous linear functional $\mu : \mathcal{C}(U) \to \mathbb{Q}_p$. We write $\mathcal{M}(U)$ for the $\mathbb{Q}_p$-Banach space $\mathrm{Hom}_{\mathrm{cts}}(\mathcal{C}(U), \mathbb{Q}_p)$.

The fundamental example is the Dirac delta. For each $v \in U$, $\delta_v \in \mathcal{M}(U)$ is defined by $\delta_v(f) := f(v)$.

When U is a lattice, the convolution of two measures $\mu, \nu \in \mathcal{M}(U)$ is defined by

$$(\mu * \nu)(f) = \int_U \left(\int_U f(v + w)d\nu(w)\right) d\mu(v).$$

As a concrete example, note that the convolution of two dirac measure δ_v and δ_w is given by $\delta_v * \delta_w = \delta_{v+w}$. If U is a lattice, then $\mathcal{M}(U)$ is a commutative ring under the convolution product, and is isomorphic to a power series ring, as we will recall shortly. In particular, $\mathcal{M}(U)$ is a domain.

The space of pseudo-measures on L_p is a subring of the field of fractions of $\mathcal{M}(L_p)$, which, in some sense, accommodates the poles (at $s = 1$) of the p-adic zeta functions we construct. Let $S \subset \mathcal{M}(L_p)$ denote the multiplicative subset generated by the set $\{\delta_0 - \delta_v : v \in L_p, v \neq 0\}$.

Definition 4.3. The space of p-adic pseudo-measures on L_p is the localization

$$\widetilde{\mathcal{M}}(L_p) := S^{-1}\mathcal{M}(L_p).$$

Note that this definition differs from some standard definitions, e.g., Coates [5].

Proposition 4.4. *The natural map $\mathcal{M}(L_p) \to \widetilde{\mathcal{M}}(L_p)$ is injective.*

Proof. This follows from the fact (due to Amice) that $\mathcal{M}(L_p)$ is isomorphic to a subring of power series over $\mathbb{Q}_p$, and thus is an integral domain. We will recall this fact in greater detail in the following section. □

The fundamental example of a pseudo-measure is the classical Kubota–Leopoldt p-adic zeta function. If we take $V = \mathbb{Q}$, then there is a unique pseudo-measure $\xi \in \mathcal{M}(\mathbb{Z}_p)$ satisfying $(\delta_0 - \delta_1) * \xi = \delta_0$. This pseudo-measure interpolates (in a sense that can be made precise) the special values of the Riemann zeta function. After a brief foray into measures on L_p, we shall generalize this to pseudo-measures associated to cones (Proposition 4.14).

4.2. The Amice transform. After choosing coordinates, all computations will reduce to the case of measures on $\mathbb{Z}_p^n$. The space of measures on $\mathbb{Z}_p^n$ can be explicitly described in terms of power series over $\mathbb{Z}_p$. We refer to [1] or [7] for proofs. When $n = 1$, Mahler's theorem gives an Banach basis of $\mathcal{C}(\mathbb{Z}_p)$ via the generalized binomial coefficients

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0. \end{cases}$$

This generalizes in a straightforward way to the case of several variables.

Proposition 4.5. *The functions $\left\{\binom{x_1}{k_1}\cdots\binom{x_n}{k_n}\right\}$ form a Banach basis of $\mathcal{C}(\mathbb{Z}_p^n)$. Concretely, every $f \in \mathcal{C}(\mathbb{Z}_p^n)$ can be written uniquely as*

$$f(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n \geq 0} a_{k_1,\ldots,k_n} \binom{x_1}{k_1} \cdots \binom{x_n}{k_n}$$

with $|a_{k_1,\ldots,k_n}|_p \to 0$ and $||f|| = \sup |a_{k_1,\ldots,k_n}|_p$.

Proof. See, for example, the remarks following Proposition 7, Section 7 in [1]. Proposition 4.5 tells us a measure μ is uniquely determined by the moments $\mu\left(\binom{x_1}{k_1}\cdots\binom{x_n}{k_n}\right)$, and that $||\mu|| = \sup_k \left|\binom{x}{k}\right|_p$. To ease notation, our convention will be to write x and k for the vectors $x = (x_1, \ldots, x_n)$ and $k = (k_1, \ldots, k_n)$. We simply write $\binom{x}{k} = \binom{x_1}{k_1}\cdots\binom{x_n}{k_n}$ when no confusion may arise.

These moments can be conveniently packaged into power series in n variables via the Amice transform. □

Definition 4.6. Given a measure $\mu \in \mathcal{M}(\mathbb{Z}_p^n)$, let $\mathcal{A}(\mu) \in \mathbb{Q}_p[[T_1, \ldots, T_n]]$ be the power series

$$\mathcal{A}(\mu)(T_1, \ldots, T_n) = \sum_{k_1, \ldots, k_n \geq 0} \left(\int_{\mathbb{Z}_p^n} \binom{x_1}{k_1} \cdots \binom{x_n}{k_n} d\mu(x) \right) T_1^{k_1} \cdots T_n^{k_n}.$$

We call $\mathcal{A}(\mu)$ the Amice transform of μ. Note that (at least formally), this is the power series

$$\mathcal{A}(\mu) = \int_{\mathbb{Z}_p^n} (1 + T_1)^{x_1} \cdots (1 + T_n)^{x_n} d\mu(x).$$

The series $\mathcal{A}(\mu)(T_1, \ldots, T_n)$ uniquely determines the measure μ. What is more, the following theorem of Amice tells us that each bounded power series corresponds to a measure.

Theorem 4.7 (Amice). *The map $\mathcal{A} : \mathcal{M}(\mathbb{Z}_p^n, \mathbb{Q}_p) \to \mathbb{Z}_p[[T_1, \ldots, T_n]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism of $\mathbb{Q}_p$-Banach algebras.*

As a map of commutative rings, the Amice transform extends to a homomorphism

$$\mathcal{A} : \widetilde{\mathcal{M}}(\mathbb{Z}_p^n) \hookrightarrow \mathrm{Frac}(\mathbb{Z}_p[[T_1, \ldots, T_n]]).$$

Remark 4.8. For each $r \in \mathbb{R}_+$ and $a \in \mathbb{C}_p$, let us write $B(a, r) \subset \mathbb{C}_p$ for the open disc $B(a, r) = \{z \in \mathbb{C}_p : |z - a|_p < r\}$. Since the power series $\mathcal{A}(\mu)(T_1, \ldots, T_n)$ has bounded denominators, the series $\mathcal{A}(\mu)(z_1 - 1, \ldots, z_n - 1)$ converges for all $z = (z_1, \ldots, z_n) \in B(1, 1)^n$ and in fact converges to $\int_{\mathbb{Z}_p^n} z_1^{x_1} \cdots z_n^{x_n} d\mu(x)$.

