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LANTERN SUBSTITUTION AND NEW SYMPLECTIC 4-MANIFOLDS WITH $b_2{}^+ = 3$

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Abstract. Motivated by the construction of H. Endo and Y. Gurtas, changing a positive relator in Dehn twist generators of the mapping class group by using lantern substitutions, we show that 4-manifold $K3\#2\overline{\mathbb{CP}}^2$ equipped with the genus two Lefschetz fibration can be rationally blown down along six disjoint copies of the configuration C_2. We compute the Seiberg–Witten invariants of the resulting symplectic 4-manifold, and show that it is symplectically minimal. Using our example, we also construct an infinite family of pairwise non-diffeomorphic irreducible symplectic and non-symplectic 4-manifolds homeomorphic to $M = 3\mathbb{CP}^2\#(19-k)\overline{\mathbb{CP}}^2$ for $1 \leq k \leq 4$.

1. Introduction

There is a beautiful interplay between the algebra and the topology when one studies a Lefschetz fibration structure on a smooth 4-manifold. To every Lefschetz fibration over $\mathbb{S}^2$, one can associate a *factorization* of the identity element as a product of positive Dehn twists in the mapping class group of the fiber, and conversely, for such a factorization in the mapping class group, there is a corresponding Lefschetz fibration over $\mathbb{S}^2$ [18]. By the remarkable work of Donaldson [10, 11], every closed symplectic 4-manifold admits a structure of Lefschetz pencil, which can be blown up at its base points to yield a Lefschetz fibration. Conversely, Gompf and Stipsicz [18] showed that if $g \geq 2$, then the total space of a genus g Lefschetz fibration admits a symplectic structure. As one changes the identity word by conjugations and relations of the mapping class group, the corresponding Lefschetz fibration also changes topologically. There are many efforts in trying to understand what all the relations in the mapping class groups mean topologically. One of the well understood relations is the lantern relation, which corresponds to the symplectic operation of rational blowdown [12, 14].

In this paper, we start with the genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$ with global monodromy given by the relation $\varrho = (t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1})^6 = 1$ in the mapping class group M_2 of a closed genus two surface, where each c_i is a simple closed curve as in Figure 5 and t_{c_i} is a right-handed Dehn twist along the curve c_i, $i = 1, \ldots, 5$. We factorize the monodromy of the given Lefschetz fibration by a series of conjugations and braid relations to get a word upon which we can perform six lantern relation substitutions. Applying these lantern substitutions change the total space of our Lefschetz fibration topologically as a six rational blowdowns on $K3\#2\overline{\mathbb{CP}}^2$. Furthermore, using the Seiberg–Witten invariants, we show that the resulting symplectic 4-manifold

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is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$ and symplectically minimal. We would like to point out that the goal of this paper is not to construct exotic smooth structures on very small 4-manifolds with ${b_2}^+ = 3$, but rather to use the lantern relation substitutions to study smooth structures on various 4-manifolds. The study of exotic 4-manifolds with small Euler characteristics has been carried out by the first author and Park in [1–3].

In the following three sections, we present some background material and recall some results which will be needed in this paper. Section 2 discusses some well-known relations in the mapping class group, which will be used in our computations and study. In Section 3, we give a brief background information on Lefschetz fibrations, provide some examples that will be used to illustrate discussions in the paper and prove various results that will be needed in the sequel. In Sections 4 and 5, we recall the rational blowdown technique of Fintushel and Stern, provide the theorem proved by H. Endo and Y. Gurtas relating the lantern substitution to the rational blowdown operation, state results of R. Gompf on rational blowdown along smooth -4 sphere and discuss the knot surgery operation of Fintushel–Stern, respectively. Finally, in Section 6, we prove our main theorems.

2. Mapping class groups

Let Σ_{g} denote a two-dimensional, closed, oriented and connected surface of genus $g > 0$ surface.

Definition 1. Let $\mathrm{Diff}^+(\Sigma_{\mathrm{g}})$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_{\mathrm{g}} \to \Sigma_{\mathrm{g}}$ and $\mathrm{Diff}_0^+(\Sigma_{\mathrm{g}})$ be the subgroup of $\mathrm{Diff}^+(\Sigma_{\mathrm{g}})$ consisting of all orientation-preserving diffeomorphisms $\Sigma_{\mathrm{g}} \to \Sigma_{\mathrm{g}}$ that are isotopic to the identity. *The mapping class group* M_{g} of Σ_{g} is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_{g}, i.e.,

$$M_{\mathrm{g}} = \mathrm{Diff}^+(\Sigma_{\mathrm{g}})\,/\mathrm{Diff}_0^+(\Sigma_{\mathrm{g}})\,.$$

Definition 2. Let α be a simple closed curve on Σ_{g}. A *right-handed Dehn twist* t_α about α is the isotopy class of a self-diffeomorphism of Σ_{g} obtained by cutting the surface Σ_{g} along α and gluing the ends back after rotating one of the ends 2π to the right.

Notice that the conjugate of a Dehn twist is again a Dehn twist. Indeed, if $f : \Sigma_{\mathrm{g}} \to \Sigma_{\mathrm{g}}$ is an orientation-preserving diffeomorphism, then it is easy to check that $f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}$. Next, we briefly mention some relations that hold in the mapping class group M_{g}. This elementary fact and relations will be used quite often in our computation in Section 6.

2.1. Commutativity and braid relation.

Let α and β be two simple closed curves on Σ_{g}.

