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ALMOST CRITICAL WELL-POSEDNESS FOR NONLINEAR WAVE EQUATIONS WITH $Q_{\mu\nu}$ NULL FORMS IN 2D

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Abstract. In this paper, we prove an optimal local well-posedness result for the 1+2 dimensional system of nonlinear wave equations (NLW) with quadratic null-form derivative nonlinearities $Q_{\mu\nu}$. The Cauchy problem for these equations is known to be ill-posed for data in the Sobolev space H^s with $s \leq 5/4$ for all the basic null forms, except Q_0, thus leaving a gap to the critical regularity of $s_c = 1$. Following Grünrock's result for the quadratic derivative NLW in three dimensions, we consider initial data in the Fourier–Lebesgue spaces $\widehat{H}^r_s$, which coincide with the Sobolev spaces of the same regularity for $r = 2$, but scale like lower regularity Sobolev spaces for $1 < r < 2$. Here we obtain local well-posedness for the range $s > \frac{3}{2r} + \frac{1}{2}$, $1 < r \leq 2$, which at one extreme coincides with $H^{\frac{5}{4}+}$ optimal Sobolev space result, while at the other extreme establishes local well-posedness for the model null-form problem for the almost critical Fourier–Lebesgue space $\widehat{H}^{1+}_2$. Using appropriate multiplicative properties of the solution spaces, and relying on bilinear estimates for the $Q_{\mu\nu}$ forms, we prove almost critical local well-posedness for the Ward wave map problem as well.

1. Introduction

Consider systems of nonlinear wave equations (NLW)

$$\Box u^I = \sum_{J,K} Q(u^J, u^K), \tag{1.1}$$

where $(u^I) : R^{1+2} \to \mathbb{R}^m$, and Q is a bilinear form inhibiting a null structure. That is, Q can be written as a linear combination of the three basic null forms of Klainerman [16] (see also [4, 17])

$$\begin{aligned} Q_0(f, g) &= \partial_t f \partial_t g - \nabla f \cdot \nabla g, \\ Q_{ij}(f, g) &= \partial_i f \partial_j g - \partial_j f \partial_i g, \\ Q_{0j}(f, g) &= \partial_t f \partial_j g - \partial_j f \partial_t g. \end{aligned} \tag{1.2}$$

Here ∂_j stands for spatial derivatives and ∇ is the spatial gradient.

We are interested in the local well-posedness (LWP) question for the system (1.1) for initial data in Fourier–Lebesgue spaces $\widehat{H}^r_s$. More precisely, we consider the Cauchy problem for (1.1) with initial conditions

$$(u^I, \partial_t u^I)|_{t=0} = (f^I, g^I) \in \widehat{H}^r_s \times \widehat{H}^r_{s-1} \tag{1.3}$$

and wish to establish local well-posedness for a range of the exponents (r, s), which achieves almost critical well-posedness stemming from scaling considerations. The

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Fourier–Lebesgue spaces $\widehat{H}^r_s$ have been previously successfully used to achieve improved regularity results for a variety of equations (see e.g. [2, 3, 11–15, 26]).

Although systems (1.1) were studied as standalone problems before, nonlinear terms with null structure also naturally arise in physical and geometric problems. One such example, which we will study in this paper in detail, is the Ward wave map system. It was introduced by Ward in [27] as a two-dimensional (2D) completely integrable system, with its linear part invariant under Lorentz transformations. The Ward system can be realized as a dimensional reduction of the anti-self-dual Yang Mills equation with split signature in $\mathbb{R}^{2+2}$, and can also be obtained from the space–time monopole equation via gauge fixing (see e.g. [7] for more details). The Ward wave map equation has the form

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0, \tag{1.4}$$

where

$$J : \mathbb{R}^{1+2} \to U(n)$$

is a $U(n)$ (or $SU(n)$) valued function, and hence $J^{-1} = J^*$, while $(x, y) \in \mathbb{R}^2$, and $[\cdot,\cdot]$ is the Lie bracket on $U(n)$. Using the product rule, (1.4) can be written as

$$J^{-1}J_{tt} - J^{-1}\Delta J + (J^{-1})_t J_t - (\nabla J^{-1})\nabla J - J^{-1}J_t J^{-1}J_y + J^{-1}J_y J^{-1}J_t = 0.$$

Multiplying the last equation by J on the left and using $(\partial J)J^{-1} = -J\partial J^{-1}$, the Ward wave map equation will become

$$\Box J + JQ_0(J^{-1}, J) + JQ_{02}(J^{-1}, J) = 0. \tag{1.5}$$

Notice that the nonlinearity in the last equation is cubic in the unknown J, and has a null structure in the terms appearing with derivatives.

System (1.1) is a particular example of the more general quadratic derivative NLW

$$\Box u = \partial u \partial u, \tag{1.6}$$

where ∂u is the space–time gradient of u. Equation (1.6) is invariant under the scaling

$$(t, x) \mapsto (\lambda t, \lambda x). \tag{1.7}$$

That is, if u is a solution to (1.6) in $\mathbb{R}^{1+n}$, then so is $u_\lambda(t, x) = u(\lambda t, \lambda x)$. Under this scaling, the homogeneous Sobolev norm of the initial data scales as

$$\|u_\lambda(0,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{s-\frac{n}{2}}\|u(0,\cdot)\|_{\dot{H}^s(\mathbb{R}^n)}$$

and $s_c = \frac{n}{2}$ is called the critical exponent, as the $\dot{H}^{s_c}$ norm of the initial data is preserved under the scaling. Under general scaling considerations, one expects local well-posedness for data in the Sobolev space H^s for $s > s_c$ (subcritical regime), global existence for small data in $\dot{H}^{s_c}$ (critical regime), and some form of ill-posedness for data in H^s for $s < s_c$ (supercritical regime).

As the critical exponent in $\mathbb{R}^{1+n}$ is $s_c = \frac{n}{2}$, it is expected that the local well-posedness must hold for data in the Sobolev space H^s, $s > \frac{n}{2}$. This, however, is known to be false in dimensions $n = 2, 3$.

In dimension $n = 3$, the almost critical local well-posedness for null-form quadratic derivative NLW was proved by Klainerman and Machedon [18], while for the general quadratic derivative NLW (1.6), the local well-posedness for $s > 2$ was proved by Ponce and Sideris [24], which is sharp in light of the counterexamples of Lindblad [21–23].

In dimension $n = 2$, the almost critical LWP for the Q_0 null-form NLW was showed by Klainerman and Selberg [19] in the context of wave maps, but it is known to be false for the other null forms, for which the best result is for data in H^s with $s > \frac{5}{4}$ by Zhou [28], who also showed that it is sharp. We also observe that the sharp bilinear $X^{s,b}$ estimates of Foschi–Klainerman [9] for the solutions of the free wave equation associated with the Cauchy problem with data in H^s are also $\frac{1}{4}$ derivative above the scaling regularity. We will see that the $\widehat{H}^r_s$ space approach with $1 < r < 2$ circumvents the counterexamples of Zhou and Foschi–Klainerman.

The best result for the general quadratic derivative NLW in dimension $n = 2$ is for $s > \frac{7}{4}$, which can be shown by the Strichartz estimates approach.

The null structure in the Ward equation (1.5) was exploited by Czubak in [5] to prove local well-posedness for the Ward wave map problem for data in $H^s(\mathbb{R}^2)$ for $s > \frac{5}{4}$. In light of Zhou's results, this result is also a $\frac{1}{4}$ derivative above the scaling prediction, since the Ward equation scales as (1.6).