The p-adic exponential map gives an inclusion $B(0, p^{1/(p-1)})^n \hookrightarrow B(1, 1)^n$ by $(t_1, \ldots, t_n) \mapsto (\exp(t_1), \ldots, \exp(t_n))$. This inclusion allows us to write the Amice transform of μ as

$$\mathcal{A}(\mu)(e^{t_1} - 1, \ldots, e^{t_n} - 1) = \int_{\mathbb{Z}_p} (e^{t_1})^{x_1} \cdots (e^{t_n})^{x_n} d\mu(x) = \int_{\mathbb{Z}_p}^{n} e^{x_1 t_1 + \cdots + x_n t_n} d\mu(x).$$

Expanding the right-hand side as a power series in $t_1, \ldots, t_n$, we see $\mathcal{A}(\mu)(e^{t_1} - 1, \ldots, e^{t_n} - 1)$ is an analytic function on $B(1, p^{-1/(p-1)})^n$ with power series

$$\sum_{k_1, \ldots, k_n \geq 0} \left(\int_{\mathbb{Z}_p^n} x_1^{k_1} \cdots x_n^{k_n} d\mu(x) \right) \frac{t_1^{k_1} \cdots t_n^{k_n}}{k_1! \cdots k_n!}.$$

We can read off the moments $\int_{\mathbb{Z}_p^n} x_1^{k_1} \cdots x_n^{k_n} d\mu(x)$ as coefficients of $\mathcal{A}(\mu)$ or, equivalently, by differentiating $\mathcal{A}(\mu)$. Let $D = \frac{\partial^{k_1}}{\partial t_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial t_n^{k_n}}$ so that $De^{x_1 t_1 + \cdots + x_n t_n} = x_1^{k_1} \cdots x_n^{k_n} e^{x_1 t_1 + \cdots + x_n t_n}$. Thus,

$$\begin{aligned} \int_{\mathbb{Z}_p^n} x_1^{k_1} \cdots x_n^{k_n} d\mu(x) &= \int_{\mathbb{Z}_p}^{n} (De^{x_1 t_1 + \cdots + x_n t_n})|_{t_1, \ldots, t_n = 0} d\mu(x) \\ &= D(\mathcal{A}(\mu)(e^{t_1} - 1, \ldots, e^{t_n} - 1))|_{t_1, \ldots, t_n = 0}. \end{aligned}$$

This formula will be useful to us in the next section.

4.3. Pseudo-measures from cones. Let C be a rational Shintani set, $f' \in \mathcal{S}(V^{(p)})$, and $U \subset L_p$ a compact open. We wish to define a pseudo-measure $\mu_{C,f',U} \in \widetilde{\mathcal{M}}(L_p)$ which makes sense of the formal sum

$$\sum_{v \in C \cap V} f' \otimes [U](v)\delta_v.$$

For example, if $f' \in \mathcal{S}(\mathbb{Q}^{(p)})$ is the test function satisfying $f' \otimes [\mathbb{Z}_p] = [\mathbb{Z}]$ and C is the cone of non-negative real numbers, one can interpret the formal sum

$$\sum_{v \in C \cap \mathbb{Q}} [\mathbb{Z}](v)\delta_v = \sum_{n \geq 0} \delta_n = \sum_{n \geq 0} \delta_1^n$$

as the formal geometric series expansion of $\xi = \frac{\delta_0}{\delta_0 - \delta_1}$, i.e., the Kubota–Leopoldt pseudo-measure.

Definition 4.9. Given a rational Shintani set C, we define the "C-upper half-plane in $V_{\mathbb{C}}^* := \operatorname{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$ as

$$\mathcal{H}_C := \{\lambda \in V_{\mathbb{C}}^* : \lambda(v) \in \mathcal{H} \text{ for all } v \in C\},$$

where $\mathcal{H} \subset \mathbb{C}$ is the usual upper half-plane. For example, let $v_1, \ldots, v_n$ be a basis of V, and let $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$ for some $r \leq n$. The basis identifies $V_{\mathbb{C}}^*$ with $\mathbb{C}^n$ by $\lambda \mapsto (\lambda(v_1), \ldots, \lambda(v_n))$ and the image of $\mathcal{H}_C$ is $\mathcal{H}^r \times \mathbb{C}^{n-r}$. Note that if C contains a line, then $\mathcal{H}_C$ is empty.

For each $v \in V$, define $q^v : V_{\mathbb{C}}^* \longrightarrow \mathbb{C}^\times$ by $q^v(\lambda) := e^{2\pi i \lambda(v)}$. The ring $\mathbb{Q}[q^v]_{v \in V \cap L_p}$, with multiplication $q^v q^w = q^{v+w}$, is simply the group ring $\mathbb{Q}[V \cap L_p]$. In particular, $\mathbb{Q}[q^v]_{v \in V \cap L_p}$ is an integral domain. We have a ring homomorphism $\mathbb{Q}[q^v]_{v \in V \cap L_p} \longrightarrow \mathcal{M}(L_p)$ by sending q^v to δ_v and extending linearly. Let $S \subset \mathbb{Q}[V \cap L_p]$ denote the multiplicative subset generated by the set $\{1 - q^v | v \in V \cap L_p\}$ and define $\mathcal{R}$ to be the localization $S^{-1}\mathbb{Q}[V \cap L_p]$. Localization gives us a homomorphism $\mathcal{R} \longrightarrow \widetilde{\mathcal{M}}(L_p)$.

Lemma 4.10. *Let $v_1, \ldots, v_r \in V \cap L_p$ be linearly independent and set $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$. Let f' be a test function away from p, and $U \subset L_p$ a compact open. There exist $a_1, \ldots, a_r \in \mathbb{Q}$, such that the infinite sum*

$$G_{C,f',U}(\lambda) := \sum_{v \in C \cap V} f' \otimes [U](v) q^v(\lambda)$$

converges in $\mathbb{C}$ for all $\lambda \in \mathcal{H}_C$. Moreover, it converges to

$$G_{C,f',U}(\lambda) = \frac{1}{1 - q^{a_1 v_1}(\lambda)} \cdots \frac{1}{1 - q^{a_r v_r}(\lambda)} \sum_{v \in \mathcal{P}} f'[U](v) q^v(\lambda), \tag{4.1}$$

where $\{\sum_{i=1}^n \lambda_i v_i : \lambda_i \in \mathbb{Q}, 0 < \lambda_i \leq a_i\}$.

Proof. We pick $a_1, \ldots, a_n \mathbb{Q}$, so that the test function $f' \otimes [U]$ is periodic with respect to $a_1 v_1, \ldots, a_n v_n$. If $\lambda \in \mathcal{H}_C$, then for all $v \in C$ $|q^v(\lambda)| < 1$. Expanding the right-hand side of 4.1 as a product of geometric series gives the result. ☐

Since $f' \otimes [U]$ is a test function, the sum $\sum_{v \in \mathcal{P}} f'[U](v) q^v$ is finite. Thus $G_{C,f,U}$ converges to an element of the ring $\mathcal{R}$.

Definition 4.11. With C, f' and U as above, we define $\mu_{C,f',U} \in \widetilde{\mathcal{M}}(L_p)$ to be the image of $G_{C,f',U}$ under this homomorphism. More generally, if the Shintani set C is the disjoint union of open cones $\coprod_{j=1}^{N} C_j$, we define $\mu_{C,f',U}$ to be the sum $\sum_{j=1}^{N} \mu_{C_j,f',U}$. When $U = L_p$, we abbreviate μ_{C,f',L_p} by $\mu_{C,f'}$.

Remark 4.12. If $\mu_{C,f'}$ is a measure and $U \subset L_p$ is a compact open, we see from the formal expansions that the restriction of $\mu_{C,f'}$ to U is just $\mu_{C,f',U}$. It follows that $\mu_{C,f',U}$ is also a measure and is supported on U.

In the case of a single open cone, we have a useful characterization of the Amice transform of $\mu_{C,f',U}$. After rescaling, we may suppose $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$, with $v_1, \ldots, v_r \in L$ linearly independent. If $r < n$, we choose $v_{r+1}, \ldots, v_n \in L$, so that $v_1, \ldots, v_n$ is a $\mathbb{Q}$-basis of V. Now let $U \subset L_p$ be the $\mathbb{Z}_p$-span of $v_1, \ldots, v_r$. We see that $\mu_{C,f',U}$ is a pseudo-measure in $\widetilde{\mathcal{M}}(U) \subset \widetilde{\mathcal{M}}(L_p)$. Our basis of U identifies U with $\mathbb{Z}_p^n$ and this identification allows us to compute the Amice transform of $\mu_{C,f',U} \in \widetilde{\mathcal{M}}(U) \cong \widetilde{\mathcal{M}}(\mathbb{Z}_p^n)$.