Lemma 3. *If α and β are disjoint, then we have the following commutativity relation: $t_\alpha t_\beta = t_\beta t_\alpha$. If α and β transversely intersect at a single point, then the corresponding Dehn twists satisfy the following braid relation: $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.*

For a proof, see [20].

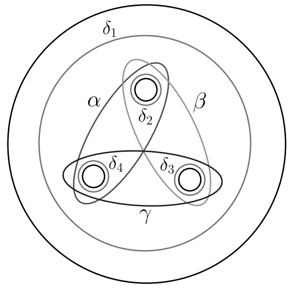

FIGURE 1. Lantern relation.

2.2. Lantern relation. Let $\Sigma_{0,4}$ be a sphere with four boundary components.

Lemma 4. *If $\delta_1, \delta_2, \delta_3$ and δ_4 are the boundary curves of $\Sigma_{0,4}$ and α, β and γ are the simple closed curves as shown in Figure 1, then we have*

$$t_\gamma t_\beta t_\alpha = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4},$$

where t_{δ_i}, $1 \leq i \leq 4$, denote the Dehn twists about δ_i.

For a proof, see [20].

This relation was known to Dehn. Later on it was rediscovered by D. Johnson and named as *lantern relation* by him [20]. For more results on lantern relation, see [21].

3. Lefschetz fibration

Let us first recall the definition of Lefschetz fibration.

Definition 5. Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a smooth map $f : X \longrightarrow \Sigma_h$, where Σ_h is a compact, oriented, smooth 2-manifold of genus h, such that f is surjective and each critical point of f has an orientation-preserving chart on which $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is given by $f(z_1, z_2) = {z_1}^2 + {z_2}^2$.

It is a corollary of the Sard's theorem that f is a smooth fiber bundle away from finitely many critical points, say $p_1, \ldots, p_k$. The genus of the regular fiber of f is defined to be the genus of the Lefschetz fibration. If a fiber passes through critical point set $p_1, \ldots, p_k$, then it is called a singular fiber which is an immersed surface with a single transverse self-intersection. A singular fiber can be described by its monodromy, an element of the mapping class group M_{g}, where g is the genus of the Lefschetz fibration. This element is a right-handed Dehn twist along a simple closed curve on Σ, called the *vanishing cycle*. If this curve is a non-separating curve, then the singular fiber is called *non-separating*, otherwise it is called *separating*. For a genus g Lefschetz fibration over S^2, the product of right-handed Dehn twists t_{α_i} along the vanishing cycles α_i, for $i = 1, \ldots, k$, gives us the global monodromy of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2}, \ldots, t_{\alpha_k} = 1$ in M_{g}. Conversely, such a relation defines a genus g Lefschetz fibration over S^2 with the vanishing cycles $\alpha_1, \ldots, \alpha_k$.

Let c_1, c_2, c_3, c_4 and c_5 be the simple closed curves as in Figure 5. For convenience, we shall denote the right-handed Dehn twists t_{c_i} along the curve c_i by c_i. It is well-known that the following relations hold in the mapping class group M_2:

$$\begin{aligned} (c_1c_2c_3c_4{c_5}^2c_4c_3c_2c_1)^2 &= 1, \\ (c_1c_2c_3c_4c_5)^6 &= 1, \\ (c_1c_2c_3c_4)^{10} &= 1. \end{aligned} \tag{3.1}$$

For each relation above, we have a corresponding genus two Lefschetz fibrations over S^2 with total spaces $\mathbb{CP}^2\#13\overline{\mathbb{CP}}^2$, $K3\#2\overline{\mathbb{CP}}^2$, and the Horikawa surface H, respectively. In this paper, we will consider genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ with global monodromy $(c_1c_2c_3c_4c_5)^6 = 1$.

The following result is well-known. For the convenience of the reader, we sketch the proof.

Lemma 6. *The genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2 with the monodromy factorization $(c_1c_2c_3c_4c_5)^6 = 1$ can be obtained as the double branched covering of $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ branched along a smooth algebraic curve B in the linear system $|6\tilde{L}|$, where $\tilde{L}$ is the proper transform of line L in $\mathbb{CP}^2$ avoiding the blown-up point.*

Proof. We closely follow the proof of Lemma 3.1 in [4], see also the discussion in [5]. Let D denote a degree d algebraic curve in $\mathbb{CP}^2$. Fix a generic projection $\pi : \mathbb{CP}^2\backslash pt \to \mathbb{CP}^1$ whose pole does not belong to D. According to the work of B. Moishezon and M. Teicher, the braid monodromy describing a degree d branch curve D in $\mathbb{CP}^2$ is given by a braid factorization. In fact, it is shown that the braid monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central element Δ^2 in B_d, can be written as the product of the monodromies about the critical points of the projection map π. More precisely, the following factorization $\Delta^2 = (\sigma_1 \ldots \sigma_{d-1})^d$ holds in the braid group B_d, where σ_i is a positive half-twist exchanging two points, and fixing the remaining $d-2$ points.

We first degenerate the smooth algebraic curve B in $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ into a union of six lines in a general position (see Figure 2). The braid group factorization corresponding to the configuration B is given by $\Delta^2 = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^6$. By lifting this braid factorization to the mapping class group of the genus two surface, we obtain that the monodromy factorization $(c_1c_2c_3c_4c_5)^6 = 1$ for the corresponding double branched covering.