Recently, Grünrock showed in [13] that the gap to the almost criticality for the general quadratic NLW (1.6) can be closed in dimension $n = 3$ by considering initial data in the Fourier–Lebesgue spaces $\widehat{H}^r_s$. These spaces are defined by their norms as

$$\|f\|_{\widehat{H}^r_s} = \|\langle\xi\rangle^s \widehat{f}\|_{L^{r'}_\xi}, \quad \frac{1}{r} + \frac{1}{r'} = 1,$$

where $\widehat{f}$ stands for the Fourier transform of f, and[1] $\langle\xi\rangle = 1 + |\xi|$. The norm of the corresponding homogeneous space is $\|f\|_{\dot{\widehat{H}}^r_s} = \||\xi|^s \widehat{f}\|_{L^{r'}_\xi}$. Under the scaling (1.7), the norm of the initial data in the homogeneous Fourier–Lebesgue spaces scales as

$$\|u_\lambda(0,\cdot)\|_{\dot{\widehat{H}}^r_s(\mathbb{R}^n)} = \lambda^{s-\frac{n}{r}} \|u(0,\cdot)\|_{\dot{\widehat{H}}^r_s(\mathbb{R}^n)}$$

so the critical exponent for these spaces is $s^r_{\mathrm{c}} = \frac{n}{r}$. Comparing how the homogeneous Sobolev and Fourier–Lebesgue norms scale, we observe the following correspondence in terms of scaling:

$$\dot{\widehat{H}}^r_s \sim \dot{H}^\sigma, \quad \text{if} \quad \sigma = s + n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Grünrock established local well-posedness for data in the space $\widehat{H}^r_s$ for $s > \frac{2}{r} + 1$, $1 < r \le 2$. Thus, his range of exponents almost reaches criticality at the endpoint $r = 1$, since for this r, $s > \frac{2}{r} + 1 = \frac{3}{r}$, which gives the critical exponent $(s, r) = (3, 1)$ in dimension $n = 3$. His approach relies on free wave interaction estimates of Foschi–Klainerman [9] that come with a factor of $||\tau| - |\xi||^{\frac{n-3}{2}}$, which becomes unbounded near the null cone $|\tau| = |\xi|$ in 2D. Thus, Grünrock's result cannot be directly generalized to a LWP result for the general quadratic NLW (1.6) in dimension $n = 2$. However, if the nonlinearity has enough cancelation along the null cone to offset this factor, then the arguments can be salvaged, leading to a LWP for these special nonlinearities for

[1] Alternatively, one can use the Japanese bracket $\langle\xi\rangle = \sqrt{1 + |\xi|^2} \simeq 1 + |\xi|$.

a range of exponents (s, r) that reach almost criticality. We follow this approach to prove the LWP for the Cauchy problem (1.1)–(1.3) for all the null forms (1.2).

The main results of this paper are the following two theorems.

Theorem 1.1 (LWP for null-form NLW)**.** *Let $1 < r \leq 2$, $s > \frac{3}{2r} + \frac{1}{2}$, then the Cauchy problem* (1.1)–(1.3) *is locally well-posed for data in the space $\widehat{H}^r_s \times \widehat{H}^r_{s-1}$.*

We also consider the Cauchy problem for the Ward equation (1.5) with data in the Fourier Lebesgue spaces and will prove the following result.

Theorem 1.2 (LWP for Ward)**.** *Let $1 < r \leq 2$, $s > \frac{3}{2r} + \frac{1}{2}$, then the Cauchy problem for the Ward equation* (1.5) *is locally well-posed for data in the space*

$$(J, \partial_t J)\big|_{t=0} \in \widehat{H}^r_s \times \widehat{H}^r_{s-1}.$$

Remark 1.3. Notice that at one extreme, $(s, r) = \left(\frac{5}{4}, 2\right)$, this result coincides with the local well-posedness for data in $H^{\frac{5}{4}+}$, which is optimal on the Sobolev scale, while at the other extreme, $(s, r) = (2, 1)$, we obtain the almost critical local well-posedness in the space $\widehat{H}^{1+}_2$.

The region for the $\left(s, \frac{1}{r}\right)$ exponents, for which the local well-posedness holds is shaded in Figure 1. Notice that the bottom and right edges of the region are not included. The dotted line segment in Figure 1 connects the sharp Sobolev result $H^{\frac{5}{4}+}$ with the almost critical $\widehat{H}^{1+}_2$. Our estimates will establish the local well-posedness result for exponents in the region above the dashed line $s > \frac{1}{r} + 1$, while the result for exponents in the triangular region below the dashed line will follow from bilinear interpolation between our estimates and those for the $H^{\frac{5}{4}+}$ result on the Sobolev

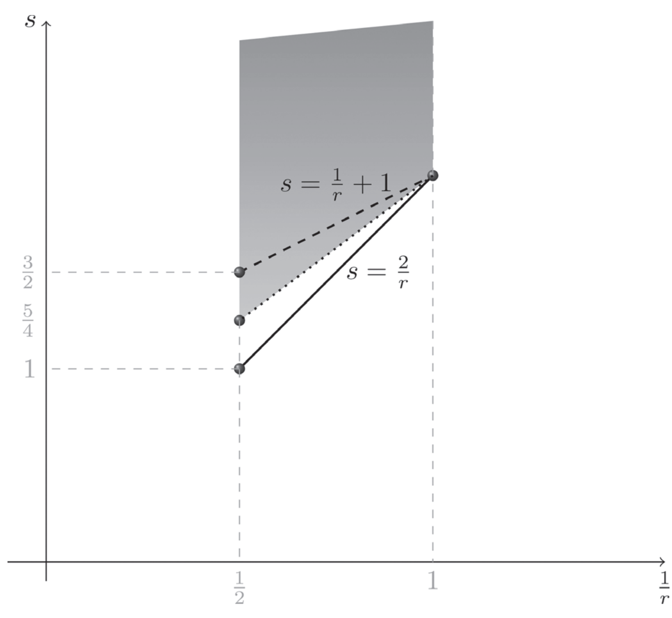

FIGURE 1. The shaded region represents the range of indices for which LWP for data in space $\widehat{H}^r_s \times \widehat{H}^r_{s-1}$ holds

scale. We also note that our approach in Section 3 can be used to establish the needed bilinear estimates for solutions of the free wave equation for the entire shaded region, however, below the dashed line, the estimates require placing the nonlinearity in the $X^r_{\sigma,b'}$ spaces for $b' < 0$. These estimates cannot be transfered to estimates for general $X^r_{s,b}$ functions with the transfer principle, Proposition A.2.

As we mentioned above, the approach via the transfer principle and free wave estimates is not well-suited for the general equation (1.6). For this equation, an upcoming joint result of the first author with Tanguay [10] improves the well-posedness range from the best-known Sobolev result. Their approach uses generalizations of bilinear estimates in the inhomogeneous norms of D'Ancona *et al.* [8] to the Fourier-$L^{r'}$ based spaces.

The rest of the paper is organized as follows. In Section 2, the solution spaces are introduced, and Theorem 1.1 is reduced to a bilinear estimate for the range of indices corresponding to the region above the dashed line segment in Figure 1. This is done via the general LWP theorem, which is stated in Appendix B, and a bilinear interpolation with the L^2 based estimate corresponding to the left boundary of the shaded region in Figure 1. In Section 3, we prove the main bilinear estimate by first reducing it to a corresponding bilinear estimate for solutions of the free wave equation via the transfer principle, the proof of which appears in Appendix A. In Section 4, using appropriate multiplicative estimates of the solutions spaces, we show that the bilinear estimates of Section 3 imply trilinear estimates for $wQ(u,v)$, leading to the almost critical LWP of the Ward wave map problem.

2. The $X^r_{s,b}$ space, reduction to a bilinear estimate

The local in time solution is obtained via a contraction principle in a suitable solution space. For this, we will use time restriction spaces based on the $X^r_{s,b}$ space, which is a Fourier-$L^{r'}$ analog of the wave-Sobolev space $X^{s,b}$ and is defined by the norm

$$\|u\|_{X^r_{s,b}} = \|\langle\xi\rangle^s\langle|\tau|-|\xi|\rangle^b\widetilde{u}\|_{L^{r'}_{\tau,\xi}},$$

where $\widetilde{u}$ denotes the time–space Fourier transform of u. The time restriction space is then

$$X^r_{s,b;T} = \left\{u = U|_{[-T,T]\times\mathbb{R}^n} : U \in X^r_{s,b}\right\}$$

with its norm defined as

$$\|u\|_{X^r_{s,b;T}} = \inf\left\{\|U\|_{X^r_{s,b}} : U|_{[-T,T]\times\mathbb{R}^n} = u\right\}.$$

For us, $b > \frac{1}{r}$, which guarantees the embeddings (see Proposition A.2)

$$X^r_{s,b} \subset C(\mathbb{R},\widehat{H}^r_s) \quad \text{and} \quad X^r_{s,b;T} \subset C([-T,T],\widehat{H}^r_s).$$

As the wave operator is of second order in time, we also need to separately estimate the time derivative of the solution. Hence, we define our solution space, $Z^r_{s,b}$, by the norm

$$\|u\|_{Z^r_{s,b}} = \|u\|_{X^r_{s,b}} + \|\partial_t u\|_{X^r_{s-1,b}}.$$

The time restriction space $Z^r_{s,b;T}$ and its norm are defined as before.