Proposition 4.13. *Suppose $U \subset L_p$ is a lattice with $\mathbb{Z}_p$-basis $v_1, \ldots, v_n$ and that C is the open cone $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$ for some $r \leq n$. Denote by ι the isomorphism $\widetilde{\mathcal{M}}(U) \longrightarrow \widetilde{\mathcal{M}}(\mathbb{Z}_p^n)$ induced by the basis $v_1, \ldots, v_n$. Then there exists p-adic units $a_1, \ldots, a_r \in \mathbb{Q} \subset \mathbb{Q}_p$, such that $\mathcal{A}(\iota(\mu_{C,f',U}))$ is equal to*

$$\frac{1}{1-(1+T_1)^{a_1}} \cdots \frac{1}{1-(1+T_r)^{a_r}} \sum_{v \in \mathcal{P}} f' \otimes [U](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)},$$

where $\mathcal{P}$ is the half-open parallelepiped $\{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{Q}, 0 < \lambda_i \leq a_i\}$.

Proof. The test function $[U]$ is periodic with respect to the vectors $v_1, \ldots, v_r$ and thus we can find $a_1, \ldots, a_r \in \mathbb{Q}$ with trivial p-adic valuation such that $f' \otimes [U]$ is periodic with respect to the vectors $a_1 v_1, \ldots, a_r v_r$. The sum $G_{C,f',U} = \sum_{v \in C \cap V} f' \otimes [U](v) q^v$ can be written as

$$\frac{1}{1-q^{a_1 v_1}} \cdots \frac{1}{1-q^{a_r v_r}} \sum_{v \in \mathcal{P}} f' \otimes [U] q^v$$

and we conclude that $\mu_{C,f',U}$ is the pseudo-measure

$$\frac{1}{\delta_0 - \delta_{a_1 v_1}} \cdots \frac{1}{\delta_0 - \delta_{a_r v_r}} \sum_{v \in \mathcal{P}} f' \otimes [U] \delta_v.$$

For each $v \in U$, applying ι to δ_v gives $\iota(\delta_v) = \delta_{(v_1^*(v), \ldots, v_n^*(v))} \in \mathcal{M}(\mathbb{Z}_p^n)$. Therefore, $\mathcal{A}(\iota(\delta_v)) = (1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)}$ and, in particular, $\mathcal{A}(\iota(\delta_{a_i v_i})) = (1+T_i)^{a_i}$. We conclude $\mathcal{A}(\iota(\mu_{C,f'}))(T_1, \ldots, T_n)$ is equal to

$$\frac{1}{1-(1+T_1)^{a_1}} \cdots \frac{1}{1-(1+T_r)^{a_r}} \sum_{v \in \mathcal{P}} f' \otimes [U](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)}$$

as claimed. ∎

Now let us suppose we have chosen an isomorphism $V_{\mathbb{R}} \cong \mathbb{R}^n$ under which the norm function $N = e_1^* \cdots e_n^*$ assumes rational values on $V \subset V_{\mathbb{R}}$. For example, when $V = F$ is a totally real field of degree n, the isomorphism will come from the usual embedding of F into $\mathbb{R}^n$ via F's n archimedean places.

Proposition 4.14. *Let C be a rational Shintani set contained in the positive orthant $(V_{\mathbb{R}})_+$, $U \subset L_p$ a compact open, and $f' \in \mathcal{S}(V^{(p)})$. If $\mu_{C,f'}$ is a measure, then*

$$\int_U N^k(v) d\mu_{C,f'}(v) = \text{ the value at } s = -k \text{ of the analytic continuation of}$$
$$\zeta(f' \otimes [U], C; s) = \sum_{v \in C} \frac{f' \otimes [U](v)}{N(v)^s}.$$

It is useful to keep in mind the following heuristic, which we emphasize is *not* mathematically meaningful

$$\begin{aligned}\int_U N^k d\mu \text{``}=\text{''} &\sum_{v \in C} f(v) \int_U N^k d\delta_v \\ \text{``}=\text{''} &\sum_{v \in C} f(v) N(v)^k \\ \text{``}=\text{''} &\zeta_{\mathrm{SH}}(f, C; s)\mid_{s=-k}.\end{aligned}$$

The key argument we need to make this meaningful are Shintani's formulas.

Proof. It suffices to consider the cases of $C = C^{\mathrm{o}}(v_1, \ldots, v_r)$ with $v_1, \ldots, v_r \in V \cap L_p$ linearly independent. Fix f', and U as above. As before, we know there exists $a_1, \ldots, a_n \in \mathbb{Q}$ with non-negative p-adic valuation, such that

$$\mu_{C,f',U} = \frac{1}{\delta_0 - \delta_{a_1 v_1}} \cdots \frac{1}{\delta_0 - \delta_{a_r v_r}} \sum_{v \in \mathcal{P}} f' \otimes [U] \delta_v.$$

Now pick a $\mathbb{Z}_p$-basis $w_1, \ldots, w_n$ of L_p and let $\iota : \widetilde{\mathcal{M}}(L_p) \longrightarrow \widetilde{\mathcal{M}}(\mathbb{Z}_p^n)$ denote the corresponding isomorphism. For each $v \in L_p$, $\iota(\delta_v) = \delta_{(w_1^*(v), \ldots, w_n^*(v))}$ and $\mathcal{A}(\iota\delta_v)(e^{t_1} - 1, \ldots, e^{t_n} - 1) = e^{w_1^*(v)t_1 + \cdots + w_n^*(v)t_n}$. Writing $v \cdot t$ for $w_1^*(v)t_1 + \cdots + w_n^*(v)t_n$, this becomes $\mathcal{A}(\iota\delta_v) = e^{v \cdot t}$. It follows that $\mathcal{A}(\iota(\mu))(e^{t_1} - 1, \ldots, e^{t_n} - 1)$ is equal to

$$\frac{1}{1 - e^{a_1 v_1 \cdot t}} \cdots \frac{1}{1 - e^{a_n v_n \cdot t}} \sum_{v \in \mathcal{P}} f' \otimes [U] e^{v \cdot t}. \tag{4.2}$$

Since μ is a measure, this is an analytic function on the disk $B(1, p^{-1/(p-1)})$. On the other hand, we can expand 4.2 into a Laurent series in $\mathbb{Q}((t_1, \ldots, t_n))$ using the identity $\frac{e^{sX}}{1-e^X} = \frac{-1}{X} \sum_{n \geq 0} B_n(s) \frac{X^n}{n!}$, which belongs to $\mathbb{Q}((X))$ when $s \in \mathbb{Q}$. Because this Laurent series defines an analytic function on $B(1, p^{-1/(p-1)})$, it follows that the series has removable singularities, and we can compute the moments of μ in terms of the differential operator $D_N e^{v \cdot t} = N(v) e^{v \cdot t}$ using Remark 4.8. Abbreviating $\mu_{C,f',U}$ by μ, we have

$$\begin{aligned}\int_U N^k(v) d\mu_{C,f'}(v) &= \left(\int_U D_N^k e^{v \cdot t} d\mu\right)\Big|_{q=1} \\ &= D_N^k \mathcal{A}(\iota(\mu))(e^{t_1} - 1, \ldots, e^{t_n} - 1)\mid_{t_1, \ldots, t_n = 0}.\end{aligned}$$