Notice that a regular fiber of the given fibration is a two-fold cover of a sphere with homology class $f = h - e_1$ branched over six points, where h denotes the hyperplane class in $\mathbb{CP}^2$. Thus, a regular fiber is a surface of genus two. The exceptional sphere e_1 in $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, which intersects $f = h - e_1$ positively at one point, gives rise to two disjoint -1 sphere sections to the given genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$. □

Below we prove key proposition, which plays an important role in the proof of our main theorems. The following proposition also gives an alternative and convenient way of thinking the genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2 with the monodromy $(c_1c_2c_3c_4c_5)^6 = 1$.

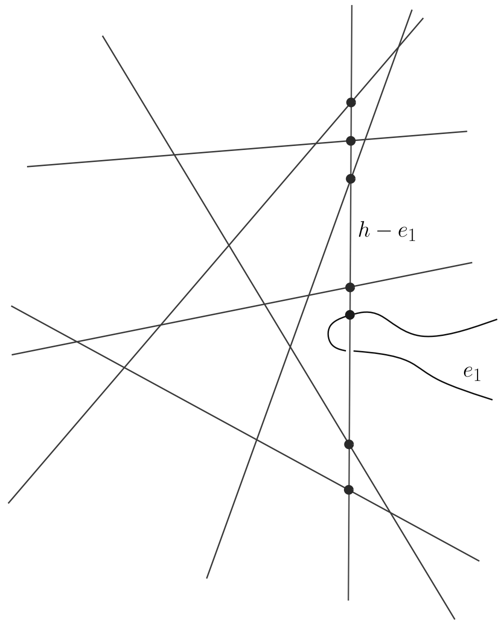

FIGURE 2. Branch locus for $K3\#2\overline{\mathbb{CP}}^2$.

Proposition 7. *The genus two Lefschetz pencil on $K3$ with two base points given above can be constructed by symplectic fiber summing two copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ along a regular torus fiber.*

Proof. Since the fiber of the elliptic fibration on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ is a blow up of a generic cubic curve in $\mathbb{CP}^2$, its homology class is equal to $f = 3h - e_1 - e_2 - \cdots - e_9$, where h denotes the hyperplane class in $\mathbb{CP}^2$ and e_i is the homology class of the exceptional sphere of the ith blow up. We first consider the pencil of lines in $\mathbb{CP}^2$ all passing through the fixed point p_1 away from the cubic curve C in $\mathbb{CP}^2$. Observe that each generic line L, which is a sphere of self-intersection 1, in this pencil intersects the cubic curve C at three distinct points by Bezout's theorem. Furthermore, by applying the Riemann–Hurwitz formula to the restriction of the map $\pi : \mathbb{CP}^2\backslash p_1 \to \mathbb{CP}^1$ to C, we compute the degree of the ramification divisor R: $\deg(R) = 2g(C)-2-3(2g(\mathbb{CP}^1)-2) = 6$. Consequently, for a generic smooth cubic C, there are exactly six lines in the pencil above that are tangent to C.

Next, we choose the regular torus fiber F along which the fiber sum of two copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ will be performed. Since the generic elliptic fiber of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ intersects the lines of the corresponding pencil at three points (see Figure 4), the generic line of the pencil in each $E(1)$ intersects the boundary of $E(1)\backslash\nu(F)$ in three disjoint circles. We choose a gluing diffeomorphism $\psi = id_F \times$ (complex conjugation), that identities these circles as in Figure 3 to obtain a pencil of genus two curves in $E(2) = K3$ surface. Since the pencils in each copy of $E(1)$ are holomorphic and the gluing map on the boundary 3-torus is identity on the elliptic fiber, the resulting genus two pencil is holomorphic as well. By holomorphically blowing up this genus two pencils at base points of the pencil p_1 and p_2 in $K3$ surface, we obtain

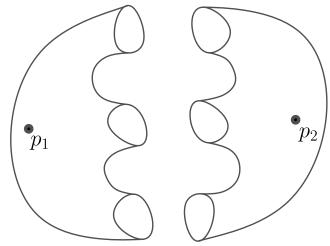

FIGURE 3. A genus two surface via fiber sums of $E(1)$.

the genus two holomorphic fibration on $K3\#2\overline{\mathbb{CP}}^2$. This holomorphic fibration has six singular fibers resulting from the gluing of six tangent lines mentioned above in each copy of $E(1)$. These singular fibers are not a Lefschetz type, and each singularity is topologically a sphere. Furthermore, using the analysis of the singular fibers, we see that each singular fiber can be perturbed into five Lefschetz type singularities with non-separating vanishing cycles (see the braid monodromy discussion in [4], p. 5). Finally, using the result of Siebert and Tian, Theorem A in [23], on holomorphicity of genus two Lefschetz fibration, we see that the fibration is isomorphic to the one given in Lemma 6. The isomorphism of fibrations also follows from the proof of Lemma 6 (see also discussion in [4], p. 5) by considering the degree, and the braid monodromy of the ramification divisor. First, we view the fiber sum of two copies of $E(1)$ as a two-fold ramified branched cover of $E(1)$ along the smooth divisor in the class $6h - 2e_1 - 2e_2 - \cdots - 2e_9 = 2F_{E(1)} = 2(-K_{E(1)})$, twice the anticanonical divisor of $E(1)$. A pencil of lines in $\mathbb{CP}^2$ with one base point gives rise to a pencil of lines in $E(1)$ with one base point assuming that we blow up the pencil nine times away from the basepoint. Since a generic line in the pencil, which has class h, intersects the ramification divisor $6h - 2e_1 - 2e_2 - \cdots - 2e_9$ at six points, it determines a genus two pencil under two-fold cover. Notice that the braid monodromy of a smooth plane curve of degree six (i.e., of a ramification divisor), can be computed by degenerating it into the union of six lines in generic position and given by $(c_1c_2c_3c_4c_5)^6 = 1$. $\square$

Remark 8. This description of the genus two pencil allows us to see rim tori and Gompf nuclei in elliptic surface $K3$ (see Example 9 below and [17] for an explanation).

