UNIQUENESS OF EQUIVARIANT COMPACTIFICATIONS OF $\mathbb{C}^n$ BY A FANO MANIFOLD OF PICARD NUMBER 1

BAOHUA FU AND JUN-MUK HWANG

ABSTRACT. Let X be an n-dimensional Fano manifold of Picard number 1. We study how many different ways X can compactify the complex vector group $\mathbb{C}^n$ equivariantly. Hassett and Tschinkel showed that when $X = \mathbb{P}^n$ with $n \geq 2$, there are many distinct ways that X can be realized as equivariant compactifications of $\mathbb{C}^n$. Our result says that projective space is an exception: among Fano manifolds of Picard number 1 with smooth VMRT, projective space is the only one compactifying $\mathbb{C}^n$ equivariantly in more than one ways. This answers questions raised by Hassett–Tschinkel and Arzhantsev–Sharoyko.

1. Introduction

Throughout, we will work over complex numbers. Unless stated otherwise, an open subset is in classical topology.

Definition 1.1. Let $G = \mathbb{C}^n$ be the complex vector group of dimension n. An *equivariant compactification* of G is an algebraic G-action $A : G \times X \to X$ on a projective variety X of dimension n with a Zariski open orbit $O \subset X$. In particular, the orbit O is G-equivariantly biregular to G. Given a projective variety X, such an action A is called an *EC-structure* on X, in abbreviation of 'equivariant compactification-structure'. Let $A_1 : G \times X_1 \to X_1$ and $A_2 : G \times X_2 \to X_2$ be EC-structures on two projective varieties X_1 and X_2. We say that A_1 and A_2 are *isomorphic* if there exist a linear automorphism $F : G \to G$ and a biregular morphism $\iota : X_1 \to X_2$ with the commuting diagram

$$
\begin{array}{ccc}
G \times X_1 & \xrightarrow{A_1} & X_1 \\
(F,\iota) \downarrow & & \downarrow \iota \\
G \times X_2 & \xrightarrow{A_2} & X_2.
\end{array}
$$

In [HT], Hassett and Tschinkel studied EC-structures on projective space $X = \mathbb{P}^n$. They proved a correspondence between EC-structures on $\mathbb{P}^n$ and Artin local algebras of length $n + 1$. By classifying such algebras, they discovered that there are many distinct isomorphism classes of EC-structures on $\mathbb{P}^n$ if $n \geq 2$ and infinitely many of them if $n \geq 6$. They posed the question, whether a similar phenomenon occurs when X is a smooth quadric hypersurface. This was answered negatively in [S], using arguments along the line of Hassett–Tschinkel's approach. A further study was made in [AS], where the authors raised the corresponding question when X is a Grassmannian. Even for simplest examples like the Grassmannian of lines on $\mathbb{P}^4$, a direct generalization of the arguments in [HT] or [S] seems hard.

Our goal is to give a uniform conceptual answer to these questions, as follows.

Theorem 1.2. *Let X be a Fano manifold of dimension n with Picard number 1, different from $\mathbb{P}^n$. Assume that X has a family of minimal rational curves whose variety of minimal rational tangents (VMRT) $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is smooth. Then all EC-structures on X are isomorphic.*

The meaning of VMRT will be explained in the next section. A direct corollary is the following.

Corollary 1.3. *Let $X \subset \mathbb{P}^N$ be a projective submanifold of Picard number 1 such that for a general point $x \in X$, there exists a line of $\mathbb{P}^N$ passing through x and lying on X. If X is different from the projective space, then all EC-structures on X are isomorphic.*

It is well known that when X has a projective embedding satisfying the assumption of Corollary 1.3, some family of lines lying on X gives a family of minimal rational curves, for which the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is smooth (e.g., by Proposition 1.5 of [H]). Thus Corollary 1.3 follows from Theorem 1.2. Corollary 1.3 answers Arzhantsev–Sharoyko's question on Grassmannians and also gives a more conceptual answer to Hassett–Tschinkel's question on a smooth quadric hypersurface, as a Grassmannian or a smooth hyperquadric can be embedded into projective space with the required property. In fact, all examples of Fano manifolds of Picard number 1 known to the authors, which admit EC-structures, can be embedded into projective space with the property described in Corollary 1.3. These include all irreducible Hermitian symmetric spaces and some non-homogeneous examples coming from Proposition 6.14 of [FH].

