Math. Res. Lett. **21** (2014), *no.* 2, 235–240

SOME UNSTABLE CRITICAL METRICS FOR THE $L^{\frac{n}{2}}$-NORM OF THE CURVATURE TENSOR

Atreyee Bhattacharya and Soma Maity

Abstract. We consider the Riemannian functional defined on the space of Riemannian metrics with unit volume on a closed smooth manifold M given by $\mathcal{R}_{\frac{n}{2}}(g) := \int_M |R(g)|^{\frac{n}{2}} dv_g$ where $R(g)$, dv_g denote the Riemannian curvature and volume form corresponding to g. We show that there are locally symmetric spaces which are unstable critical points for this functional.

1. Introduction

Let M be a closed smooth manifold of dimension $n \geq 3$. In this paper, we study the following Riemannian functional defined on the space of Riemannian metrics:

$$\mathcal{R}_p(g) := \int_M |R(g)|^p dv_g,$$

where $R(g)$ and dv_g denote the Riemannian curvature tensor and volume form corresponding to the Riemannian metric g on M, $p \in [2, \infty)$. If $p \neq \frac{n}{2}$, then $\mathcal{R}_p$ is not scale invariant. For p not equal to $\frac{n}{2}$, we always restrict $\mathcal{R}_p$ to the space of metrics with unit volume to study its critical points, infimum and local minimizing properties. Let $S^2(T^*M)$ be the space of symmetric two tensors on M and $\mathcal{W}$ be the subspace of $S^2(T^*M)$ orthogonal to $\mathbb{R}g$ and the tangent space of the orbit of g under the action of the group of diffeomorphisms of M at g. By a standard technique, one proves that every irreducible locally symmetric space is a critical point of $\mathcal{R}_p$. Let H denote the Hessian of $\mathcal{R}_p$. For definitions of critical metric and Hessian, we refer to Section 2.

Definition 1.1. Let g be a critical point for $\mathcal{R}_p$. g is *stable* for $\mathcal{R}_p$ if there is an $\epsilon > 0$ such that for every element h in $\mathcal{W}$

$$H(h,h) \geq \epsilon \|h\|^2, \tag{1.1}$$

where $\|.\|$ denotes the L^2-norm on $S^2(T^*M)$ defined by g.

g is called *weakly stable* if H is non-negative definite. In [6], it is proved that spherical space forms are stable for $\mathcal{R}_p$ for $p \geq 2$ and hyperbolic manifolds are stable for $p \geq \frac{n}{2}$. Viewing these results it is natural to expect that every irreducible locally symmetric space is stable for this functional. However, we show in this paper that this is not always the case.

Received by the editors August 11, 2013.
Key words and phrases. Riemannian functional, stability.

Theorem 1. *Let (M, g) be an irreducible locally symmetric space of compact type. If the universal cover of M is one of the following then (M, g) is not weakly stable for $\mathcal{R}_{\frac{n}{2}}$.*

$$SU(q)(q \geq 3), \quad Sp(q)(q \geq 2), \quad \mathrm{Spin}(5), \quad \mathrm{Spin}(6), \quad SU(2q+2)/Sp(q+1),$$
$$Sp(q+l)/Sp(q) \times Sp(l)(l, q \geq 1), \quad E_6/F_4, \quad F_4/\mathrm{Spin}(9).$$

As a corollary, we see that the examples of locally symmetric spaces mentioned in the above theorem are not local minima for $\mathcal{R}_{\frac{n}{2}}$. On the other hand, the Cheeger–Gromov [2, 3] theory on collapsing Riemannian manifolds with bounded curvature implies that if a manifold admits an F-structure of positive rank then the infimum of $\mathcal{R}_p$ is zero for any $p > 0$. In particular, if a compact manifold admits a free S^1-action then the conclusion holds. Every Lie group and $SU(2q+2)/Sp(q+1)$ admit free isometric S^1 actions (for the latter, see [4]). Therefore the infimum of $\mathcal{R}_p$ is zero in these cases.