Example 9. In this example, we study $K3$ surfaces $E(2) = E(1)\#_{\mathbb{T}^2}E(1)$ in some detail. As a consequence of our discussion, we will also derive some useful facts about the genus two Lefschetz pencil on $K3$ with two base points given above. Let us think of $K3$ surface as the fiber sum of two copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ along a torus fiber as in Proposition 7. We choose the following basis for the intersection form of $E(1)$: $< f = 3h - e_1 - \cdots - e_9,\ e_9,\ e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_7 - e_8,\ -h + e_6 + e_7 + e_8 >$. Note that the last eight classes can be represented by spheres of self-intersection -2 and generate the intersection matrix $-E_8$, where E_8 the matrix corresponding to the Dynkin diagram of the exceptional Lie algebra E_8. The class f is fiber of an elliptic fibration on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$ and e_9 is a sphere section of self-intersection -1. When we perform the fiber sum along torus to obtain $E(2)$, it is not hard to see the surfaces that generate the intersection form $2(-E_8) \oplus 3H$ for $E(2)$, where H is a hyperbolic pair. The two copies of the Milnor fiber $\Phi(1) \in E(1)$ are in $E(2)$, providing 16 spheres

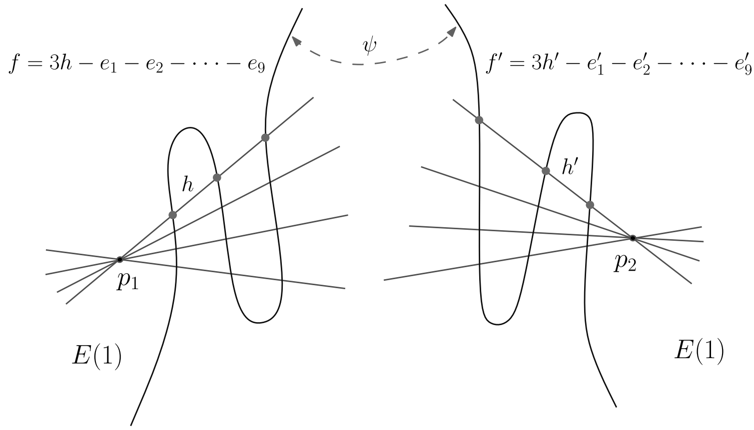

FIGURE 4. The pencil of genus two curves in $K3$ surface.

of self-intersection -2 (corresponding to the classes $\{e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_7 - e_8,\ -h + e_6 + e_7 + e_8\}$ and $\{e_1{}' - e_2{}',\ e_2{}' - e_3{}',\ \ldots,\ e_7{}' - e_8{}',\ -h' + e_6{}' + e_7{}' + e_8{}'\}$ mentioned above), realize two copies of $-E_8$. One copy of hyperbolic pair H comes from an identification of the torus fibers f and f', and a sphere section σ of self-intersection -2 obtained by sewing the sphere sections e_9 and $e_9{}'$, i.e., from the Gompf's nucleus $N(2)$ in $E(2)$. The remaining two copies of H come from 2 rim tori and their dual -2 spheres (see discussion in [18], p. 73). These 22 classes (19 spheres and 3 tori) generate H_2 of $E(2)$. Since $c_1(E(n)) = (2-n)f$, $E(2)$ has a trivial canonical class. As we can see from Figure 4, the class of the genus two surface of square 2 of the genus two Lefschetz pencil on $K3$ is $h + h'$, we simply add the homology classes of the surfaces. As a consequence, the class of the genus two fiber in $K3\#2\overline{\mathbb{CP}}^2$ is given by $h + h' - E_1 - E_2$, where E_1 and E_2 are the homology classes of the exceptional spheres of the blow ups at the points p_1 and p_2. We can also verify the symplectic surface Σ, given by the class $h + h' - E_1 - E_2$, has genus two by applying the adjunction formula to $(K3\#2\overline{\mathbb{CP}}^2, \Sigma)$: $g(\Sigma) = 1 + 1/2(K_{K3\#2\overline{\mathbb{CP}}^2} \cdot [\Sigma] + [\Sigma]^2) = 1 + ((E_1 + E_2) \cdot (h + h' - E_1 - E_2) + (h + h' - E_1 - E_2)^2)/2 = 1 + (2+0)/2 = 2$. The reader can notice from the intersection form of $K3$ that both rim tori has no intersections with the genus surface in the pencil given by the homology class $h + h'$. Thus, the genus two fiber Σ is disjoint from the rim tori that descend to $K3\#2\overline{\mathbb{CP}}^2$.

4. Rational blowdown and lantern relations

The basic idea of the rational blowdown surgery is that if a smooth 4-manifold X contains a particular configuration C_p of transversally intersecting 2-spheres whose boundary is the lens space $L(p^2, 1-p)$, then one can replace C_p with rational ball B_p to construct a new manifold X_p. If one knows the Seiberg–Witten invariants of the original manifold X, then one can determine the same invariants of X_p. Below we briefly discuss the rational blowdown. We refer the reader to [14, 22] for full details.