Once it is correctly formulated, the proof of Theorem 1.2 is a simple consequence of the Cartan–Fubini type extension theorem in [HM2], as we will explain in the next section.

It is natural to ask whether an analogue of Theorem 1.2 holds for equivariant compactifications of other linear algebraic groups. Part of our argument can be generalized easily. We need, however, an essential feature of the vector group in Proposition 2.4: any linear automorphism of the tangent space at the origin comes from an automorphism of the group. New ideas are needed to generalize this part of the argument to other groups.

2. Proof of Theorem 1.2

Let us recall the definition of VMRT (see [HM1, H] for details). Let X be a Fano manifold. For an irreducible component $\mathcal{K}$ of the normalized space $\mathrm{RatCurves}^n(X)$ of rational curves on X (see [K] for definition), denote by $\mathcal{K}_x \subset \mathcal{K}$ the subscheme parameterizing members of $\mathcal{K}$ passing through $x \in X$. We say that $\mathcal{K}$ is a family of minimal rational curves if $\mathcal{K}_x$ is nonempty and projective for a general point $x \in X$. By a celebrated result of Mori, any Fano manifold has a family of minimal rational curves. Fix a family $\mathcal{K}$ of minimal rational curves. Denote by $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$ the universal family. Then ρ is a $\mathbb{P}^1$-bundle and for a general point $x \in X$, the fibre $\mu^{-1}(x)$ is complete. By associating a smooth point x of a curve in X to its tangent direction in $\mathbb{P}T_x(X)$, we have a rational map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$. The proper image of

τ is denoted by $\mathcal{C} \subset \mathbb{P}T(X)$ and called the total space of VMRT. For a general point $x \in X$, the fibre $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is called the VMRT of $\mathcal{K}$ at x. In other words, $\mathcal{C}_x$ is the closure of the union of tangent directions to members of $\mathcal{K}_x$ smooth at x.

The concept of VMRT is useful to us via the Cartan–Fubini type extension theorem, Theorem 1.2 in [HM2]. We will quote a simpler version, Theorem 6.8 of [FH], where the condition $\mathcal{C}_x \neq \mathbb{P}T_x(X)$ is used instead of our condition "X_1, X_2 different from projective space". It is well known that these two conditions are equivalent from [CMS].

Theorem 2.1. *Let X_1, X_2 be two Fano manifolds of Picard number 1, different from projective spaces. Let $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be a family of minimal rational curves on X_1 (resp. X_2). Assume that for a general point $x \in X_1$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X_1)$ is irreducible and smooth. Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be connected open subsets. Suppose there exists a biholomorphic map $\varphi : U_1 \to U_2$ such that for a general point $x \in U_1$, the differential $d\varphi_x : \mathbb{P}T_x(U_1) \to \mathbb{P}T_{\varphi(x)}(U_2)$ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}_{\varphi(x)}$. Then there exists a biregular morphism $\Phi : X_1 \to X_2$ such that $\varphi = \Phi|_{U_1}$.*

The irreducibility of $\mathcal{C}_x$ is automatic in our case. We prove a slightly more general statement.

Proposition 2.2. *Let X be a Fano manifold. Let H be a connected closed subgroup of a connected algebraic group G. Suppose that there exists a G-action on X with an open orbit $O \subset X$ such that H is the isotropy subgroup of a point $o \in O$. Then for any family of minimal rational curves on X, the VMRT $\mathcal{C}_x$ at $x \in O$ is irreducible.*

Proof. For a family $\mathcal{K}$ of minimal rational curves on X, let $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$ be the universal family. Since G is connected, we have an induced G-action on $\mathcal{K}$ and $\mathcal{U}$ such that ρ and μ are G-equivariant. For $\alpha \in \mu^{-1}(x)$, denote by $G \cdot \alpha$ the G-orbit of α and by $\mu_\alpha : G \cdot \alpha \to O$ the restriction of μ to the orbit. Then μ_α is surjective. Note that $\mu_\alpha^{-1}(x) = H_x \cdot \alpha$. Since $H_x \simeq H$ is connected, we see that μ_α has irreducible fibres.