As for the other examples in Theorem 1, we observe that $Sp(q+l)/Sp(q) \times Sp(l)$ $(l, q \geq 1)$ and $F4/\mathrm{Spin}(9)$ do not admit F-structures of positive rank, since their Euler characteristics are non-zero (for the latter fact, see [5]). The results of Yang in [8] imply that if a compact manifold does not admit an F-structure of positive rank then the infimum of $\mathcal{R}_p$ is positive for any $p > \frac{n}{2}$. However, it is not known if the infimum is positive for the same for $p = \frac{n}{2}$. It was conjectured by Gromov that a compact Riemannian manifold with sufficiently small $L^{\frac{n}{2}}$-norm of the curvature tensor admits an F-structure of positive rank.

It is interesting to note that the proof of Theorem 1 actually gives the instability of $\mathcal{R}_p$ at the symmetric metrics of $Sp(q+l)/Sp(q) \times Sp(l)$ $(l, q = 1)$ and $F4/\mathrm{Spin}(9)$ for $p \in (n/2, n/2 + \epsilon)$ for some small ϵ. For such p, it would be interesting to study the existence and geometry of minimizers of $\mathcal{R}_p$ on these spaces. Even the question of existence and geometry of minimizers of $\mathcal{R}_p$ restricted to the conformal class of symmetric metrics might be of some interest.

The main theorem in this paper follows by restricting H to the space of conformal variations of g and then using an estimate for the first positive eigenvalue of the Laplacian. Let (M, g) be a simply connected irreducible symmetric space of compact type. Let λ_1 denote the first positive eigenvalue of the Laplacian and s be the scalar curvature of (M, g). We prove that g is stable for $\mathcal{R}_{\frac{n}{2}}$ restricted to the space of conformal variations of g if and only if $\frac{\lambda_1}{s} \geq \frac{2}{n}$.

2. Proof

Let $\{e_i\}$ be an orthonormal basis at a point of M. $\check{R}$ is a symmetric 2-tensor defined by

$$\check{R}(x, y) = \sum R(x, e_i, e_j, e_k) R(y, e_i, e_j, e_k).$$

Let D and D^* be the Riemannian connection and its formal adjoint.
$d^D : S^2(T^*M) \to \Gamma(T^*M \otimes \Lambda^2 M)$ and its formal adjoint δ^D are defined by

$$d^D\alpha(x, y, z) := (D_y\alpha)(x, z) - (D_z\alpha)(x, y),$$
$$\delta^D(A)(x, y) = \sum \{D_{e_i} A(x, y, e_i) + D_{e_i} A(y, x, e_i)\},$$

where $\Lambda^2 M$ and $\Gamma(T^*M \otimes \Lambda^2 M)$ denote the alternating two forms and the sections of $T^*M \otimes \Lambda^2 M$, respectively. Let g_t be a one-parameter family of metrics with $\frac{d}{dt}(g_t)_{|t=0} = h$ and $T(t)$ be a tensor depending on g_t. Then $\frac{d}{dt}T(t)_{|t=0}$ is denoted by $T'_g(h)$. Define $\Pi_h(x,y) = \frac{d}{dt}D_x y_{|t=0}$ where x, y are two fixed vector fields. The suffix h will be omitted when there is no ambiguity. Consider any $f \in C^\infty(M)$. Then

$$g(\Pi_{fg}(x,y),z) = \tfrac{1}{2}[D_x fg(y,z) + D_y fg(x,z) - D_z fg(x,y)] \tag{2.1}$$
$$= \tfrac{1}{2}[df(x)g(y,z) + df(y)g(x,z) - df(z)g(x,y)].$$

Let Δ denote the Laplace operator defined on $C^\infty(M)$. We use the following definition:

$$\Delta f = -\mathrm{tr}(Ddf).$$

The point-wise inner product, point-wise norm, global inner product and global norm induced by g are denoted by $(,)$, $|.|$, $\langle,\rangle$, $\|.\|$, respectively.