In the sequel, we also make use of the notation $\widehat{L}^r_{t,x} = X^r_{0,0}$, and $\widehat{L}^r = \widehat{H}^r_0$, where the last norm can be taken either with respect to the time or space variables.

By the general well-posedness, Theorem B.2, Theorem 1.1 will follow from the following two estimates:

$$\|Q(u,v)\|_{X^r_{s-1,b+\epsilon-1}} \lesssim \|u\|_{Z^r_{s,b}} \|v\|_{Z^r_{s,b}} \tag{2.1}$$

and

$$\begin{aligned}\|Q(u,v) - Q(U,V)\|_{X^r_{s-1,b+\epsilon-1}} \lesssim &\left(\|u\|_{Z^r_{s,b}} + \|v\|_{Z^r_{s,b}} + \|U\|_{Z^r_{s,b}} + \|V\|_{Z^r_{s,b}}\right) \\ &\times \left(\|u-U\|_{Z^r_{s,b}} + \|v-V\|_{Z^r_{s,b}}\right).\end{aligned}$$

The second estimate will trivially follow from the first one, as Q is bilinear and, hence

$$Q(u,v) - Q(U,V) = Q(u-U,v) + Q(U,v-V).$$

Thus, the proof of Theorem 1.1 reduces to proving estimate (2.1) for all the null forms (1.2).

The proof of estimate (2.1) for the range $s > \frac{3}{2r} + \frac{1}{2}$, $1 < r \le 2$ and for some $b \in \left(\frac{1}{r}, 1\right)$ and $\epsilon \in (0, 1-b)$ will be done in two steps. The first step involves proving the estimate for a reduced range of the exponents, which corresponds to the region above the dashed line segment in Figure 1.

Proposition 2.1. *Let $s_0 > \frac{1}{r_0} + 1$, $1 < r_0 \le 2$, then the estimate*

$$\|Q(u,v)\|_{X^{r_0}_{s_0-1,b_0+\epsilon_0-1}} \lesssim \|u\|_{X^{r_0}_{s_0,b_0}} \|v\|_{X^{r_0}_{s_0,b_0}} \tag{2.2}$$

holds for some $b_0 \in \left(\frac{1}{r_0}, 1\right)$ and $\epsilon_0 \in (0, 1-b_0)$.

The proof of this proposition appears in the next section. Notice that we can trivially replace the norms on the right-hand side by the $Z^r_{s,b}$ norms. The second step toward the proof of estimate (2.1) for the full range of exponents (s,r) involves interpolating between estimate (2.2) and the analogous estimate for $r = 2$, used in the proof of the $H^{\frac{5}{4}+}$ sharp Sobolev result. For the proof of this result, Zhou [28] used solution spaces $N_{s,b}$ defined by the norms

$$\|u\|_{N_{s,b}} = \|\langle|\tau| + |\xi|\rangle^s \langle|\tau| - |\xi|\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}}$$

and established the bilinear estimate (see [28, Theorem 7])

$$\|Q_{\mu\nu}(u,v)\|_{N_{s-1,b-1}} \lesssim \|u\|_{N_{s,b}} \|v\|_{N_{s,b}}$$

for $\frac{5}{4} < s < \frac{3}{2}$ and $b = s - \frac{1}{2}$. However, an inspection of Zhou's proof of the last estimate shows that it can be easily modified to be placed in the context of $X_{s,b} = X^2_{s,b}$ spaces. This was earlier observed by Czubak in [5], who established the following (see also [6, Section 5.2.1]).

Proposition 2.2 (cf. Section 2 in [5])**.** *Let $s > \frac{5}{4}$, then*

$$\|Q_{\mu\nu}(u,v)\|_{X^2_{s_1-1,b_1+\epsilon_1-1}} \lesssim \|u\|_{Z^2_{s_1,b_1}} \|v\|_{Z^2_{s_1,b_1}} \tag{2.3}$$

for some $b_1 \in \left(\frac{1}{2}, 1\right)$ and $\epsilon_1 \in (0, 1 - b_1)$.

The analogous estimate to (2.3) for Q_0 follows from the stronger estimate of Selberg [25, Section 4.1.3] (see also [20, Section 7.2]). Notice that the range of s_1 corresponds to the left boundary of the shaded region in Figure 1.

Complex bilinear interpolation (see e.g. [1, Section 4.4]) between (2.2) and (2.3) will establish estimate (2.1) for

$$\begin{aligned} \frac{1}{r} &= \frac{1-\theta}{r_0} + \frac{\theta}{2}, \\ s &= (1-\theta)s_0 + \theta s_1, \\ b &= (1-\theta)b_0 + \theta b_1, \end{aligned}$$

where $0 \le \theta \le 1$. This will clearly give the entire range for the exponents $\left(s, \frac{1}{r}\right)$ shaded in Figure 1. Thus, to finish the proof of Theorem 1.1, it remains to prove Proposition 2.1.

3. Proof of Proposition 2.1

In the sequel, we neglect the subindices of the exponents s, r, b and ϵ of Proposition 2.1. Since $b + \epsilon - 1 < 0$, estimate (2.2) will follow from

$$\|Q(u,v)\|_{X^r_{s-1,0}} \lesssim \|u\|_{X^r_{s,b}} \|v\|_{X^r_{s,b}}. \tag{3.1}$$

We denote the symbol of Q by m, that is,

$$\widetilde{Q(u,v)}(\tau,\xi) = \int\limits_{\alpha+\beta=\tau} \int\limits_{\eta+\zeta=\xi} m(\alpha,\eta;\beta,\zeta)\widetilde{u}(\alpha,\eta)\widetilde{v}(\beta,\zeta)\, d\alpha\, d\beta\, d\eta\, d\zeta.$$

Denoting the corresponding bilinear operator with the space-normalized symbol

$$\frac{m(\alpha,\eta;\beta,\zeta)}{|\eta||\zeta|}$$

by q, we will have

$$|\mathcal{F}_{t,x} Q(u,v)| \lesssim |\mathcal{F}_{t,x} q(\partial_x u, \partial_x v)|,$$

where $\mathcal{F}_{t,x}$ is the space–time Fourier transform operator. Then estimate (3.1) will follow from

$$\|\Lambda^\sigma q(u,v)\|_{\widehat{L}^r_{t,x}} \lesssim \|u\|_{X^r_{\sigma,b}} \|v\|_{X^r_{\sigma,b}}, \tag{3.2}$$

where $\sigma = s-1$, and the operator Λ^σ has the multiplier $\langle\xi\rangle^\sigma$, that is,

$$\widehat{\Lambda^\sigma\phi}(\xi) = \langle\xi\rangle^\sigma\widehat{\phi}(\xi).$$

By the transfer principle[2], Proposition A.2, estimate (3.2) will follow from the following bilinear estimate for free waves:

$$\|\Lambda^\sigma q(u_\pm, v_{[\pm]})\|_{\widehat{L}^r_{t,x}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma}\|v_0\|_{\widehat{H}^r_\sigma}, \tag{3.3}$$

where $[\pm]$ denotes an independent choice of signs, and

$$u_\pm(t) = e^{\pm itD}u_0, \quad v_\pm(t) = e^{\pm itD}v_0.$$

The rest of this section is dedicated to proving (3.3) in which we follow the method of Grünrock [13], while relying on calculations of Foschi–Klainerman [9].

We first observe that by symmetry we only need to consider the $(++)$ and $(+-)$ cases in (3.3). Defining $P_\pm(\eta) = |\xi-\eta| \pm |\eta|$ with $\nabla P_\pm(\eta) = \frac{\eta-\xi}{|\eta-\xi|} \pm \frac{\eta}{|\eta|}$, and using the properties of the δ-distribution, we have (see [9, Sections 3, 4])

$$\mathcal{F}u_+v_\pm(\xi,\tau) \simeq \int_{P_\pm(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|}\widehat{u_0}(\eta)\widehat{v_0}(\xi-\eta).$$

The set $\{P_+(\eta)=\tau\}$ is an ellipsoid of rotation, while the set $\{P_-(\eta)=\tau\}$ is a hyperboloid of rotation, so we refer to the $(++)$ and $(+-)$ cases as elliptic and hyperbolic, respectively.