Now to prove the irreducibility of $\mathcal{C}_x$, it suffices to prove the irreducibility of $\mathcal{K}_x$. This is equivalent to showing that the fibres of $\mu : \mathcal{U} \to X$ are connected. Suppose not. Let

$$\mathcal{U} \xrightarrow{\eta} \mathcal{U}' \xrightarrow{\nu} X$$

be the Stein factorization, i.e., η has connected fibres and ν is a finite surjective morphism. By our assumption, ν is not birational. The image $\eta(G \cdot \alpha) \subset \mathcal{U}' \to X$ is a constructible subset of dimension equal to $\mathcal{U}'$. Thus it contains a Zariski dense open subset of $\mathcal{U}'$. Since $\mu_\alpha : G \cdot \alpha \to O$ has irreducible fibres, we see that $\nu|_{\eta(G \cdot \alpha)}$ is birational over O. It follows that ν is birational, a contradiction. $\square$

Although the next lemma is straightforward, this property of the vector group is essential for our proof of Theorem 1.2.

Lemma 2.3. *Let G be the vector group and let f be a linear automorphism of the tangent space $T_o(G)$ at the identity $o \in G$. Then there exists a group automorphism $F : G \to G$ such that $dF_o = f$ as endomorphisms of $T_o(G)$. In particular, denoting by $r_h : G \to G$ the multiplication (=translation) by $h \in G$, we have $dF_h \circ dr_h = dr_{F(h)} \circ dF_o$ as homomorphisms from $T_o(G)$ to $T_{F(h)}(G)$.*

Proof. The group automorphism F is just the linear automorphism f viewed as an automorphism of G via the "exponential map" $T_o(G) \cong G$. We can rewrite the group automorphism property as $F \circ r_h(g) = r_{F(h)} \circ F(g)$ for any $g, h \in G$. By taking differentials on both sides, we get $dF_h \circ dr_h = dr_{F(h)} \circ dF_o$. $\qquad \square$

Proposition 2.4. *Let G be the vector group. Let X_1, X_2 be two projective manifolds equipped with EC-structures $A_i : G \times X_i \to X_i, i = 1, 2$. Fix a general point $x_i \in X_i$ such that the orbit $O_i := A_i(G, x_i)$ is Zariski open in X_i. Let $a_i : G \to O_i$ be the biregular morphism given by $A_i(\cdot, x_i)$ and let $a_i^{-1} : O_i \to G$ be its inverse. Suppose we have $\mathcal{Z}_i \subset \mathbb{P}T(X_i), i = 1, 2$, closed subvarieties dominant over X_i such that*

- (1) *$\mathcal{Z}_i$ is invariant under the G-actions A_i, i.e., for any $g \in G$, the differential $dg_{x_i} : \mathbb{P}T_{x_i}(X_i) \to \mathbb{P}T_{g \cdot x_i}(X_i)$ sends $\mathcal{Z}_i \cap \mathbb{P}T_{x_i}(X_i)$ isomorphically to $\mathcal{Z}_i \cap \mathbb{P}T_{g \cdot x_i}(X_i)$; and*
- (2) *the two projective varieties $\mathcal{Z}_1 \cap \mathbb{P}T_{x_1}(X_1)$ and $\mathcal{Z}_2 \cap \mathbb{P}T_{x_2}(X_2)$ are projectively isomorphic.*

Then there exists a group automorphism F of G such that the biholomorphic map $\varphi : O_1 \to O_2$ defined by $\varphi = a_2 \circ F \circ a_1^{-1}$ satisfies

- (i) *$\varphi(x_1) = x_2$;*
- (ii) *$\varphi(g \cdot x_1) = F(g) \cdot \varphi(x_1)$ for any $g \in G$; and*
- (iii) *the differential*
$$d\varphi_u : \mathbb{P}T_u(X_1) \to \mathbb{P}T_{\varphi(u)}(X_2)$$
sends $\mathcal{Z}_1 \cap \mathbb{P}T_u(X_1)$ isomorphically to $\mathcal{Z}_2 \cap \mathbb{P}T_{\varphi(u)}(X_2)$ for all $u \in O_1$.