We make the change of variables $t_i = e_1^*(w_i)x_i + \cdots + e_n^*(w_i)x_n$ for $i = 1, \ldots, n$, so that $v \cdot t = w_1^*(v)t_1 + \cdots + w_n^*(v)t_n = e_1^*(v)x_1 + \cdots + e_n^*(v)x_n$, which we denoted $v \cdot x$

in Section 3.2. In these coordinates, $\mathcal{A}(\iota(\mu))$ is equal to

$$G(x_1,\ldots,x_n) = \frac{1}{1-e^{a_1v_1\cdot x}}\cdots\frac{1}{1-e^{a_nv_n\cdot x}}\sum_{v\in\mathcal{P}} f'\otimes[U](v)e^{v\dot{x}}.$$

One also sees that $D_N^k = \frac{\partial^{nk}}{\partial^k x_1\cdots\partial^k x_n}$. Because $G(x_1,\ldots,x_n)$ has no singularities along the axes $x_1,\ldots,x_n = 0$, Shintani's theorem gives

$$D_N^k\mathcal{A}(\iota(\mu))\mid_{x_1,\ldots,x_n=0} = \frac{\partial^{nk}}{\partial^k x_1\cdots\partial^k x_n}G(x_1,\ldots,x_n)|_{x_1,\ldots,x_n=0} = \zeta_{\mathrm{SH}}(f'\otimes[U],C;-k).$$

□

4.4. Measure criteria. In light of Proposition 4.14 it is natural to ask, "when is $\mu_{C,f'}$ a measure?" The main result of this section, and the technical heart of the paper, is an exact criterion for $\mu_{C,f'}$ to be a measure. Roughly speaking, this happens whenever the test function f' has vanishing average in the directions of the extremal rays of C. To make precise this vague statement, we introduce some notation.

For each non-zero $w\in V$ and any $v\in V$, write $\pi_{v,w}:\mathcal{S}(V_\ell)\to\mathcal{S}(\mathbb{Q}_\ell)$ for the map which sends a test function $f\in\mathcal{S}(V_\ell)$ to the function $\pi_{v,w}f:\mathbb{Q}_\ell\to\mathbb{Z}$

$$(\pi_{v,w}f)(x) := f(v+xw), \text{ for all } x\in\mathbb{Q}_\ell.$$

A fortiori, $\pi_{v,w}f$ is indeed a test function on $\mathbb{Q}_\ell$. Similarly, we define $\pi_{v,w}:\mathcal{S}(V^{(p)})\to\mathcal{S}(\mathbb{Q}^{(p)})$ and $\pi_{v,w}:\mathcal{S}(V)\to\mathcal{S}(\mathbb{Q})$.

The Haar measure on $\mathcal{S}(V)$, normalized with respect to L, can be defined in two equivalent ways. First, for each $f\in\mathcal{S}(V)$, there exist a lattice L_f for which f is periodic: $\forall\ell\in L_f$ and $v\in V$, $f(v+\ell)=f(v)$. One can define (see [18]) the global Haar measure h_V by putting

$$h_V(f) := \frac{1}{[L:L_f]}\sum_{v\in V/L_f} f(v). \tag{4.3}$$

Since f has bounded supported, the sum is finite, and it is easy to see that it is independent of choice of L_f.

On the other hand, we have at each local component V_ℓ, a local Haar measure h_ℓ normalized so that $h_\ell([L_\ell])=1$. Given $f=\bigotimes_\ell f_\ell\in\mathcal{S}(V)$, we can also define

$$h'_V(f) := \prod_\ell h_\ell(f_\ell) \tag{4.4}$$

and extend to all of $\mathcal{S}(V)$ by linearity.

Lemma 4.15. *The measures 4.3 and 4.4 both define the same Haar measure on V.*

Proof. See [18], Section 3.3. □

We define the Haar measure of test functions away from p by defining $h^{(p)}(f') := h_V(f'\otimes[L_p])$. If f' is factorizable (i.e., $f'=\bigotimes_\ell f_\ell\in\mathcal{S}(V)$), then $h^{(p)}(f')=\prod_{\ell\neq p}h_\ell(f')$.

Definition 4.16 (Vanishing Hypothesis)**.** Let $w\in V$ be a non-zero vector. We will say a test function f' *satisfies the Vanishing Hypothesis for* w if $h^{(p)}(\pi_{v,w}f')=0$ for all $v\in V$.

We remark that $h^{(p)}(\pi_{v,w}f')$ depends only on $v \mod \langle w\rangle$ by the translation invariance of Haar.

Lemma 4.17. *If a test function $f' \in \mathcal{S}(V^{(p)})$ satisfies the vanishing hypothesis for w, then for all compact open $U \subset V_p$ and $v \in V$,*

$$h_{\mathbb{Q}}(\pi_{v,w}(f' \otimes [U])) = 0.$$

Proof. The vectors v, w embed $\mathbb{Q} \hookrightarrow V$ and $\mathbb{Q}_p \hookrightarrow V_p$ via the inclusion $\lambda \mapsto v + \lambda w$. Define $W = \{x \in \mathbb{Q}_p | v + xw \in U\}$. Then, $\pi_{v,w}(f' \otimes [U]) = (\pi_{v,w}f') \otimes (\pi_{v,w}[U]) = (\pi_{v,w}f') \otimes [W]$, and $h_V(\pi_{v,w}(f' \otimes [U])) = h^{(p)}(\pi_{v,w}f')h_p([W]) = 0$. □

One may interpret this lemma as saying a test function f' satisfies the vanishing hypothesis for w then the average value of $f' \otimes f_p$ is 0 along all lines parallel to w, for all $f_p \in \mathcal{S}(V_p)$. While this hypothesis may seem odd, it is in fact easy to verify in important cases. Indeed, we will show that the vanishing hypothesis is satisfied when f' comes from the data of a "smoothed" ray class zeta function of a totally real field. In the next section, we show how the construction of p-adic L-functions of totally real fields is a corollary of our main theorem.

We begin with a special case of the criteria, from which we shall deduce the full result. Let $v_1, \dots, v_r$ be linearly independent vectors in L_p, and put $C = C^{\mathrm{o}}(v_1, \dots, v_r)$. Extending $v_1, \dots, v_r$ to a $\mathbb{Q}_p$-basis by $v_{r+1}, \dots, v_n \in L_p$, put $U = \mathbb{Z}_p v_1 + \cdots + \mathbb{Z}_p v_n$. For each $v_0 \in L_p$, consider the pseudo-measure $\mu_{f',C,v_0+U} \in \widetilde{\mathcal{M}}(L_p)$.

Proposition 4.18. *With $C = C^{\mathrm{o}}(v_1, \dots, v_r)$, a test function $f' \in \mathcal{S}(V^{(p)})$ satisfies the vanishing hypothesis for $v_1, \dots, v_r$ if and only if μ_{f',C,v_0+U} is a measure.*

Proof. We only record the proof of the case $v_0 = 0$. The general case is virtually identical, modulo a few change of variables that we will indicate following the proof.