$\nabla\mathcal{R}_p(g)$ in $S^2(T^*M)$ is called the *gradient* of $\mathcal{R}_p$ at g if for every $h \in S^2(T^*M)$,

$$\frac{d}{dt}_{|t=0}\mathcal{R}_p(g+th) = \mathcal{R}'_{p|g}.h = \langle \nabla\mathcal{R}_p(g), h\rangle$$

g is called a *critical point* for $\mathcal{R}_p$ if the component of $\nabla\mathcal{R}_p(g)$ along the tangent space of the space of Riemannian metrics with unit volume at g is zero. The Hessian of $\mathcal{R}_p$ at a critical point is given by

$$H(h_1,h_2) = \langle (\nabla\mathcal{R}_p)'_g(h_1), h_2\rangle \quad \forall\, h_1,h_2 \in S^2(T^*M).$$

Proposition 2.1. *Let (M,g) be a compact irreducible symmetric space and $f \in C^\infty(M)$. Then*

$$H(fg,fg) = p|R|^{p-2}[a\|\Delta f\|^2 - b\|df\|^2 + c\|f\|^2],$$

where, a, b and c is given by

$$a = n-1+(p-2)\frac{4s^2}{n^2|R|^2},$$
$$b = 4(p-1)\frac{s}{n},$$
$$c = \left(p-\frac{n}{2}\right)|R|^2.$$

Let (M,g) be a closed irreducible symmetric space and $h_1,h_2 \in S^2(T^*M)$. From [6] (4.1), we have

$$\begin{aligned} H(h_1,h_2) = &-p|R|^{p-2}(\langle \delta^D(D^*)'_g(h_1)R, h_2\rangle + \langle D^*R'_g(h_1), d^D h_2\rangle) \\ &- p|R|^{p-2}\langle \check{R}'_g(h_1), h_2\rangle - p\langle (|R|^{p-2})'_g(h_1)R, Dd^D h_2\rangle \\ &- \frac{p}{n}|R|^2\langle (|R|^{p-2})'_g(h_1)g, h_2\rangle + \frac{1}{2}\langle (|R|^p)'_g(h_1)g, h_2\rangle \\ &+ \frac{p}{n}\|R\|^p\langle h_1,h_2\rangle. \end{aligned} \tag{2.2}$$

Next we compute each term of the above equation for conformal variations to obtain the proposition.

Lemma 2.1. $\langle \delta^D(D^*)'(fg)R, fg\rangle = 4\frac{s}{n}\|df\|^2$.

Proof. Let $\tilde{g}(t)$ be an one-parameter family of metrics with $\tilde{g}(0) = g$ and T be a $(0,4)$ tensor independent of t. Expressing D^* in a local coordinate chart and differentiating each term in the expression, we have

$$(D^*T)(x,y,z) = \tilde{g}^{kj}[T_{\Pi_{kj}xyz} + T_{j\Pi_{kx}yz} + T_{jx\Pi_{ky}z} + T_{jxy\Pi_{kz}}] - (\tilde{g}^{kj})'(D_kT)_{jxyz}.$$

Note that, Π acting on two vector fields gives a vector field. Now evaluating $(D^*)'_g(h)(R)$ on an orthonormal basis, we have

$$(D^*)'_g(h)(R)_{jkl} = R_{\Pi_{ii}jkl} + R_{i\Pi_{ij}kl} + R_{ij\Pi_{ik}l} + R_{ijk\Pi_{il}}.$$

From the definition of d^D, we have

$$d^D fg(x,y,z) = D_y fg(x,z) - D_z fg(x,y) = df(y)g(x,z) - df(z)g(x,y).$$

Combining the above two equations, we have

$$\sum (D^*)'(fg)_{jkl} d^D(fg)_{jkl} = 2\sum [R_{\Pi_{ii}jkj} + R_{i\Pi_{ij}kj} + R_{ij\Pi_{ik}j}]df_k.$$