Let $p \geq 2$ and C_p be the smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere according to the following linear diagram:

$$\underset{u_{p-1}}{\overset{-(p+2)}{\bullet}} \longrightarrow \underset{u_{p-2}}{\overset{-2}{\bullet}} \longrightarrow \cdots\cdots \longrightarrow \underset{u_1}{\overset{-2}{\bullet}}$$

where each vertex u_i of the linear diagram represents a disk bundle over 2-sphere with the given Euler number.

According to Casson and Harer [6], the boundary of C_p is the lens space $L(p^2, 1-p)$ which also bounds a rational ball B_p with $\pi_1(B_p) = \mathbb{Z}_p$ and $\pi_1(\partial B_p) \to \pi_1(B_p)$ surjective. If C_p is embedded in a 4-manifold X then the rational blowdown manifold X_p is obtained by replacing C_p with B_p, i.e., $X_p = (X \backslash C_p) \cup B_p$. If X and $X \backslash C_p$ are simply connected, then so is X_p.

Lemma 10. *$b_2^+(X_p) = {b_2}^+(X)$, $\sigma(X_p) = \sigma(X) + (p-1)$, ${c_1}^2(X_p) = {c_1}^2(X) + (p-1)$ and $\chi_h(X_p) = \chi_h(X)$.*

Proof. Notice that the manifold C_p is negative definite, we have ${b_2}^+(X_p) = {b_2}^+(X)$ and ${b_2}^-(X_p) = {b_2}^-(X) - (p-1)$. Thus, $\sigma(X_p) = \sigma(X) + (p-1)$. Using the formulas ${c_1}^2 = 3\sigma + 2e$ and $\chi_h = (\sigma + e)/4$, we have ${c_1}^2(X_p) = 3\sigma(X_p) + 2e(X_p) = 3(\sigma(X) + (p-1)) + 2(e(X) - (p-1)) = {c_1}^2(X) + (p-1)$ and $\chi_h(X_p) = (\sigma(X) + (p-1) + e(X) - (p-1))/4 = \chi_h(X)$. $\square$

Theorem 11 ([14, 22])**.** *Suppose X is a smooth 4-manifold with $b_2^+(X) > 1$ which contains a configuration C_p. If L is a SW basic class of X satisfying $L \cdot u_i = 0$ for any i with $1 \leq i \leq p-2$ and $L \cdot u_{p-1} = \pm p$, then L induces a SW basic class $\bar{L}$ of X_p such that $SW_{X_p}(\bar{L}) = SW_X(L)$.*

Theorem 12 ([14, 22])**.** *If a simply connected smooth 4-manifold X contains a configuration C_p, then the SW-invariants of X_p are completely determined by those of X. That is, for any characteristic line bundle $\bar{L}$ on X_p with $SW_{X_p}(\bar{L}) \neq 0$, there exists a characteristic line bundle L on X such that $SW_X(L) = SW_{X_p}(\bar{L})$.*

In this paper, we only use the rational blowdown surgery along configuration C_2, i.e., the rational blowdowns along the -4 spheres.

In Section 6, we shall use the following theorem of H. Endo and Y. Gurtas.

Theorem 13. *Let ϱ, ϱ' be positive relators of $\mathcal{M}_g$ and $M_\varrho, M_{\varrho'}$ the corresponding Lefschetz fibrations over S^2, respectively. If ϱ' is obtained by applying a lantern substitution to ϱ, then the 4-manifold $M_{\varrho'}$ is a rational blowdown of M_ϱ along a configuration $C_2 \subset M_\varrho$.*

We will also need the following lemmas, which are due to R. Gompf, to analyze the symplectic 4-manifolds constructed in Section 6. For the proof, we refer the reader to [8,17]. See also the work of Dorfmeister [7,8] who gives a related criteria on symplectic minimality and how the symplectic Kodaira dimension changes under the rational blowdown along a symplectic -4 sphere (see also related work in [9]).

Lemma 14. *Let (X, V_X) be a relatively minimal smooth pair with V_X an embedded -4 sphere. If X contains a smoothly embedded exceptional sphere transversely intersecting the hypersurface V_X in a single positive point, then the manifold obtained under -4 blowdown of V_X is diffeomorphic to the blowdown of X along this sphere.*

Lemma 15. *Let (X, V_X) be a relatively minimal smooth pair with V_X an embedded -4 sphere. If X contain two disjoint smoothly embedded exceptional spheres each transversely intersecting the hypersurface V_X in a single positive point, then the manifold obtained under -4 blowdown of V_X is diffeomorphic to the blowdown of X along one of these spheres.*

5. Knot surgery

Let X be a 4-manifold with $b_2{}^+(X) > 1$ and contain a homologically essential torus T of self-intersection 0. Let $N(K)$ be a tubular neighborhood of K in S^3 and let $T \times D^2$ be a tubular neighborhood of T in X. The knot surgery manifold X_K is defined by $X_K = (X\backslash(T \times D^2)) \cup (S^1 \times (S^3\backslash N(K))$ where two pieces are glued in a way that the homology class of $[pt \times \partial D^2]$ is identified with $[pt \times \lambda]$ where λ is the class of the longitude of knot K. Fintushel and Stern [15] proved the theorem that shows Seiberg–Witten invariants of X_K can be completely determined by the Seiberg–Witten invariant of X and the Alexander polynomial of K. Furthermore, if X and $X\backslash T$ are simply connected, then so is X_K.

Theorem 16. *Suppose that $\pi_1(X) = \pi_1(X\backslash T) = 1$ and T lies in a cusp neighborhood in X. Then X_K is homeomorphic to X and Seiberg–Witten invariants of X_K is $SW_{X_K} = SW_X \cdot \Delta_K(t^2)$, where $t = t_T$ and Δ_K is the symmetrized Alexander polynomial of K. If the Alexander polynomial $\Delta_K(t)$ of knot K is not monic, then X_K admits no symplectic structure. Moreover, if X is symplectic and K is a fibered knot, then X_K admits a symplectic structure.*

We refer the reader to [15] for the details.