We will prove estimate (3.3) for the form q_{12} corresponding to the form Q_{12}. The proofs for q_0 and q_{0j} are similar, and will be outlined in remarks. Before proceeding, we observe the following bounds for the symbol of the bilinear operator q_{12} (see [9, Lemma 13.2]):

$$\frac{\eta\wedge\zeta}{|\eta||\zeta|} = \frac{\eta\wedge(\xi-\eta)}{|\eta||\xi-\eta|} \lesssim \frac{|\xi|^{\frac12}(|\eta|+|\xi-\eta|-\xi|)^{\frac12}}{|\eta|^{\frac12}|\xi-\eta|^{\frac12}}, \tag{3.4}$$

$$\frac{\eta\wedge\zeta}{|\eta||\zeta|} = \frac{\eta\wedge(\xi-\eta)}{|\eta||\xi-\eta|} \lesssim \frac{|\xi|^{\frac12}(|\xi|-||\eta|-|\xi-\eta||)^{\frac12}}{|\eta|^{\frac12}|\xi-\eta|^{\frac12}}. \tag{3.5}$$

The first estimate above will be useful in the elliptic case, while the second estimate will be useful in the hyperbolic case.

3.1. The elliptic case. We choose $s_{1,2}$, such that $s_1 = 0, s_2 = \frac1r$ or $s_2 = 0, s_1 = \frac1r$, and use Hölder's inequality to get

$$|\mathcal{F}q_{12}(u_+,v_+)| \lesssim \left(\int_{P_+(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|}\left|\frac{\eta\wedge(\xi-\eta)}{|\eta||\xi-\eta|}\right|^r |\eta|^{-s_1r}|\xi-\eta|^{-s_2r}\right)^{\frac1r}$$
$$\times\left(\int_{P_+(\eta)=\tau}\frac{dS_\eta}{|\nabla P_+(\eta)|}\left|\widehat{\Lambda^{s_1}u_0}(\eta)\widehat{\Lambda^{s_2}v_0}(\xi-\eta)\right|^{r'}\right)^{\frac{1}{r'}}.$$

[2]Here Λ^σ in front of the bilinear form will be taken care of via a simple Leibniz-type estimate on the Fourier side, so its presence does not effect applicability of the transfer principle.

Using the bound (3.4), we have for the first factor above

$$\begin{aligned} I :&= \int_{P_+(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|} \left| \frac{\eta \wedge (\xi - \eta)}{|\eta||\xi - \eta|} \right|^r |\xi - \eta|^{-s_1 r} |\eta|^{-s_2 r} \\ &\lesssim \int_{P_+(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|} |\xi|^{\frac{r}{2}} |\eta|^{-\left(s_1+\frac{1}{2}\right)r} |\xi - \eta|^{-\left(s_2+\frac{1}{2}\right)r} (|\eta| + |\xi - \eta| - |\xi|)^{\frac{r}{2}} \\ &= |\tau - |\xi||^{\frac{r}{2}} |\xi|^{\frac{r}{2}} \int \frac{\delta(\tau - |\eta| - |\xi - \eta|)}{|\eta|^{\left(s_1+\frac{1}{2}\right)r} |\xi - \eta|^{\left(s_2+\frac{1}{2}\right)r}} \, d\eta. \end{aligned}$$

Under our assumptions on $s_{1,2}$,

$$\max\left\{ \left(s_1 + \tfrac{1}{2}\right) r, \left(s_2 + \tfrac{1}{2}\right) r \right\} = 1 + \tfrac{r}{2} > \tfrac{3}{2}.$$

Now using [9, Proposition 4.3], we get

$$I \lesssim |\tau - |\xi||^{\frac{r}{2}} |\xi|^{\frac{r}{2}} \tau^A (\tau - |\xi|)^B,$$

where

$$\begin{aligned} A &= \max\left\{ \left(s_1 + \tfrac{1}{2}\right) r, \left(s_2 + \tfrac{1}{2}\right) r, \tfrac{3}{2} \right\} - \left(s_1 + \tfrac{1}{2}\right) r - \left(s_2 + \tfrac{1}{2}\right) r = -\tfrac{r}{2}, \\ B &= 1 - \max\left\{ \left(s_1 + \tfrac{1}{2}\right) r, \left(s_2 + \tfrac{1}{2}\right) r, \tfrac{3}{2} \right\} = -\tfrac{r}{2}. \end{aligned}$$

Thus,

$$I \lesssim \frac{|\tau - |\xi||^{\frac{r}{2}} |\xi|^{\frac{r}{2}}}{|\tau - |\xi||^{\frac{r}{2}} \tau^{\frac{r}{2}}} \lesssim 1$$

since $\tau = |\xi - \eta| + |\eta| \geq |\xi|$.

But then for $\sigma > \frac{1}{r}$

$$\|q_{12}(u_+, v_+)\|_{\widehat{L}^r_{x,t}} \lesssim \|\Lambda^\sigma u_0\|_{\widehat{L}^r_x} \|v_0\|_{\widehat{L}^r_x}.$$

We can trivially exchange u_0 and v_0 in the above estimate. On the other hand, the convolution constraint $\xi = \eta + (\xi - \eta)$ implies $\langle \xi \rangle^\sigma \lesssim \langle \eta \rangle^\sigma + \langle \xi - \eta \rangle^\sigma$, which gives the following.

Lemma 3.1. *Let $1 < r \leq 2$ and $\sigma > \frac{1}{r}$, then*

$$\|q_{12}(u_+, v_+)\|_{X^r_{\sigma,0}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma} \|v_0\|_{\widehat{H}^r_\sigma}.$$

3.2. The hyperbolic case. We again choose $s_{1,2} \in \{0, \frac{1}{r}\}$ with $s_1 + s_2 = \frac{1}{r}$.

We further split the hyperbolic case into the *low-frequency* and *high-frequency* cases, respectively

$$|\eta| + |\xi - \eta| \leq 2|\xi|, \tag{3.6}$$

$$|\eta| + |\xi - \eta| > 2|\xi|. \tag{3.7}$$

The low-frequency case (3.6) is similar to the elliptic case. Indeed, using (3.5) and Hölder's estimate as before, we will have

$$|\mathcal{F}q_{12}(u_+,v_+)| \lesssim I^{\frac{1}{r}} \times \left(\int_{P_-(\eta)=\tau} \frac{dS_\eta}{|\nabla P_-(\eta)|} \left| \widehat{\Lambda^{s_1}u_0}(\eta)\widehat{\Lambda^{s_2}v_0}(\xi-\eta) \right|^{r'} \right)^{\frac{1}{r'}},$$

where

$$\begin{aligned} I &= \int_{P_-(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|} \left| \frac{\eta\wedge(\xi-\eta)}{|\eta||\xi-\eta|} \right|^r |\eta|^{-s_1 r}|\xi-\eta|^{-s_2 r} \\ &\lesssim ||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}} \int_{|\eta|+|\xi-\eta|\le 2|\xi|} \frac{\delta(\tau-|\eta|+|\xi-\eta|)}{|\eta|^{\left(s_1+\frac{1}{2}\right)r}|\xi-\eta|^{\left(s_2+\frac{1}{2}\right)r}}\, d\eta. \end{aligned}$$

Now using [9, Proposition 4.5], we obtain for the case $s_1=0, s_2=\frac{1}{r}$ (the other case is similar)

$$I \lesssim ||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}}|\xi|^A(||\xi|-|\tau||)^B,$$

where in the region $0\le\tau\le|\xi|$

$$\begin{aligned} A &= \max\left\{\left(s_2+\tfrac{1}{2}\right)r, \tfrac{3}{2}\right\} - \left(s_1+\tfrac{1}{2}\right)r - \left(s_2+\tfrac{1}{2}\right)r = -\tfrac{r}{2}, \\ B &= 1-\max\left\{\left(s_2+\tfrac{1}{2}\right)r, \tfrac{3}{2}\right\} = -\tfrac{r}{2} \end{aligned}$$

and in the region $-|\xi|\le\tau\le 0$

$$\begin{aligned} A &= \max\left\{\left(s_1+\tfrac{1}{2}\right)r, \tfrac{3}{2}\right\} - \left(s_1+\tfrac{1}{2}\right)r - \left(s_2+\tfrac{1}{2}\right)r = \tfrac{1}{2}-r, \\ B &= 1-\max\left\{\left(s_1+\tfrac{1}{2}\right)r, \tfrac{3}{2}\right\} = -\tfrac{1}{2}. \end{aligned}$$