Let μ be the Einstein constant of (M,g)

$$\begin{aligned} 2\sum R_{\Pi_{ii}jkj}df_k &= 2\sum g(\Pi(e_i,e_i),e_m)R(e_m,e_j,e_k,e_j)df(e_k) \\ &= -(n-2)\mu|df|^2. \end{aligned} \tag{2.3}$$

Similarly,

$$2\sum R_{ij\Pi_{ikj}}df_k = n\mu|df|^2 \quad \text{and} \quad 2\sum R_{i\Pi_{ij}kj}df_k = 2\mu|df|^2. \tag{2.4}$$

Combining equations (2.3) and (2.4) the proof of the lemma follows. □

Lemma 2.2. $\langle D^*R'_g(fg), d^D fg\rangle = -(n-1)\|\Delta f\|^2 + (n-4)\frac{s}{n}\|df\|^2.$

Proof. By a simple calculation, we have

$$D^2_{x,y} fg(u,v) = Ddf(x,y)g(u,v) \tag{2.5}$$

and

$$Dd^D fg(x,y,z,w) = Ddf(x,z)g(y,w) - Ddf(x,w)g(y,z). \tag{2.6}$$

From [1] 1.174(c) and using (2.5), we have

$$\begin{aligned} R'_g(fg)(x,y,z,u) = &-\frac{1}{2}[Ddf(y,z)g(x,u) + Ddf(x,u)g(y,z) \\ &- Ddf(x,z)g(y,u) - Ddf(y,u)g(x,z)] \\ &+ fR(x,y,z,u). \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} (R'_g(fg), Dd^D fg) &= Ddf_{ik}R'_g(fg)_{ijkj} - Ddf_{il}R'_g(fg)_{ijjl} \\ &= -(n-2)|Ddf|^2 - |\Delta f|^2 - 2\mu|df|^2. \end{aligned}$$

Using Bochner–Weitzenbök formula on the space of one forms, we have

$$\Delta df = D^*Ddf + \mu df.$$

Hence the lemma follows. □

Lemma 2.3. $\langle \check{R}'_g(fg), fg\rangle = 4\frac{s}{n}\|df\|^2 - |R^2|\|f\|^2.$

Proof.

$$\check{R}_{pq} = \tilde{g}^{i_1 i_2}\tilde{g}^{j_1 j_2}\tilde{g}^{k_1 k_2} R_{p i_1 j_1 k_1} R_{q i_2 j_2 k_2}.$$

Differentiating it with respect to t and evaluating on an orthonormal basis, we have

$$\begin{aligned}(\check{R}_g.h)'_{pq} &= -h_{mn}\left(R_{pmij}R_{qnij} + R_{pimj}R_{qinj} + R_{pijm}R_{qijn}\right)\\ &\quad + (R'_g.h)_{pijk}R_{qijk} + R_{pijk}(R'_g.h)_{qijk}.\end{aligned}$$

Therefore,

$$\langle \check{R}'_g(fg), fg\rangle = -3|R|^2\|f\|^2 + 2\langle (R'_g.fg), fR\rangle.$$

Using (2.7), we have

$$\begin{aligned}(\check{R}'_g(fg), R) &= \frac{1}{2}\sum [Ddf_{jk}R_{ijki} + Ddf_{il}R_{ijjl} - Ddf_{ik}R_{ijkj} - Ddf_{jl}R_{ijil}]\\ &\quad + f|R|^2\\ &= 2\mu\Delta f + f|R|^2.\end{aligned}$$

Hence the lemma follows. □

Lemma 2.4. $(|R|^p)'(fg) = 2\frac{s}{n}p|R|^{p-2}\Delta f - pf|R|^p$.

Proof.

$$\begin{aligned}(|R|^p)'(fg) &= p|R|^{p-2}(R, R'_g.fg) - 2p|R|^{p-2}(\check{R}, fg)\\ &= p|R|^{p-2}(R, R'_g.fg) - 2pf|R|^{p-2}\mathrm{tr}(\check{R})\\ &= p|R|^{p-2}(2\mu\Delta f + f|R|^2) - 2pf|R|^p\\ &= 2\mu p|R|^{p-2}\Delta f - pf|R|^p.\end{aligned}$$

□

Using (2.6), we have $(Dd^D fg, R) = 2(Ddf, r) = -2\mu\Delta f$. Now the Proposition (2.1) follows from the above lemma and equation.