The basis $v_1, \dots, v_n$ of U gives an isomorphism $\iota : \widetilde{\mathcal{M}}(U) \longrightarrow \widetilde{\mathcal{M}}(\mathbb{Z}_p^n)$, and proposition 4.13, tells us $\mathcal{A}(\iota(\mu_{C,f',U}))$ is equal to

$$\frac{1}{1-(1+T_1)^{a_1}} \cdots \frac{1}{1-(1+T_r)^{a_r}} \sum_{v \in \mathcal{P}} f' \otimes [U](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)},$$

where $a_1, \dots, a_r$ are p-adic units. The fact that a_i is a unit implies $(1-(1+T_i)^{a_i})/T_i$ is a unit in $\mathbb{Z}_p[[T_1, \dots, T_n]]$. Thus, $\mathcal{A}(\iota(\mu_{C,f',U}))$ is an element of $\mathbb{Z}_p[[T_1, \dots, T_n]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ if and only if $T_1 \cdots T_r$ divides

$$F(T_1, \dots, T_n) := \sum_{v \in \mathcal{P}} f' \otimes [U](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)},$$

which is equivalent to each of $T_1, \dots, T_r$ dividing F.

Claim. For each $i \in 1, \dots, r$, $T_i | F$ if and only if f' satisfies the vanishing hypothesis for v_i.

For notational simplicity, we focus on the case $i = 1$. It is easy to see that T_1 divides $F(T_1, \dots, T_r) \in \mathbb{Z}_p[[T_1, \dots, T_n]]$ if and only if $F(0, T_2, \dots, T_n) = 0$, i.e.,

$$F(0, T_2, \dots, T_n) = \sum_{v \in \mathcal{P}} f' \otimes [U](v)(1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)} = 0.$$

We carefully rearrange the sum as

$$F(0,T_2,\ldots,T_n)=\sum_{v\in\mathcal{P}\cap v_1^\perp}\left(\sum_{x\in(0,a_1]}f'\otimes[U](v+xv_1)\right)(1+T_2)^{v_2^*(v)}\cdots(1+T_n)^{v_n^*(v)}.$$

The coefficient in the parenthesis can be rewritten as

$$\sum_{x\in(0,a_1]}f'\otimes[U](v+xv_1)=a_1\frac{1}{[\mathbb{Z}:a_1\mathbb{Z}]}\sum_{x\in(0,a_1]}f'\otimes[U](v+xv_1)=a_1h_{\mathbb{Q}}(\pi_{v,v_1}(f'\otimes[U])).$$

This shows that $F(0,T_2,\ldots,T_n)=0$, if and only if $h_{\mathbb{Q}}(\pi_{v,v_1}f'\otimes[U])=0$ for all $v\in V$. Lemma 4.17 tells us this is equivalent to the vanishing hypothesis for v_1, so T_1 divides F if and only if f' satisfies the vanishing hypothesis for v_1. Similarly, we conclude T_i divides F if and only if f' satisfies the vanishing hypothesis for v_i giving the result.

More generally, the formal sum associated to f', C, and v_0+U is

$$\sum_{v\in C}f'\otimes[v_0+U](v)q^v=\sum_{v\in-v_0+C}f'(v_0+v)[U](v)q^{v_0+v}=q^{v_0}\sum_{v\in-v_0+C}f'(v_0+v)[U](v)q^v.$$

The vanishing hypothesis is translation invariant, so we may replace $f'(-v_0+v)$ with $f'(v)$ which reduces us to showing

$$q^{v_0}\sum_{v\in-v_0+C}f'\otimes[U](v)q^v$$

represents a measure when f' satisfies the vanishing hypotheses for $v_1,\ldots,v_r$. This is the convolution by δ_{v_0} of a pseudo-measure representing $\sum_{v\in-v_0+C}f'\otimes[U](v)q^v$ and the above argument, with $\mathcal{P}$ replaced by $-v_0+\mathcal{P}$, shows $\sum_{v\in-v_0+C}f'\otimes[U](v)q^v$ represents a measure if and only if f' satisfies the vanishing hypothesis for $v_1,\ldots,v_r$. □

Theorem 4.19. *Let C be a Shintani set with extremal rays $v_1,\ldots,v_r$. The pseudo-measure $\mu_{C,f'}\in\widetilde{\mathcal{M}}(L_p)$ is a measure if f' satisfies the vanishing hypothesis for $v_1,\ldots,v_r$.*

Proof. Since $C(\lambda_1v_1,\ldots,\lambda_rv_r)=C(v_1,\ldots,v_r)$ for $\lambda_i\in\mathbb{Q}_+$, we may assume without loss of generality that $v_1,\ldots,v_r$ are contained in L_p. Furthermore, we may write C as the disjoint union of open cones $C=\coprod_{j=1}^d C_j$, with each C_j open and generated by a subset of $v_1,\ldots,v_r$. We pick $v_{r+1},\ldots,v_n$, also in L_p, so that $v_1,\ldots,v_n$ is a $\mathbb{Q}_p$-basis of V_p. Note that the lattice $U=\mathbb{Z}_pv_1+\cdots+\mathbb{Z}_pv_n$ is contained in the lattice L_p with finite index. The sum representing μ_{C,f',L_p} breaks up as

$$\begin{aligned}\sum_{j=1}^d\sum_{v\in C_j}f'\otimes[L_p](v)q^v&=\sum_{j=1}^d\sum_{v\in C_j}\sum_{x\in L_p/U}f'\otimes[x+U](v)q^v\\&=\sum_{j=1}^d\sum_{w\in L_p/U}\sum_{v\in C_j}f'\otimes[w+U](v)q^v,\end{aligned}$$

where the sum over L_p/U is the sum over distinct cosets $x+V$. Therefore, μ_{C,f',L_p} is equal to

$$\sum_{j=1}^{d} \sum_{x \in L_p/U} \mu_{C_j,f',w+U}.$$

We conclude that $\mu_{C,f'}$ is a measure when each summand is a measure. Thus, by Proposition 4.18, $\mu_{C,f'}$ is a measure when f' satisfies the vanishing hypothesis for $v_1, \ldots, v_r$ □

5. The Shintani cocycle

5.1. Hill's construction. We recall Section 3 of [13], slightly modifying Hill's conventions and construction. Hill's construction takes as input a choice of basis for V, so fix $\{w_1, \ldots, w_n\}$ a basis.

Write $\mathcal{K}_V^o$ for the abelian group of functions $V_{\mathbb{R}} \backslash \{0\} \to \mathbb{Z}$ generated by the characteristic function of rational open cones. We write $\mathcal{K}_V$ for the group of functions $V_{\mathbb{R}} \to \mathbb{Z}$ whose restrictions to $V_{\mathbb{R}} \backslash \{0\}$ are in $\mathcal{K}_V^o$. The group $\mathrm{GL}(V)$ acts on $\mathcal{K}_V$ by

$$(\gamma \cdot \kappa)(v) = \mathrm{sign}(\det \gamma) \kappa(\gamma^{-1} v). \tag{5.1}$$

If κ_1, κ_2 are cone functions, then we will say $\kappa_1 \leq \kappa_2$ if the support of κ_1 is contained in the support of κ_2.

The constant functions $V_{\mathbb{R}} \backslash \{0\} \to \mathbb{Z}$ form a submodule of $\mathcal{K}_V$, and we write $\mathcal{L}_V$ for the quotient $\mathcal{K}_V/\mathbb{Z}$.

For example, if $v_1, \ldots, v_n$ are linearly independent vectors of V, the rational open cone $C^{\mathrm{o}}(v_1, \ldots, v_n)$ is the set $\{\sum_{i=1}^n \alpha_i v_i : \alpha_i \in \mathbb{R}_+\}$. Then the characteristic function of this open cone, denoted $[C^{\mathrm{o}}(v_1, \ldots, v_n)]$, is an element of $\mathcal{K}_V$.

Definition 5.1. If $\alpha_1 w_1, \ldots, \alpha_n w_1$ are linearly independent, we will say $(\alpha_1, \ldots, \alpha_n)$ is non-degenerate. Degenerate will refer to the case that $\alpha_1 w_1, \ldots, \alpha_n w_1$ are linearly dependent.