But then in the region $0\le\tau\le|\xi|$

$$I \lesssim \frac{||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}}}{||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}}} \lesssim 1,$$

while in the region $-|\xi|\le\tau\le 0$

$$I \lesssim \frac{||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}}}{||\xi|-|\tau||^{\frac{1}{2}}|\xi|^{r-\frac{1}{2}}} \lesssim 1.$$

In the high-frequency case (3.7), we will instead use [9, Lemma 4.4][3], which gives

$$\begin{aligned} I &= \frac{||\xi|-|\tau||^{\frac{r}{2}}|\xi|^{\frac{r}{2}}}{(\tau^2-|\xi|^2)^{\frac{1}{2}}} \int_2^\infty |x|\xi|+\tau|^{-\left(s_1+\frac{1}{2}\right)r+1}|x|\xi|-\tau|^{-\left(s_2+\frac{1}{2}\right)r+1}(x^2-1)^{-\frac{1}{2}}\,dx \\ &\lesssim \frac{||\xi|-|\tau||^{\frac{r}{2}-\frac{1}{2}}|\xi|^{\frac{r}{2}}}{(|\xi|+|\tau|)^{\frac{1}{2}}}|\xi|^{1-r}\int_2^\infty |x+\tfrac{\tau}{|\xi|}|^{-\left(s_1+\frac{1}{2}\right)r+1}|x-\tfrac{\tau}{|\xi|}|^{-\left(s_2+\frac{1}{2}\right)r+1}(x^2-1)^{-\frac{1}{2}}\,dx \\ &\lesssim \int_2^\infty x^{2-(1+r)-1}\,dx = \int_2^\infty x^{-r}\,dx \lesssim 1. \end{aligned}$$

Proceeding similar to the elliptic case, we will obtain the following.

[3] An examination of the proof of [9, Lemma 4.4] shows that the lower bound of the resulting integral in the case $|\eta|+|\xi-\eta|\ge c|\xi|$ is exactly c.

Lemma 3.2. *Let* $1 < r \leq 2$ *and* $\sigma > \frac{1}{r}$, *then*

$$\|q_{12}(u_+, v_-)\|_{X^r_{\sigma,0}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma} \|v_0\|_{\widehat{H}^r_\sigma}.$$

The main estimate (3.3) now follows from Lemmas 3.1 and 3.2.

Remark 3.3. We observe that if instead of Q_{12} one has Q_{0j} for $j = 1, 2$, in (1.1), then instead of bounds (3.4) and (3.5), one will have the following bounds for the normalized multiplier of Q_{0j} (see [9, Lemma 13.2] for details):

$$\frac{||\eta|(\xi-\eta) - |\xi-\eta|\eta|}{|\eta||\xi-\eta|} \approx \frac{(|\eta| + |\xi-\eta|)^{\frac{1}{2}}(|\eta| + |\xi-\eta| - \xi|)^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{1}{2}}},$$

$$\frac{||\eta|(\xi-\eta) - |\xi-\eta|\eta|}{|\eta||\xi-\eta|} \approx \frac{|\xi|^{\frac{1}{2}}(|\xi| - ||\eta| - |\xi-\eta||)^{\frac{1}{2}}}{|\eta|^{\frac{1}{2}}|\xi-\eta|^{\frac{1}{2}}}.$$

The estimates for the Q_{0j} will then differ from those for Q_{12} only in having a factor of $\tau^{\frac{r}{2}}$ instead of $|\xi|^{\frac{r}{2}}$ in the elliptic case, which can be taken care of just as before.

Remark 3.4. For Q_0, one has the following bounds for the normalized multiplier (see [9, Lemma 13.2] for details):

$$\frac{|\eta||(\xi-\eta)| - \eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|} \approx \frac{(|\eta| + |\xi-\eta|)(|\eta| + |\xi-\eta| - \xi|)}{|\eta||\xi-\eta|},$$

$$\frac{|\eta||(\xi-\eta)| + \eta\cdot(\xi-\eta)}{|\eta||\xi-\eta|} \approx \frac{|\xi|(|\xi| - ||\eta| - |\xi-\eta||)}{|\eta||\xi-\eta|}.$$

Using these bounds, the bilinear estimate for Q_0 will follow from the uniform bound on

$$I = |\tau - |\xi||^r \tau^r \int \frac{\delta(\tau - |\eta| - |\xi-\eta|)}{|\eta|^{(s_1+1)r}|\xi-\eta|^{(s_2+1)r}}\, d\eta$$

in the elliptic case, and

$$I = |\tau - |\xi||^r |\xi|^r \int \frac{\delta(\tau - |\eta| + |\xi-\eta|)}{|\eta|^{(s_1+1)r}|\xi-\eta|^{(s_2+1)r}}\, d\eta$$

in the hyperbolic case. These uniform bounds can be shown in exactly the same way as before.

4. The Ward wave map problem: trilinear estimates

In order to obtain the local well-posedness for the Ward wave map Cauchy problem, we need to establish the trilinear analog of the bilinear estimate (2.1) to take care of the cubic nonlinearity in (1.5). Thus, we need to show the trilinear estimate

$$\|wQ(u,v)\|_{X^r_{s-1,b+\epsilon-1}} \lesssim \|w\|_{Z^r_{s,b}} \|u\|_{Z^r_{s,b}} \|v\|_{Z^r_{s,b}}$$

for $s > \frac{3}{2r} + \frac{1}{2}$, $1 < r \leq 2$, and some $b \in \left(\frac{1}{r}, 1\right)$, $\epsilon \in (0, 1-b)$, where $Q(u,v)$ is any of the three basic null forms. Similar to the proof of Theorem 1.1, we will only establish this trilinear estimate for the reduced range $s > \frac{1}{r} + 1$. A trilinear interpolation between this estimate and a corresponding L^2 based estimate

$$\|wQ(u,v)\|_{X^2_{s_1-1,b_1+\epsilon-1}} \lesssim \|w\|_{Z^2_{s_1,b_1}} \|u\|_{Z^2_{s_1,b_1}} \|v\|_{Z^2_{s_1,b_1}}$$

will again give the full range. This last estimate will follow from (2.3) and the following embedding:

$$X^2_{s_1,b_1} \cdot X^2_{s_1-1,b_1-1+\epsilon_1} \hookrightarrow X^2_{s_1-1,b_1+1-\epsilon_1}$$

for $s_1 > \frac{5}{4}$ and some $b_1 \in \left(\frac{1}{2}, s_1 - \frac{1}{2}\right)$, which is a special case of [25, Proposition 12] (see also [20, Theorem 7.2]).

Thus, it is enough to show the estimate

$$\|wQ(u,v)\|_{X^r_{s-1,b+\epsilon-1}} \lesssim \|w\|_{X^r_{s,b}} \|u\|_{X^r_{s,b}} \|v\|_{X^r_{s,b}}$$

for $s > \frac{1}{r} + 1$. But the last estimate will follow from (3.1) and the following multiplicative estimate:

$$\|wQ(u,v)\|_{X^r_{s-1,0}} \lesssim \|w\|_{X^r_{s,b}} \|Q(u,v)\|_{X^r_{s-1,0}}.$$

So it suffices to prove

$$X^r_{s,b} \cdot X^r_{s-1,0} \hookrightarrow X^r_{s-1,0} \tag{4.1}$$

provided $1 < r \leq 2$, $s > \frac{1}{r} + 1$, $b > \frac{1}{r}$.