Proof of Theorem 1. Let (M, g) is a simply connected irreducible symmetric space of compact type which is not a sphere. Then

$$R = \frac{s}{n(n-1)}I + W,$$

where I is the curvature of standard sphere with sectional curvature 1 and W is the Weyl curvature of (M, g). From the above expression, we have

$$\frac{s^2}{|R|^2} < \frac{2}{n(n-1)}.$$

Let λ_1 be the first positive eigenvalue of the Laplacian of (M, g) and f be an eigenfunction corresponding to λ_1. Then from Proposition 2.1, we have

$$\begin{aligned}H(fg, fg) &= s\lambda_1 p|R|^{p-2}\left[a\left(\frac{\lambda_1}{s}\right) - \frac{b}{s}\right]\|f\|^2\\ &\le s\lambda_1 p|R|^{p-2}\left[\left(n - 1 + 4\frac{n-4}{n^3(n-1)}\right)\frac{\lambda_1}{s} - 2\frac{n-2}{n}\right]\|f\|^2.\end{aligned}$$

From Tables A.1 and A.2 in [7], we have

$$H(fg, fg) < 0.$$

This completes the proof. □

Remark: If we choose a sufficiently large eigenvalue λ_i such that $a\left(\frac{\lambda_i}{s}\right) - \frac{b}{s} < 0$. Let $\tilde{f}$ be an eigenfunction corresponding to λ_i. Then we have $H(\tilde{f}g, \tilde{f}g) > 0$. This makes (M, g) a saddle point for $\mathcal{R}_{\frac{n}{2}}$.

Theorem 2. *Let (M, g) be either a compact quotient of an irreducible symmetric space of non-compact type or a compact symmetric space which is not one of the types in Theorem 1. Then g is stable for $\mathcal{R}_p$ restricted to the space of conformal variations of g for $p \geq \frac{n}{2}$.*

Proof. If (M, g) is a compact quotient of an irreducible symmetric of non-compact type then the theorem is an immediate consequence of Proposition 2.1. Otherwise from Tables A.1 and A.2 in [7], we observe that $\frac{\lambda_1}{s} \geq \frac{2}{n}$. Using this estimate and Proposition 2.1, the proof of the theorem follows. □

Remark: Let (M, g) be one of the critical metrics of $\mathcal{R}_p$ mentioned in Theorem 2. This is an immediate consequence of the above theorem that g is a local minimizer for $\mathcal{R}_p(p \geq \frac{n}{2})$ restricted to the space of metrics conformal to g.

It is interesting to note that if (M, g) is a simply connected irreducible symmetric space of compact type which is not a sphere, then (M, g) is stable for $\mathcal{R}_{\frac{n}{2}}$ if and only if it is stable for the identity map as a harmonic map.

Acknowledgment

We would like to thank Harish Seshadri for his valuable comments and suggestions on this paper.

References

[1] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 10, Springer-Verlag, Berlin, 1987.
[2] J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded. I*, J. Differential Geom. **23** (1986), 309–346.
[3] J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded. II*, J. Differential Geom. **32** (1990), 269–298.
[4] J.-H. Eschenburg, A. Kollross and K. Shankar, *Free, isometric circle actions on compact symmetric spaces*, Geom. Dedicata **102** (2003), 35–44.
[5] H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment. Math. Helv. **13** (1941), 240–251.
[6] S. Maity, *On the stability of the L^p-norm of curvature tensor*, `arXiv:1201.1691`.
[7] H. Urakawa, *The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold*, Compos. Math. **59**(1) (1986), 57–71.
[8] D. Yang, *Riemannian manifolds with small integral norm of curvature*, Duke Math. J. **65** (1992), 501–510.

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India
E-mail address: soma123maity@gmail.com
E-mail address: atreyee@math.iisc.ernet.in