Hill's cocycle is a $\mathrm{GL}(V)$-equivariant map $\sigma_{\mathrm{Hill}} : \mathrm{GL}(V)^n \to \mathcal{K}_V$ which, after quotienting out the constant functions, satisfies

$$\sum_{i=0}^{n} (-1)^n \sigma_{\mathrm{Hill}}(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n) = 0. \tag{5.2}$$

Naively, one might try to define a $\mathrm{GL}(V)$ cocycle by sending the tuple $(\alpha_1, \ldots, \alpha_n)$ to the cone function $[C^{\mathrm{o}}(\alpha_1 w_1, \ldots, \alpha_n w_1)]$. However, one must decided what to do in the degenerate cases. Even after solving this problem, the resulting cocycle will no longer satisfying the cocycle condition 5.2: the "edges" of cones are missing, so they do not glue together. In the case of $V = \mathbb{Q}^2$, Solomon [16] solves these problems by giving the edges weight 1/2. His cocycle (in Hill's language) is defined by

$$\sigma_{\mathrm{Solomon}}(\alpha, \beta) = \mathrm{sign}\ \det(\alpha w_1, \beta w_1) \left([C^{\mathrm{o}}(\alpha w_1, \beta w_1)] + \frac{1}{2}[C^{\mathrm{o}}(\alpha w_1)] + \frac{1}{2}[C^{\mathrm{o}}(\beta w_1)] \right)$$

with the convention that $\mathrm{sign}\, 0 = 0$. However, it is not clear how to extend this to higher dimensions. Hu and Solomon [14] define a cocycle on $\mathrm{PGL}_3(\mathbb{Q})$, but their methods to not extend to higher dimension. Hill's construction, which we briefly recall, elegantly side-steps these problems.

First, Hill notes that if $\{v_1, \ldots, v_n\}$ is a basis of V, then the cone function $[C^o(v_1, \ldots, v_n)]$ is given by

$$[C^o(v_1, \ldots, v_n)](w) = \begin{cases} 1 & \text{if } v_1^*(w), \ldots, v_n^*(w) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Next, Hill "deforms" $\alpha_1 w_1, \ldots, \alpha_n w_1$ to a linearly independent set of vectors. Let $\mathbb{F} = \mathbb{Q}((\varepsilon_1)) \cdots ((\varepsilon_n))$. Every element of $f \in F$ can be expressed as a sum of monomials

$$f = \sum_{\mathbf{r}=(r_1,\ldots,r_n)\in\mathbb{Z}^n} a_{\mathbf{r}} \varepsilon_1^{r_1} \cdots \varepsilon_n^{r_n}.$$

Ordering the indices $\mathbf{r} \in \mathbb{Z}^n$ lexicographically, Hill defines the leading term of (a non-zero) f to be the non-zero monomial $a_{\mathbf{r}}\varepsilon^{\mathbf{r}}$ for which $\mathbf{r}$ is smallest. For distinct $f, g \in \mathbb{F}$, Hill declares $f > g$ if the leading term of $f - g$ has positive coefficient, thus endowing $\mathbb{F}$ with the structure of an ordered field. Note that every positive power of ε_j is smaller than every positive power of ε_{j-1}, and every positive power of ε_1 is smaller than every rational number.

Now consider the vector space $V_{\mathbb{F}} := V \otimes_{\mathbb{Q}} \mathbb{F}$ over $\mathbb{F}$. For each $i \in \{1, \ldots, n\}$, define $b_i = w_1 + \varepsilon_i w_2 + \cdots + \varepsilon_i^{n-1} w_n$. This forms an $\mathbb{F}$-basis for $V_{\mathbb{F}}$ over $\mathbb{F}$, and in fact:

Lemma 5.2 ([13], Lemma 1)**.** *For any $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}(V)$ the vectors $\alpha_1 b_1, \ldots, \alpha_n b_n$ form a basis of $V \otimes_{\mathbb{Q}} \mathbb{F}$ over $\mathbb{F}$.*

Thus, for any $\alpha_1, \ldots, \alpha_n \in \mathrm{GL}(V)$, we have a natural cone function (on $V_{\mathbb{F}}$) by putting

$$[C^o(\alpha_1 b_1, \ldots, \alpha_n b_n)](w) = \begin{cases} 1 & \text{if } (\alpha_1 b_1)^*(w), \ldots, (\alpha_n b_n)^*(w) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The key result is that this cone function on $V_{\mathbb{F}}$ restricts to a cone function on $V \subset V_{\mathbb{F}}$.

Theorem 5.3 ([13], Theorem 1)**.** *The cone function $[C^o(\alpha_1 b_1, \ldots, \alpha_n b_n)] : V_{\mathbb{F}} \to \mathbb{Z}$ restricts to a rational cone function $[C^o(\alpha_1 b_1, \ldots, \alpha_n b_n] : V \to \mathbb{Z}$.*

Definition 5.4. Let

$$\sigma_{\mathrm{Hill}}(\alpha_1, \ldots, \alpha_n) = \mathrm{sign}\ \det(\alpha_1 b_1, \ldots, \alpha_n b_n)[C^o(\alpha_1 b_1, \ldots \alpha_n b_n)]|_V.$$

By Theorem 5.3, σ_{Hill} is valued the module $\mathcal{K}_V$. It is not hard to see that it is $\mathrm{GL}(V)$-equivariant, but moreover it satisfies the cocycle condition (22).

Theorem 5.5 (Hill)**.** *The map $\sigma_{\mathrm{Hill}} : \mathrm{GL}(V)^n \to \mathcal{K}_V$ is, modulo constant functions, an $n-1$ cocycle for $\mathrm{GL}(V)$. Moreover, if $(\alpha_1, \ldots, \alpha_n)$ is non-degenerate, there exists a Shintani set C with extremal rays $\alpha_1 w_1, \ldots \alpha_n w_1$ and*

$$C^o(\alpha_1 w_1, \ldots, \alpha_n w_1) \subset C \subset C(\alpha_1 w_1, \ldots \alpha_n w_1), \tag{5.3}$$

such that $\sigma_{\mathrm{Hill}}(\alpha_1, \ldots, \alpha_n) = \pm[C]$.

Proof. A calculation shows 5.3 — for details see [17], Lemma 3.5. □

5.2. The $\widetilde{\mathcal{M}}$- valued cocycle. Now fix $f' \in \mathcal{S}(V^{(p)})$. Write $\Gamma \subset \mathrm{GL}(V)$ for the stabilizer of $f' \otimes [L_p]$. We apply our methods to Hill's cocycle, constructing a pseudo-measure valued cocycle and describe when its specializations are measures.

Let us write $\Gamma \subset \mathrm{GL}(V)$ for the stabilizer of $f' \otimes [L_p]$. Note that Γ acts on L_p (on the left), and hence by duality, Γ has a right action on $\mathcal{C}(L_p)$ and a left action on $\mathcal{M}(L_p)$. This extends to a left action of Γ on $\widetilde{\mathcal{M}}(L_p) = \mathcal{M}(L_p) \otimes_{\mathbb{Z}[L_p]} \mathbb{Z}[L_p]_S$ by acting on each term, endowing $\widetilde{\mathcal{M}}(L_p)$ with the structure of a $\mathbb{Z}[\Gamma]$-module.