6. Construction of exotic 4-manifolds via lantern substitution

In this section, we first construct a simply connected, minimal symplectic 4-manifold X homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$ starting from $K3\#2\overline{\mathbb{CP}}^2$ and applying the sequence of six rational blowdowns via lantern substitutions. Next, by performing knot surgery on a homologically essential torus of self-intersection 0, we obtain an infinite family of simply connected, symplectic and non-symplectic 4-manifolds all homeomorphic but not diffeomorphic to X. Using Seiberg–Witten invariants, we distinguish their smooth structures.

Theorem 17. *There exists an irreducible symplectic 4-manifold X homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$ that can be obtained using the genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2 with global monodromy given by the relation $\varrho = (c_5c_4c_3c_2c_1)^6 = 1$ in the mapping class group M_2 by applying six lantern substitutions.*

In order to prove this theorem, we first prove the following two lemmas.

Lemma 18. *The word $c_5c_4c_3c_2c_1c_5c_4c_3c_2c_1$ in the mapping class group M_2 can be conjugated to contain the lantern relation in three different ways.*

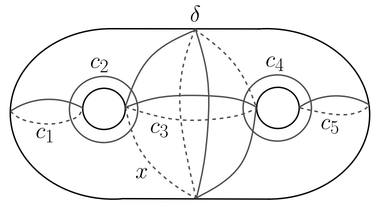

FIGURE 5. The curves c_1, c_2, c_3, c_4 and c_5.

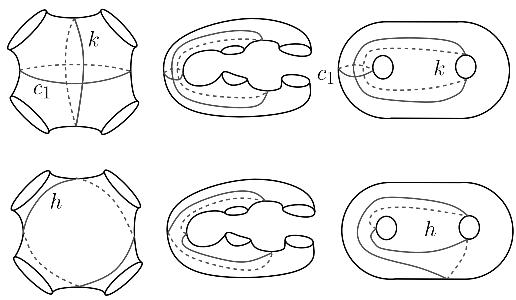

FIGURE 6. The loops c_1, c_2, c_3, c_4 and c_5.

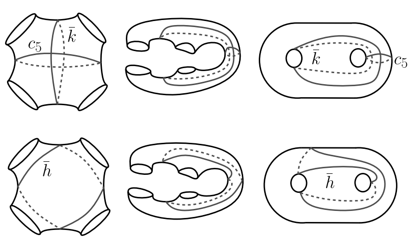

FIGURE 7. Special curves k, h.

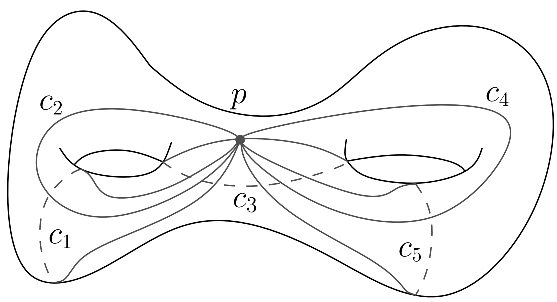

FIGURE 8. Special curves $\bar{k}$, $\bar{h}$.

Proof. Below we denote the lantern relation substitution, the braid relation substitution, the conjugation and the arrangement using the commutativity by $\xrightarrow{L}$, $\xrightarrow{B}$, $\xrightarrow{C}$, $\sim$, respectively. For the convenience of the reader, we have highlighted the changes that occur in each step.

Let us consider the following cases:

- Making the lantern substitution $c_5c_1^2c_5$ for $c_3\delta x$.

$$\begin{aligned} c_5c_4c_3c_2c_1c_5c_4c_3c_2c_1 &\sim {}_{c_5}(c_4)\cdot c_3c_2c_5c_1c_5c_1c_4c_3\cdot {}_{c_1^{-1}}(c_2) \\ &\sim {}_{c_5}(c_4)\cdot c_3c_2\cdot c_5^2c_1^2\cdot c_4c_3\cdot {}_{c_1^{-1}}(c_2) \\ &\xrightarrow{L} {}_{c_5}(c_4)\cdot c_3c_2\cdot c_3\delta x\cdot c_4c_3\cdot {}_{c_1^{-1}}(c_2) \end{aligned}$$

- Making the lantern substitution $c_1c_3c_1c_3$ for $\bar{k}\bar{h}c_5$.

$$\begin{aligned} c_5c_4c_3c_2c_1c_5c_4c_3c_2c_1 &\sim c_5c_4c_5c_3c_2c_4c_1c_3c_2c_1 \\ &\sim c_5c_4c_5\cdot {}_{c_3}(c_2c_4)\cdot c_1^2c_3^2\cdot {}_{c_1^{-1}}(c_2) \\ &\xrightarrow{L} c_5c_4c_5\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2) \end{aligned}$$

- Making the lantern substitution $c_3c_5^2c_3$ for c_1kh.

$$\begin{aligned} c_5c_4c_3c_2c_1c_5c_4c_3c_2c_1 &\sim {}_{c_5}(c_4)\cdot c_5c_3c_2c_5c_3\cdot (c_4)_{c_3}\cdot c_1c_2c_1 \\ &\sim {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_3^2c_5^2\cdot {}_{c_3^{-1}}(c_4)\cdot c_1c_2c_1 \\ &\xrightarrow{L} {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_1c_2c_1 \end{aligned}$$

□

Applying the above lemma several times, we next show how the word $\varrho = (c_5c_4c_3c_2c_1)^6 = 1$ can be conjugated to contain six lantern relation.