Using the triangle inequality on the frequency side, it is not hard to see the following estimate:

$$\Lambda^\alpha(\phi\psi) \precsim (\Lambda^\alpha \phi)\psi + \phi \Lambda^\alpha \psi, \quad \forall \alpha > 0$$

for all ϕ and ψ with $\widetilde{\phi}, \widetilde{\psi} \geq 0$, where $u \precsim v$ denotes the pointwise estimate $\widetilde{u}(\tau,\xi) \leq \widetilde{v}(\tau,\xi)$. Then, the proof of (4.1) reduces to the following two embeddings:

$$X^r_{s,b} \cdot \widehat{L}^r \hookrightarrow \widehat{L}^r, \tag{4.2}$$

$$X^r_{1,b} \cdot X^r_{s-1,0} \hookrightarrow \widehat{L}^r. \tag{4.3}$$

The first of these, (4.2), follows trivially from Young's inequality and appropriate Hölder's inequalities to show that

$$\widehat{L}^\infty_{t,x} \hookrightarrow X^r_{s,b}.$$

The second embedding (4.3), is equivalent to the estimate

$$\|fg\|_{\widehat{L}^r_{t,x}} \lesssim \|f\|_{X^r_{1,b}} \|g\|_{X^r_{s-1,0}}$$

which we now prove. Using the definition of $\widehat{L}^r$, we have by Young's inequality

$$\|fg\|_{\widehat{L}^r} = \|\widetilde{f} *_{\tau,\xi} \widetilde{g}\|_{L^{r'}} \lesssim \left\| \|\widetilde{f}\|_{L^1_\tau} *_\xi \|\widetilde{g}\|_{L^{r'}_\tau} \right\|_{L^{r'}_\xi} \lesssim \left\| \|\widetilde{f}\|_{L^1_\tau} \right\|_{L^p_\xi} \left\| \|\widetilde{g}\|_{L^{r'}_\tau} \right\|_{L^q_\xi}$$

for some p, q, satisfying

$$\frac{1}{r'} + 1 = \frac{1}{p} + \frac{1}{q}. \tag{4.4}$$

Using Hölder's inequality, we can bound the previous by

$$\lesssim \left\| \langle \xi \rangle \| \widetilde{f} \|_{L^1_\tau} \right\|_{L^{r'}_\xi} \| \langle \xi \rangle^{-1} \|_{L^m} \left\| \langle \xi \rangle^{s-1} \| \widetilde{g} \|_{L^{r'}_\tau} \right\|_{L^{r'}_\xi} \| \langle \xi \rangle^{-(s-1)} \|_{L^l},$$

where

$$\begin{aligned} \frac{1}{p} &= \frac{1}{r'} + \frac{1}{m}, \\ \frac{1}{q} &= \frac{1}{r'} + \frac{1}{l}. \end{aligned} \tag{4.5}$$

Notice that, since $b > \frac{1}{r}$

$$\begin{aligned} \left\| \langle \xi \rangle \| \widetilde{f} \|_{L^1_\tau} \right\|_{L^{r'}_\xi} &\lesssim \left\| \langle \xi \rangle \| \langle |\tau| - |\xi| \rangle^b \widetilde{f} \|_{L^{r'}_\tau} \| \langle |\tau| - |\xi| \rangle^{-b} \|_{L^r_\tau} \right\|_{L^{r'}_\xi} \\ &\lesssim \left\| \langle \xi \rangle \| \langle |\tau| - |\xi| \rangle^b \widetilde{f} \|_{L^{r'}_\tau} \right\|_{L^{r'}_\xi} = \| f \|_{X^r_{1,b}} \end{aligned}$$

and

$$\left\| \langle \xi \rangle^{s-1} \| \widetilde{g} \|_{L^{r'}_\tau} \right\|_{L^{r'}_\xi} = \| g \|_{X^r_{s-1,0}}.$$

But then it only remains to show that there is an appropriate choice of m and l in (4.5), which insures that

$$\| \langle \xi \rangle^{-1} \|_{L^m}, \| \langle \xi \rangle^{-(s-1)} \|_{L^l} \lesssim 1. \tag{4.6}$$

For this, we need $m > 2$ and $l(s-1) > 2$. The later will follow from $l > 2r$, since $s - 1 > \frac{1}{r}$. So let us choose

$$l = 2r + \epsilon.$$

From (4.4) and (4.5), we will then have

$$\frac{1}{r'} + 1 = \frac{1}{r'} + \frac{1}{m} + \frac{1}{r'} + \frac{1}{l}$$

from which we can solve for

$$\frac{1}{m} = 1 - \frac{1}{r'} - \frac{1}{l} = \frac{1}{r} - \frac{1}{2r + \epsilon}.$$

If $r = 2$, then obviously $\frac{1}{m} < \frac{1}{2}$, which would imply that $m > 2$. On the other hand, if $r = 1 + \delta$ for some $0 < \delta < 1$, then

$$\frac{1}{m} = \frac{1}{1+\delta} - \frac{1}{2(1+\delta) + \epsilon} < \frac{1}{2}$$

provided

$$2(1+\delta) + \epsilon < \frac{1}{\frac{1}{1+\delta} - \frac{1}{2}} = \frac{2(1+\delta)}{1-\delta}$$

or

$$0 < \epsilon < \frac{2(1+\delta)}{1-\delta} - 2(1+\delta).$$

Since the right-hand side of the last inequality is positive, due to $0 < \delta < 1$, there is an ϵ, for which $l > 2r$ and $m > 2$, thus guaranteeing (4.6) and finishing the proof of (4.3).

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Appendix A. The transfer principle

In this appendix, we state and prove the transfer principle for elements of $X^r_{s,b}$ spaces. This is very similar to the L^2 case, although, due to the nature of our spaces, we stick to the frequency side, when proving the transfer principle. We closely follow [25, Section 3.2] for the first half of the appendix. See also [20, Propositions 3.6 and 3.7] and [11, Lemma 2.1].

Given a function $u \in X^r_{s,b}$, it can be uniquely decomposed as

$$u = u_+ + u_-,$$

where $\widetilde{u_+}$ is supported in $[0,\infty) \times \mathbb{R}^n$ and $\widetilde{u_-}$ is supported in $(-\infty, 0] \times \mathbb{R}^n$. Clearly $u_\pm \in X^r_{s,b}$, and

$$\|u\|^{r'}_{X^r_{s,b}} = \|u_+\|^{r'}_{X^r_{s,b}} + \|u_-\|^{r'}_{X^r_{s,b}}.$$

When $b > \frac{1}{r}$, one has the following characterization of $X^r_{s,b}$.

Proposition A.1. *If $b > \frac{1}{r}$, then*

(a) $X^r_{s,b} \subset C_b(\mathbb{R}, \widehat{H}^r_s)$ *in the sense that any tempered distribution $u \in X^r_{s,b}$ has a unique representative*

$$t \mapsto u(t) \quad \textit{in} \quad C_b(\mathbb{R}, \widehat{H}^r_s)$$

and

$$\|u(t)\|_{\widehat{H}^r_s} \leq C\|u\|_{X^r_{s,b}} \quad \textit{for all } t \in \mathbb{R},$$

where C depends only on b and r.

(b) $u \in X^r_{s,b}$ *iff there exist $f_+, f_- \in L^{r'}(\mathbb{R}, \widehat{H}^r_s)$, such that*

$$\widehat{f_+}(\rho)(\xi) = 0 \quad \textit{for } |\xi| < -\rho,$$
$$\widehat{f_-}(\rho)(\xi) = 0 \quad \textit{for } |\xi| < \rho$$

and

$$u_\pm(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(\rho \pm D)} f_\pm(\rho)}{(1+|\rho|)^b}\, d\rho. \tag{A.1}$$

Moreover,

$$\|u_\pm\|_{X^r_{s,b}} = \|f_\pm\|_{L^{r'}(\mathbb{R}, \widehat{H}^r_s)}.$$

Remark A.1. The representation (A.1) is equivalent to a decomposition with respect to the foliation of the Fourier space by the two families of cones

$$\mathcal{N}_+(\rho) : \tau = |\xi| + \rho, |\xi| > -\rho \quad \text{and} \quad \mathcal{N}_-(\rho) : \tau = -|\xi| + \rho, |\xi| > \rho,$$

where the family parameter $\rho \in (-\infty, \infty)$. Indeed, we can write

$$\tag{A.2} \begin{aligned} \widetilde{u_\pm}(\tau, \xi) &= \int \widetilde{u_\pm}(\rho \pm |\xi|, \xi)\delta(\rho - (\tau \mp |\xi|))\, d\rho \\ &= \int \frac{\widehat{f_\pm}(\rho)(\xi)}{(1 + |\rho|)^b}\delta(\rho - (\tau \mp |\xi|))\, d\rho, \end{aligned}$$

where

$$\widehat{f_\pm}(\rho)(\xi)\delta(\rho - (\tau \mp |\xi|)) = \widetilde{u_\pm}(\tau, \xi)(1 + |\rho|)^b\delta(\rho - (\tau \mp |\xi|))$$

define $W^{s,r'}$ measures on the cones $\mathcal{N}_\pm(\rho)$, and as such, represent translations of the Fourier transforms of $\widehat{H}^r_s$ solutions of the free wave equation restricted, respectively, to the upper and lower Fourier half-spaces.