If $\kappa \in \mathcal{K}_V$ is a cone function, then it is a finite $\mathbb{Z}$-linear combination of characteristic functions of pointed open cones. Each open cone, with f', gives us a pseudo-measure $\mu_{C,f'}$. After taking appropriate linear combinations, we have a pseudo-measure $\mu_{\kappa,f'} \in \widetilde{\mathcal{M}}(L_p)$. In this way, we have a (well-defined!) map $\mathcal{K}_V \to \widetilde{\mathcal{M}}(L_p)$, and in fact:

Lemma 5.6. *The map $\mathcal{K}_V \to \widetilde{\mathcal{M}}(L_p)$ is a homomorphism of $\mathbb{Z}[\Gamma]$-modules.*

Proof. See Hu and Solomon [14], Lemma 2.1, or [13], Section 2. □

Lemma 5.7. *The image of the constant functions is $\mathbb{Z}\delta_0$.*

Proof. Let $\kappa : V_{\mathbb{R}}\backslash\{0\} \to \mathbb{Z}$ be a constant function, $f' \in \mathcal{S}(V^{(p)})$, and denote by μ the pseudo-measure $\mu_{\kappa,f'}$. We claim $\mu + f'(0)\delta_0 = 0$, so that $\mu = -f'(0)\delta_0$. To see this, pick w a non-zero vector for which $f' \otimes [L_p]$ is periodic. Note that the formal sum $\sum_{v\in V} f' \otimes [L_p](v)\delta_v$ is invariant under convolution by δ_w, so that $(\delta_0 - \delta_w)(\mu + f'(0)\delta) = 0$. The claim follows from the fact that $\widetilde{\mathcal{M}}(L_p)$ is an integral domain. □

It follows that the homomorphism $\mathcal{K}_V \to \widetilde{\mathcal{M}}(L_p)$ induces a homomorphism $\mathcal{L}_V \to \widetilde{\mathcal{M}}(L_p)/\delta_0$.

Definition 5.8. The p-adic Shintani cocycle attached to the data of (V, L, f') is the composition

$$\Phi_{f'} : \Gamma^n \xrightarrow{\sigma_{\mathrm{Hill}}} \mathcal{L}_V \to \widetilde{\mathcal{M}}(L_p)/\mathbb{Z}\delta_0.$$

Since σ_{Hill} is a $n-1$ cocycle valued in $\mathcal{L}_V$, $\Phi_{f'}$ is an $n-1$ cocycle for Γ valued in $\widetilde{\mathcal{M}}(L_p)/\mathbb{Z}\delta_0$.

Our main theorem states:

Theorem 5.9. *Suppose f' satisfies the vanishing hypothesis for w_1. Then, $\Phi_{f'}(\alpha_1, \ldots, \alpha_n)$ is a measure on L_p for all non-degenerate $(\alpha_1, \ldots, \alpha_n) \in \Gamma^n$.*

Before proceeding the proof, we record an elementary lemma.

Lemma 5.10. *For all non-zero $w \in V$, $\gamma \in \Gamma_f$, and $v \in V$:*

$$\pi_{v,\gamma w} f(x) = f(v + x\gamma w) = f(\gamma(\gamma^{-1}v + xw)) = \pi_{\gamma^{-1}v,w} f|\gamma = \pi_{\gamma^{-1}v,w} f.$$

Now we are ready to prove the main result.

Proof. If f' satisfies the vanishing hypothesis for w_1, then by Lemma 5.10 it satisfies the vanishing hypothesis for all $v \in \Gamma w_1$. If $(\alpha_1, \ldots, \alpha_n) \in \Gamma^n$ is non-degenerate, then equation (5.3) implies $\sigma_{\mathrm{Hill}}(\alpha_1, \ldots, \alpha_n)$ is (up to sign) the characteristic function of open cones C_i generated by $\alpha_1 w_1, \ldots, \alpha_n w_1$. The vanishing criterion implies μ_{f',C_i} is a measure, so

$$\Phi_{f'}(\alpha_1, \ldots, \alpha_n) = \pm \sum \mu_{f',C_i}$$

is a measure. □

Corollary 5.11. *Suppose* $\dim_{\mathbb{Q}}(V) = 2$*, and* f' *satisfies the vanishing hypothesis for* w_1*. Then* $\Phi_{f'}$ *is a measure-valued cocycle for* Γ

Proof. Thanks to Theorem 5.9, we only have to verify that $\Phi_{f'}(\alpha_1, \alpha_2)$ is a measure in the degenerate case. Since $\Gamma \subset \mathrm{GL}(L)$ is a finite index subgroup, and $\mathrm{GL}(L) \equiv \mathrm{SL}_2(\mathbb{Z})$ acts transitively on $L \subset V$, we can find $\gamma \in \Gamma$, such that γw_1 is not in the line spanned by $\alpha_1 w_1, \alpha_2 w_1$. The cocycle condition tells us

$$\Phi_{f'}(\alpha_1, \alpha_2) - \Phi_{f'}(\alpha_1, \gamma) + \Phi_{f'}(\alpha_2, \gamma) \equiv 0 \pmod{\mathbb{Z}\delta_0}.$$

Our choice of γ implies $\Phi_{f'}(\alpha_1, \gamma)$ and $\Phi_{f'}(\alpha_2, \gamma)$ are measures, again by Theorem 5.9. Thus $\Phi_{f'}(\alpha_1, \alpha_2)$ is a measure. □

While this proof does not generalize to higher dimension, we believe that the conclusion should hold. That is, we believe the Shintani cocycle $\Phi_{f'}$ should be *measure-valued* whenever f' satisfies the vanishing hypothesis for w_1. However, Hill's cocycle becomes unwieldy in higher dimensional degenerate cases and our methods depend on knowledge of the generators of the cones. Even though we cannot conclude all specializations are p-adic measures, all cases of arithmetic interest are non-degenerate and fit within the framework of our results. Of particular interest is the case $V = F$, a totally real field of degree n.

6. p-adic L-functions

Now fix F a totally real field, $\ell \neq p$ a rational prime which splits completely in F, $\mathfrak{c}$ a prime in F above ℓ, $\mathfrak{f}$ an integral ideal of F prime to $\mathfrak{c}$ and p, and m a non-negative integer. For all fractional ideals $\mathfrak{a}$ prime to $\mathfrak{f}p^m$, define for $s \in \mathbb{C}$

$$\zeta^*([\mathfrak{a}]_{\mathfrak{f}p^m}, s) = \sum_{\substack{0 \neq \mathfrak{b} \subset \mathcal{O}_F \\ [\mathfrak{b}]_{\mathfrak{f}p^m} = [\mathfrak{a}]_{\mathfrak{f}p^m} \\ (\mathfrak{b}, p) = 1}} \frac{1}{\mathrm{N}(\mathfrak{b})^s}$$

and

$$\zeta^*_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, s) := \zeta^*([\mathfrak{a}]_{\mathfrak{f}p^m}, s) - \mathrm{N}(\mathfrak{c})^{1-s} \zeta^*([\mathfrak{a}\mathfrak{c}^{-1}]_{\mathfrak{f}p^m}, s).$$

Note that if $m > 0$, $\zeta^*([\mathfrak{a}]_{\mathfrak{f}p^m}, s) = \zeta([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$. By Cebotarev, we may assume, without loss of generality, that $\mathfrak{a}$ is relatively prime to p and $\mathfrak{c}$.

Let $\mathcal{X}$ denote weight space, the rigid analytic variety $\mathcal{X} := \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{G}_m)$. We embed $\mathbb{Z} \hookrightarrow \mathcal{X}(\mathbb{Q}_p)$ by $k \mapsto (t \mapsto t^k)$ (note that we do not project t to $1 + p\mathbb{Z}_p$). For arbitrary elements $s \in \mathcal{X}(\mathbb{C}_p)$, $t \in \mathbb{Z}_p^\times$, we will write t^s for the image $s(t)$.