Lemma 19. *The global monodromy of genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2 given by the relation $\varrho = (c_5c_4c_3c_2c_1)^6 = 1$ can be conjugated and braid substituted to contain six lantern relations.*

Proof. We start with the identity word: $(c_5c_4c_3c_2c_1)^6 = 1$.

By applying three identities of Lemma 18, we can convert the word $(c_5c_4c_3c_2c_1)^6 = ((c_5c_4c_3c_2c_1)^2)^3$ into the following word:

$\sim {}_{c_5}(c_4)\cdot c_3c_2\cdot c_3\delta x\cdot c_4c_3\cdot {}_{c_1^{-1}}(c_2)\cdot c_5c_4c_5\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_1c_2c_1$

$\sim {}_{c_5}(c_4)\cdot c_3c_2\cdot {}_{c_3}(\delta x)\cdot {}_{c_3}(c_4)\cdot c_3^2\cdot {}_{c_1^{-1}}(c_2)\cdot c_5^2\cdot {}_{c_5^{-1}}(c_4)\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_1c_2c_1$

$\xrightarrow{B} {}_{c_5}(c_4)\cdot c_3c_2\cdot {}_{c_3}(\delta x)\cdot {}_{c_3}(c_4)\cdot c_3^2\cdot {}_{c_1^{-1}}(c_2)\cdot c_5^2\cdot {}_{c_5^{-1}}(c_4)\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_2c_1c_2$

$\sim {}_{c_5}(c_4)\cdot c_3c_2\cdot {}_{c_3}(\delta x)\cdot {}_{c_3}(c_4)\cdot c_3^2c_5^2\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5^{-1}}(c_4)\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_2c_1c_2$

$\xrightarrow{L} {}_{c_5}(c_4)\cdot c_3c_2\cdot {}_{c_3}(\delta x)\cdot {}_{c_3}(c_4)\cdot c_1kh\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5^{-1}}(c_4)\cdot {}_{c_3}(c_2c_4)\cdot \bar{k}\bar{h}c_5\cdot {}_{c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)\cdot c_2c_1c_2$

$$\sim {}_{c_5}(c_4) \cdot c_3c_2c_1 \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2c_1c_2$$

$$\xrightarrow{C} c_1c_2 \cdot {}_{c_5}(c_4) \cdot c_3c_2c_1 \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\sim {}_{c_5}(c_4) \cdot c_1c_2c_3c_2c_1 \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\xrightarrow{B} {}_{c_5}(c_4) \cdot c_1c_3c_2c_3c_1 \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\sim {}_{c_5}(c_4) \cdot c_1^2c_3^2 \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\xrightarrow{L} {}_{c_5}(c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5^{-1}}(c_4) \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\sim {}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot c_4c_5 \cdot {}_{c_3}(c_2c_4) \cdot \bar{k}\bar{h}c_5 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\sim {}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2) \cdot c_4 \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h}) \cdot c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\sim c_4 \cdot {}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h}) \cdot c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2$$

$$\xrightarrow{C} {}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h}) \cdot c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_2c_4$$

$$\sim {}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h}) \cdot c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_4c_2$$

$$\xrightarrow{C} c_2 \cdot {}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5} \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h}) \cdot c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_4$$

$$\sim {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h})\big) \cdot c_2c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot {}_{c_3^{-1}}(c_4) \cdot c_4$$

$$\xrightarrow{B} {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4) \cdot \bar{k}\bar{h} \cdot {}_{c_1^{-1}c_3^{-1}}(c_2) \cdot {}_{c_3}(\delta x) \cdot {}_{c_3c_5}(c_4) \cdot kh \cdot {}_{c_1^{-1}}(c_2)\big) \cdot {}_{c_3c_5}(c_2c_4) \cdot {}_{c_5}(\bar{k}\bar{h})\big) \cdot c_2c_5^2 \cdot {}_{c_1^{-1}}(c_2) \cdot {}_{c_5}(c_4) \cdot {}_{c_3}(c_2) \cdot c_1kh \cdot c_4c_3$$

$$\sim \ {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4)\cdot \bar{k}\bar{h}\cdot {}_{c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_3}(\delta x)\cdot {}_{c_3c_5}(c_4)\cdot kh\cdot {}_{c_1^{-1}}(c_2)\big)\cdot {}_{c_3c_5}(c_2c_4)\cdot {}_{c_5}(\bar{k}\bar{h})\big)\cdot c_5^2\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot c_2\cdot {}_{c_3}(c_2)\cdot c_1kh\cdot c_4c_3$$

$$\xrightarrow{B} \ {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4)\cdot \bar{k}\bar{h}\cdot {}_{c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_3}(\delta x)\cdot {}_{c_3c_5}(c_4)\cdot kh\cdot {}_{c_1^{-1}}(c_2)\big)\cdot {}_{c_3c_5}(c_2c_4)\cdot {}_{c_5}(\bar{k}\bar{h})\big)\cdot c_5^2\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5}(c_4)\cdot c_3c_2\cdot c_1kh\cdot c_4c_3$$