Taking the time inverse Fourier transforms of (A.2), we have

$$\widehat{u_\pm}(\xi) = \frac{1}{2\pi} \int \frac{e^{it(\rho \pm |\xi|)}\widehat{f_\pm}(\rho)(\xi)}{(1 + |\rho|)^b}\, d\rho.$$

Taking the space inverse Fourier transform of this identity will result in (A.1).

Proof. The existence of part (a) follows from part (b). Furthermore, from (A.1), Hölder's inequality and the fact that $b > \frac{1}{r}$

$$\begin{aligned} \|u(t)\|_{\widehat{H}^r_s} &\le \int \frac{\|f_+(\rho)\|_{\widehat{H}^r_s}}{(1 + |\rho|)^b}\, d\rho + \int \frac{\|f_-(\rho)\|_{\widehat{H}^r_s}}{(1 + |\rho|)^b}\, d\rho \\ &\le C \left(\|f_+\|_{L^{r'}(\mathbb{R}, \widehat{H}^r_s)} + \|f_-\|_{L^{r'}(\mathbb{R}, \widehat{H}^r_s)} \right) \\ &\le C\|u\|_{X^r_{s,b}}, \end{aligned}$$

where $C = 2^{r'} \left(\int (1 + |\rho|)^{-rb}\, d\rho \right)^{\frac{1}{r}}$.

The uniqueness of part (a) is straightforward, so it remains to prove part (b).

Let us define the multipliers

$$\begin{aligned} \widehat{\Lambda u} &= \langle \xi \rangle \widehat{u}, \\ \widetilde{\Lambda_- u} &= \langle |\tau| - |\xi| \rangle \widetilde{u}. \end{aligned}$$

Using these, we define the isometry

$$u \longleftrightarrow F = \mathcal{F}(\Lambda^s \Lambda^b_- u), \quad X^r_{s,b} \longleftrightarrow L^{r'}$$

under which u_+ and u_- correspond to

$$F_+ = \chi_{[0,\infty) \times \mathbb{R}^n} F \quad \text{and} \quad F_- = \chi_{(-\infty,0] \times \mathbb{R}^n} F,$$

respectively. We define another isometry

$$F_\pm \longleftrightarrow f_\pm, \quad L^{r'}(\mathbb{R}^{1+n}) \longleftrightarrow L^{r'}(\mathbb{R}, \widehat{H}^r_s)$$

by

$$(1+|\xi|)^s \widehat{f_+(\rho)}(\xi) = F_+(\rho+|\xi|,\xi),$$
$$(1+|\xi|)^s \widehat{f_-(\rho)}(\xi) = F_-(\rho-|\xi|,\xi).$$

Notice that under the composition of the isometries

$$\widehat{f_\pm}(\rho)(\xi) = \frac{1}{(1+|\xi|)^s} F_\pm(\rho\pm|\xi|,\xi) = (1+|\rho|)^b \widetilde{u_\pm}(\rho\pm|\xi|,\xi).$$

It is easy to check that $f_\pm$ is in $L^{r'}(\mathbb{R},\widehat{H}^r_s)$, iff $F_\pm$ is in $L^{r'}(\mathbb{R}^{1+n})$. Thus, $u\in X^r_{s,b}$, iff $f_+, f_- \in L^{r'}(\mathbb{R},\widehat{H}^r_s)$, and we set

$$v_+(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(\rho+D)} f_+(\rho)}{(1+|\rho|)^b}\,d\rho \quad\text{and}\quad v_-(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{it(\rho-D)} f_-(\rho)}{(1+|\rho|)^b}\,d\rho$$

for $t\in\mathbb{R}$. By the dominated convergence theorem $v_+, v_- \in C(\mathbb{R},\widehat{H}^r_s)$, and $v_\pm$ are tempered distributions, as the $\widehat{H}^r_s$ norm of $v_\pm(t)$ is bounded uniformly in t. We next prove that $u_+ = v_+$ in the sense of distributions. The proof of $u_- = v_-$ is similar.

We have for $\phi\in\mathcal{S}(\mathbb{R}^{1+n})$

$$\begin{aligned}
\langle v_+,\phi\rangle &= \int \langle v_+(t),\phi(t)\rangle\,dt \\
&= \int \left\langle \frac{1}{2\pi}\int \frac{e^{it(\rho+D)} f_+(\rho)}{(1+|\rho|)^b}\,d\rho, \phi(t)\right\rangle dt \\
&= \iint \left\langle \frac{1}{2\pi}\frac{e^{it(\rho+D)} f_+(\rho)}{(1+|\rho|)^b}, \phi(t)\right\rangle d\rho dt \\
&= \iiint \frac{1}{2\pi}\frac{e^{it(\rho+|\xi|)} \widehat{f_+}(\rho)(\xi)}{(1+|\rho|)^b} \mathcal{F}^{-1}_\xi \phi(t)(\xi)\,d\xi\,d\rho\,dt \\
&= \iiint \frac{1}{2\pi} e^{it(\rho+|\xi|)} \widetilde{u_+}(\rho+|\xi|,\xi)\mathcal{F}^{-1}_\xi \phi(t)(\xi)\,d\xi\,d\rho\,dt \\
&= \iint \widetilde{u_+}(\tau,\xi)\left(\frac{1}{2\pi}\int e^{it\tau}\mathcal{F}^{-1}_\xi\phi(t)(\xi)\,dt\right) d\tau\,d\xi \\
&= \iint \widetilde{u_+}(\tau,\xi)\mathcal{F}^{-1}_{\tau,\xi}\phi(\tau,\xi)\,d\tau\,d\xi \\
&= \langle u_+,\phi\rangle.
\end{aligned}$$

□

Using the integral representation (A.1), we will prove a useful corollary, which allows one to transfer multilinear estimates involving solutions of the free wave equation to corresponding estimates for elements of $X^r_{s,b}$ spaces with $b>\frac{1}{r}$. This result is appropriately called the *transfer principle.*

Proposition A.2. *Assume that* $T:\widehat{H}^r_{s_1}(\mathbb{R}^n)\times\cdots\times\widehat{H}^r_{s_k}(\mathbb{R}^n)\to\widehat{H}^r_\sigma(\mathbb{R}^n)$ *is a* k*-linear operator, and let* $b>\frac{1}{r}$.

(a) *If*

$$\|T(e^{\lambda_1 itD}f_1,\dots,e^{\lambda_k itD}f_k)\|_{\widehat{L}^p_t(\widehat{L}^q_x)} \le C\|f_1\|_{\widehat{H}^r_{s_1}}\cdots\|f_k\|_{\widehat{H}^r_{s_k}}, \tag{A.3}$$

where λ is a fixed k-tuple in $\{-1,1\}$, then

$$\|T(u_1,\dots,u_k)\|_{\widehat{L}^p_t(\widehat{L}^q_x)} \le C\|u_1\|_{X^r_{s_1,b}}\cdots\|u_k\|_{X^r_{s_k,b}} \tag{A.4}$$

for all $(u_1,\dots,u_k)\in X^r_{s_1,b}\times\cdots\times X^r_{s_k,b}$, such that

$$\operatorname{supp}\widehat{u_j}\subseteq\begin{cases}[0,\infty)\times\mathbb{R}^n, & \text{if } \lambda_j=1\\ (-\infty,0]\times\mathbb{R}^n, & \text{if } \lambda_j=-1.\end{cases} \tag{A.5}$$

(b) *If* (A.3) *holds for all $\lambda\in\{-1,1\}^k$, then* (A.4) *holds for all $(u_1,\dots,u_k)\in X^r_{s_1,b}\times\cdots\times X^r_{s_k,b}$.*