Theorem 6.1 (Deligne–Ribet, Cassou Noguès, Barsky)**.** *There exists a* p*-adic analytic function* $\zeta_{\ell,p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$*,* $s \in \mathcal{X}(\mathbb{C}_p)$*, such that*

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k) = \zeta^*_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k)$$

for all integers $k \geq 0$*.*

The theorem will follow by taking $V = F$ and considering the test function

$$f' = \bigotimes_{q \nmid p\ell} [1 + \mathfrak{a}^{-1}\mathfrak{f}\mathcal{O}_{F,q}] \bigotimes \left([\mathcal{O}_{F,\ell}] - \ell[\mathfrak{c}\mathcal{O}_{F,\ell}]\right).$$

Let us write $\mathcal{O}_{F,p}$ for the lattice $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Note that $f' \otimes [1 + p^m \mathcal{O}_{F,p}] = [1 + \mathfrak{a}^{-1}\mathfrak{f}p^m\mathcal{O}_F] - \ell[c + \mathfrak{a}^{-1}\mathfrak{f}p^m\mathfrak{c}\mathcal{O}_F]$, where $c \in \mathfrak{c}$ is prime to $\mathfrak{f}$ and is $\equiv 1 \pmod{\mathfrak{f}}$. In what follows, it will be convenient to take $c \in \mathbb{Q}$. First, we verify the vanishing hypothesis for f':

Lemma 6.2. *The test function f' satisfies the vanishing hypothesis for $w_1 = 1$.*

Proof. Fix $\alpha \in F$. The projection $\pi_{\alpha,1} f' \in \mathcal{S}(\mathbb{Q}^{(p)})$ factors as

$$\pi_{\alpha,1} f' = \bigotimes_{q \nmid p\ell)} \pi_{\alpha,1}[1 + \mathfrak{a}^{-1}\mathfrak{f}\mathcal{O}_{F,q}](\alpha + x) \bigotimes \pi_{\alpha,1}\left([\mathcal{O}_{F,\ell}] - \ell[\mathfrak{c}\mathcal{O}_{F,\ell}]\right),$$

and so it suffices to show $h_\ell(\pi_{\alpha,1}[\mathcal{O}_{F,\ell}] - \pi_{\alpha,1}\ell[\mathfrak{c}\mathcal{O}_{F,\ell}]) = 0$. Since ℓ splits completely in F, we may choose coordinates identifying $\mathcal{O}_{F,\ell}$ with $\mathbb{Z}_\ell^n$, and $\mathfrak{c}\mathcal{O}_{F,\ell}$ with $\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^{n-1}$. Then, if $\alpha \in \mathcal{O}_{F,\ell}$,

$$\pi_{\alpha,1}([\mathbb{Z}_\ell^n] - \ell[\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^{n-1}]) = [-a + \mathbb{Z}_\ell] - \ell[-a + \ell\mathbb{Z}_\ell],$$

where $\alpha \equiv a \pmod{\mathfrak{c}}$, which clearly has Haar measure 0. If $\alpha \notin \mathcal{O}_{F,\ell}$, then the projection is 0, which also has Haar measure 0. Thus f' satisfies the vanishing hypothesis for 1. □

Since $E(\mathfrak{f}\mathfrak{c}) \subset \Gamma$, pairing our cocycle with non-degenerate elements of $H_{n-1}(E(\mathfrak{f}\mathfrak{c}), \mathbb{Z})$ gives us measures, and by picking out the right units we can recover zeta values as moments of our measure. The exact element we need to pair our cocycle is provided by Lemma 2.2 of [6], but it is not a priori clear that this will give us the correct zeta values. The problem is that Hill's cocycle, a priori, does not evaluate to Shintani domains when the degree of the field is greater than 2. However, Spiess has shown that Hill's construction does indeed recover Shintani domains.

Proposition 6.3 (Spiess)**.** *Let $\eta \in \mathbb{Z}[E(\mathfrak{f}\mathfrak{c})^n]$ be a generator of $H_{n-1}(E(\mathfrak{f}\mathfrak{c}), \mathbb{Z}) \cong \mathbb{Z}$. Then the cone function $\sigma_{\mathrm{Hill}}(\eta)$ is $\pm$ the characteristic function of a Shintani domain.*

Proof. This is Proposition 3.7 of [17]. □

Proposition 6.4. *$\Phi_{f'} \cap \eta$ is a measure.*

Proof. Let $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be fundamental units of $E(\mathfrak{f}\mathfrak{c})$. From Remark 2.1(c) of [17], $\eta = \pm \sum_{\tau \in S_{n-1}} \operatorname{sign}(\tau)[\varepsilon_{\tau(1)}|\cdots|\varepsilon_{\tau(n-1)}]$, where $[\varepsilon_{\tau(1)}|\cdots|\varepsilon_{\tau(n-1)}]$ represents the cycle

$$(1, \varepsilon_{\tau(1)}, \varepsilon_{\tau(1)}\varepsilon_{\tau(2)}, \ldots, \varepsilon_{\tau(1)}\cdots\varepsilon_{\tau(n-1)}) \in \mathbb{Z}[\Gamma^n].$$

By Lemma 2.1 of [6], this is non-degenerate. Using our Lemma 6.2 and Theorem 5.9, we deduce that $\Phi_{f'} \cap \eta$ is a measure. □

Now we are ready to prove Theorem 6.1.

Proof. Fix $k \geq 0$ an integer, and let $\kappa = \sigma_{\mathrm{Hill}}(\eta)$. By Proposition 6.3, $\sigma_{\mathrm{Hill}}(\eta)$ is $\pm$ the characteristic function of a Shintani domain for $E(\mathfrak{f})$. In particular, κ is supported on the positive orthant $\mathbb{R}_+^n$. Let μ be the measure $\mu = \pm\Phi_{f'}(\eta)$, where the sign is the sign of κ. By Proposition 4.14, the moments of μ are given by

$$\int_{1+p^m\mathcal{O}_{F,p}} \mathrm{N}(\alpha)^k d\mu(\alpha) = \zeta_{\mathrm{SH}}(f' \otimes [1 + p^m\mathcal{O}_{F,p}], \kappa; -k)$$

and

$$\int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} \mathrm{N}(\alpha)^k d\mu(\alpha) = \zeta_{\mathrm{SH}}(f' \otimes [\mathrm{N}^{-1}(\mathbb{Z}_p^\times)], \kappa; -k).$$

By equation (3.2),

$$\int_{1+p^m\mathcal{O}_{F,p}} \mathrm{N}(\alpha)^k d\mu(\alpha) = \mathrm{N}(\mathfrak{a})^k \zeta_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k) \tag{6.1}$$

and

$$\int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} \mathrm{N}(\alpha)^k d\mu(\alpha) = \mathrm{N}(\mathfrak{a})^k \zeta_{\mathfrak{c}}^*([\mathfrak{a}]_{\mathfrak{f}}, -k) \tag{6.2}$$

If $m > 0$, we define $\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$ to be the analytic function

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s) := \mathrm{N}(\mathfrak{a})^s \int_{1+p^m\mathcal{O}_{F,p}} N(\alpha)^{-s} d\mu(\alpha),$$

where $\mathrm{N}(\alpha)^{-s} := s(\mathrm{N}(\alpha)^{-1})$. If $m = 0$,

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}}, s) := \mathrm{N}(\mathfrak{a})^s \int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} N(\alpha)^{-s} d\mu(\alpha).$$

By equations (6.1) and (6.2), $\zeta_{\mathfrak{c},p}$ has the correct interpolation property. □

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