Proof. To simplify the notation, let us write $Z_x := \mathcal{Z}_i \cap \mathbb{P}T_x(X_i)$ when $x \in X_i$. Denote by $da_i : T(G) \to T(O_i)$ the isomorphism of vector bundles given by the differential of a_i. By the assumption (2), there exists a linear automorphism $f \in \mathbf{GL}(T_o(G))$ such that
$$da_2 \circ f \circ da_1^{-1} : T_{x_1}(X_1) \to T_{x_2}(X_2)$$
sends Z_{x_1} isomorphically to Z_{x_2}, i.e.,
$$f(da_1^{-1}(Z_{x_1})) = da_2^{-1}(Z_{x_2}).$$

Let F be the group automorphism of G induced by f in Lemma 2.3. It is clear that $\varphi = a_2 \circ F \circ a_1^{-1}$ is a biregular map satisfying (i). From G-equivariance of a_1 and a_2,
$$\varphi(g \cdot x_1) = a_2 \circ F \circ a_1^{-1}(g \cdot x_1) = a_2 \circ F(g \cdot o) = a_2(F(g) \cdot o) = F(g) \cdot a_2(o) = F(g) \cdot x_2.$$
This proves (ii). For any $u \in O_1$, let $h = a_1^{-1}(u)$, so that $u = h \cdot x_1$. Then using Lemma 2.3 and the condition (2),
$$\begin{aligned}
d\varphi_u(Z_u) &= da_2 \circ dF \circ da_1^{-1}(Z_{h \cdot x_1}) \\
&= da_2 \circ dF \circ dr_h \circ da_1^{-1}(Z_{x_1}) \\
&= da_2 \circ dr_{F(h)} \circ dF \circ da_1^{-1}(Z_{x_1}) \\
&= da_2 \circ dr_{F(h)} \circ f(a_1^{-1}(Z_{x_1})) \\
&= da_2 \circ dr_{F(h)} \circ da_2^{-1}(Z_{x_2}) \\
&= Z_{F(h) \cdot x_2} = Z_{\varphi(u)}.
\end{aligned}$$

The last equality is from $F(h) \cdot x_2 = a_2 \circ F(h) = a_2 \circ F(a_1^{-1}(u)) = \varphi(u)$. This shows (iii). $\qquad\square$

Proof of Theorem 1.2. Let $A_i : G \times X \to X, i = 1, 2$, be two EC-structures on X with open orbits $O_i \subset X$. Fix a family of minimal rational curves on X with the total space of VMRT $\mathcal{C} \subset \mathbb{P}T(X)$. Set $X_1 = X_2 = X$ and $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathcal{C}$. The conditions (1) and (2) of Proposition 2.4 are satisfied. We obtain two open subsets $O_1, O_2 \subset X$ with a biholomorphism $\varphi : O_1 \to O_2$ whose differential sends $\mathcal{C}_u$ to $\mathcal{C}_{\varphi(u)}$ for all $u \in O_1$. By Proposition 2.2, we can apply Theorem 2.1 to obtain an automorphism $\Phi : X \to X$ satisfying $\Phi|_{O_1} = \varphi$. From the property (ii) of φ in Proposition 2.4,

$$\Phi \circ A_1|_{G \times O_1} = A_2 \circ (F \times \Phi)|_{G \times O_1}.$$

Then the equality $\Phi \circ A_1 = A_2 \circ (F \times \Phi)$ must hold on the whole $G \times X$. Thus the two EC-structures A_1 and A_2 are isomorphic. $\qquad\square$

Acknowledgments

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Institute of Mathematics, AMSS, Chinese Academy of Sciences, 55 ZhongGuanCun East Road, Beijing 100190, China
E-mail address: bhfu@math.ac.cn

Korea Institute for Advanced Study, Hoegiro 87, Seoul 130-722, Korea
E-mail address: jmhwang@kias.re.kr