Proof. We assume that T is bilinear to keep the notation simple, the general case is similar. Let us denote $U_\pm=e^{\pm itD}$. From (A.1),

$$\widetilde{u_j}(\tau,\xi)=\frac{1}{2\pi}\int\frac{\mathcal{F}_{t,x}(e^{it\rho}U_jf_j(\rho))}{(1+|\rho|)^b}\,d\rho=\frac{1}{2\pi}\int\frac{\widetilde{U_jf_j(\rho)}(\tau-\rho,\xi)}{(1+|\rho|)^b}\,d\rho,$$

where $U_j=U_+$ or $U_j=U_-$, depending on whether $\lambda_j=1$, or $\lambda_j=-1$. Denoting the multiplier of T by m, and using $\Xi=(\tau,\xi)$, we have

$$\begin{aligned}
&\mathcal{F}T(u_1,u_2)(\Xi)\\
&\quad=\iint m(\Xi_1,\Xi_2)\widetilde{u_1}(\Xi_1)\widetilde{u_2}(\Xi_2)\delta(\Xi-\Xi_1-\Xi_2)\,d\Xi_1d\Xi_2\\
&\quad=\iint m(\Xi_1,\Xi_2)\frac{1}{2\pi}\int\frac{\widetilde{U_1f_1(\rho_1)}(\tau_1-\rho_1,\xi_1)}{(1+|\rho_1|)^b}\,d\rho_1\\
&\qquad\qquad\times\frac{1}{2\pi}\int\frac{\widetilde{U_2f_2(\rho_2)}(\tau_2-\rho_2,\xi_2)}{(1+|\rho_2|)^b}\,d\rho_2\,\delta(\Xi-\Xi_1-\Xi_2)\,d\Xi_1d\Xi_2\\
&\quad=\frac{1}{4\pi^2}\iint\frac{\mathcal{F}T(U_1f_1(\rho_1),U_2f_2(\rho_2))(\tau+\rho_1+\rho_2,\xi)}{(1+|\rho_1|)^b(1+|\rho_2|)^b}\,d\rho_1d\rho_2,
\end{aligned}$$

where we used Fubini's theorem in the last step. Then using Minkowski's inequality, (A.3), Hölder's inequality and Proposition A.1,

$$\begin{aligned}
\|T(u_1,u_2)\|_{\widehat{L}^p_t(\widehat{L}^q_x)}&\lesssim\iint\frac{\|T(U_1f_1(\rho_1),U_2f_2(\rho_2))\|_{\widehat{L}^p_t(\widehat{L}^q_x)}}{(1+|\rho_1|)^b(1+|\rho_2|)^b}\,d\rho_1d\rho_2\\
&\lesssim\iint\frac{\|f_1(\rho_1)\|_{\widehat{H}^r_{s_1}}\|f_2(\rho_2)\|_{\widehat{H}^r_{s_k}}}{(1+|\rho_1|)^b(1+|\rho_2|)^b}\,d\rho_1d\rho_2\\
&\lesssim\|f_1\|_{L^{r'}(\widehat{H}^r_{s_1})}\|f_2\|_{L^{r'}(\widehat{H}^r_{s_2})}\\
&=\|u_1\|_{X^r_{s_1,b}}\|u_2\|_{X^r_{s_2,b}}.
\end{aligned}$$

□

Appendix B. The general well-posedness theorem

In this appendix, we will state the general well-posedness theorem for NLW with data in $\widehat{H}^r_s\times\widehat{H}^r_{s-1}$. We do not include the proof here, as, with minor differences, it is similar to the proof of the analogous theorem in the L^2 case, for which we refer the reader to [25, Theorem 4.1] (see also [20, Theorem 5.3]).

The proof of the general well-posedness theorem follows from appropriate estimates for the solution to the linear wave equation in the relevant solution spaces, which we state first. This result is again similar to the analogous result in the L^2 case.

Consider the following Cauchy problem for the linear wave equation:

$$\Box u = F(t,x), \quad (t,x) \in \mathbb{R}^{1+n}, \tag{B.1}$$

$$(u, \partial_t u)_{t=0} = (f,g) \in \widehat{H}^r_s \times \widehat{H}^r_{s-1}. \tag{B.2}$$

Theorem B.1 (cf. Theorem 13 in [25]). *Assume $s \in \mathbb{R}$, $\frac{1}{r} < b < 1$, $\epsilon \in (0, 1-b)$, $F \in X^r_{s-1,b+\epsilon-1}$ and*

$$\chi \in C^\infty_c(\mathbb{R}), \quad \chi = 1 \text{ on } [-1,1], \quad \operatorname{supp} \chi \subseteq (-2,2).$$

Let $0 < T < 1$ and define

$$u(t) = \chi(t) u_0 + \chi(t/T) u_1 + u_2,$$

where

$$\begin{aligned} u_0 &= \cos(tD) \cdot f + D^{-1} \sin(tD) g, \\ u_1 &= \int_0^t D^{-1} \sin((t-t')D) \cdot F_1(t')\, dt', \\ u_2 &= \Box^{-1} F_2 \end{aligned}$$

and

$$F = F_1 + F_2 = \phi(T^{1/2}\Lambda_-)F + (1 - \phi(T^{1/2}\Lambda_-))F$$

with

$$\phi \in C^\infty_c(\mathbb{R}), \quad \phi = 1 \text{ on } [-2,2], \quad \operatorname{supp} \phi \subseteq (-4,4).$$

Then

$$\|u\|_{Z^r_{s,b}} \le C \left(\|f\|_{\widehat{H}^r_s} + \|g\|_{\widehat{H}^r_{s-1}} + T^{\epsilon/2} \|F\|_{X^r_{s-1,b+\epsilon-1}} \right),$$

where C depends only on χ and b. Furthermore, u is the unique solution of the Cauchy problem (B.1) *and* (B.2) *such that $u \in C(0,T], \widehat{H}^r_s) \cap C^1[(0,T], \widehat{H}^r_{s-1})$.*

Using this theorem, one can tackle the local well-posedness of a nonlinear Cauchy problem via an iteration argument.

Consider the Cauchy problem

$$\Box u = F(u, \partial u), \quad (t,x) \in \mathbb{R}^{1+n}, \tag{B.3}$$

$$(u, \partial_t u)|_{t=0} = (f,g) \in \widehat{H}^r_s \times \widehat{H}^r_{s-1}, \tag{B.4}$$

where ∂u is the space–time gradient of u, and F is a smooth function with $F(0) = 0$.

Theorem B.2 (c.f Theorem 14 in [25]). *Assume $s \in \mathbb{R}$, $\frac{1}{r} < b < 1$, $\epsilon \in (0, 1-b)$. If*

$$\|F(u, \partial u)\|_{X^r_{\sigma-1,b+\epsilon-1}} \le A_\sigma(\|u\|_{Z^r_{s,b}}) \|u\|_{Z^r_{\sigma,b}}$$

for all $\sigma \ge s$, and

$$\|F(u,\partial u) - F(v, \partial v)\|_{X^r_{\sigma-1,b+\epsilon-1}} \le A_s(\|u\|_{Z^r_{s,b}} + \|v\|_{Z^r_{s,b}}) \|u - v\|_{Z^r_{\sigma,b}}$$

for all $u, v \in Z^r_{s,b}$, where $A_\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and locally Lipschitz for every $\sigma \ge s$, then

- *(Existence) There exists* $u \in Z^r_{s,b}$*, which solves* (B.3) *and* (B.4) *on* $[0,T]\times\mathbb{R}^n$*, where* $T = T(\|f\|_{\widehat{H}^r_s} + \|g\|_{\widehat{H}^r_s}) > 0$ *depends continuously on* $\|f\|_{\widehat{H}^r_s} + \|g\|_{\widehat{H}^r_s}$*.*
- *(Uniqueness) The solution is unique, in the class* $Z^r_{s,b}$*, i.e., if* $u, v \in Z^r_{s,b}$ *both solve* (B.3) *and* (B.4) *on* $[0,T]\times\mathbb{R}^n$ *for some* $T > 0$*, then*

$$u(t) = v(t) \quad \text{for } t \in [0,T].$$

- *(Lipschitz) The solution map*

$$(f,g) \mapsto u, \quad \widehat{H}^r_s \times \widehat{H}^r_{s-1} \to Z^r_{s,b}$$

is locally Lipschitz.
- *(Higher regularity) If the data has higher regularity*

$$f \in \widehat{H}^r_\sigma, g \in \widehat{H}^r_{\sigma-1} \quad \text{for } \sigma > s,$$

then $u \in C([0,T],\widehat{H}^r_\sigma) \cap C^1([0,T],\widehat{H}^r_{\sigma-1})$ *for any* $T > 0$ *for which* u *solves* (B.3) *and* (B.4)*. In particular, if* $(f,g) \in \mathcal{S}$*, then* $u \in C^\infty([0,T]\times\mathbb{R}^n)$*.*

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