$$\sim \ {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4)\cdot \bar{k}\bar{h}\cdot {}_{c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_3}(\delta x)\cdot {}_{c_3c_5}(c_4)\cdot kh\cdot {}_{c_1^{-1}}(c_2)\big)\cdot {}_{c_3c_5}(c_2c_4)\cdot {}_{c_5}(\bar{k}\bar{h})\big)\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5^3}(c_4)\cdot c_5^2c_3c_2\cdot c_1kh\cdot c_4c_3$$

$$\sim \ {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4)\cdot \bar{k}\bar{h}\cdot {}_{c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_3}(\delta x)\cdot {}_{c_3c_5}(c_4)\cdot kh\cdot {}_{c_1^{-1}}(c_2)\big)\cdot {}_{c_3c_5}(c_2c_4)\cdot {}_{c_5}(\bar{k}\bar{h})\big)\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5^3}(c_4)\cdot c_5^2c_3^2\cdot {}_{c_3^{-1}}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)$$

$$\xrightarrow{L} \ {}_{c_2}\big({}_{c_4^{-1}}\big({}_{c_5}(c_4)\cdot \bar{k}\bar{h}\cdot {}_{c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_3}(\delta x)\cdot {}_{c_3c_5}(c_4)\cdot kh\cdot {}_{c_1^{-1}}(c_2)\big)\cdot {}_{c_3c_5}(c_2c_4)\cdot {}_{c_5}(\bar{k}\bar{h})\big)\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5^3}(c_4)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4).$$

After suitable conjugations, we simplify the long conjugations highlighted above to acquire the following word with 24 right-handed Dehn twists.

$$\sim \ {}_{c_4^{-1}c_5}(c_4)\cdot {}_{c_2c_4^{-1}}(\bar{k}\bar{h})\cdot {}_{c_2c_4^{-1}c_1^{-1}c_3^{-1}}(c_2)\cdot {}_{c_2c_4^{-1}c_3}(\delta x)\cdot {}_{c_2c_4^{-1}c_3c_5}(c_4)\cdot {}_{c_2c_4^{-1}}(kh)\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_2c_3c_5}(c_2c_4)\cdot {}_{c_2c_5}(\bar{k}\bar{h})\cdot {}_{c_2c_1^{-1}}(c_2)\cdot {}_{c_5^3}(c_4)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_2)\cdot c_1kh\cdot {}_{c_3^{-1}}(c_4)$$

Using a hyperelliptic signature formula for genus two Lefschetz fibration or by simple checking, we determine that 18 of these Dehn twists are along non-separating and 6 are along separating curves. Notice that each lantern relation introduces one separating vanishing cycle. □

Now we give a proof of main Theorem 17.

Proof. Let X be the symplectic 4-manifold obtained from $K3\#2\overline{\mathbb{CP}}^2$ by applying six lantern relation as in Lemma 19 above. Using Lemma 10, we compute that

$$\begin{aligned} e(X) &= e(K3\#2\overline{\mathbb{CP}}^2) - 6 = 26 - 6 = 20,\\ \sigma(X) &= \sigma(K3\#2\overline{\mathbb{CP}}^2) + 6 = (-18) + 6 = -12. \end{aligned}$$

Freedman's theorem (cf. [13]) implies that X is homeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$, once we show that $\pi_1(X) = 1$.

Let us denote a regular fiber of the genus two Lefschetz fibration on X as Σ_2 and the generators of fundamental group of Σ_2 as c_1, c_2, c_3, c_4 and c_5. We continue to use the notation of Figure 5, by choosing a base point p as in Σ_2 as in Figure 6. From the long exact homotopy sequence (see, for example [18], p. 290), we deduce that

$$\pi_1(\Sigma_2) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S^2) = 1.$$

Moreover, $\pi_1(X)$ is finitely presented group and generated by the images of the standard generators, which we again denote by c_1, c_2, c_3, c_4 and c_5, under the surjection map. Furthermore, the loops $\beta_1, \beta_2, \beta_3, \ldots, \beta_{23}, \beta_{24}$ on the regular fiber are all null-homotopic in $\pi_1(X)$, where $\beta_1, \beta_2, \beta_3, \ldots, \beta_{23}, \beta_{24}$ are the curves corresponding to 24 Dehn twists D_i of the genus two Lefschetz fibration on X given below:

$$\begin{gathered}
D_1 = {}_{c_4^{-1}c_5}(c_4), \quad D_2 = {}_{c_2c_4^{-1}}(\bar{k}), \quad D_3 = {}_{c_2c_4^{-1}}(\bar{h}), \quad D_4 = {}_{c_2c_4^{-1}c_1^{-1}c_3^{-1}}(c_2), \\
D_5 = {}_{c_2c_4^{-1}c_3}(\delta), \quad D_6 = {}_{c_2c_4^{-1}c_3}(x), \quad D_7 = {}_{c_2c_4^{-1}c_3c_5}(c_4), \quad D_8 = {}_{c_2c_4^{-1}}(k), \\
D_9 = {}_{c_2c_4^{-1}}(h), \quad D_{10} = {}_{c_2c_1^{-1}}(c_2), \quad D_{11} = {}_{c_2c_3c_5}(c_2), \quad D_{12} = {}_{c_2c_3c_5}(c_4), \\
D_{13} = {}_{c_2c_5}(\bar{k}), \quad D_{14} = {}_{c_2c_5}(\bar{h}), \quad D_{15} = {}_{c_2c_1^{-1}}(c_2), \quad D_{16} = {}_{c_5^3}(c_4), \\
D_{17} = c_1, \quad D_{18} = k, \quad D_{19} = h, \quad D_{20} = {}_{c_3^{-1}}(c_2), \quad D_{21} = c_1, \\
D_{22} = k, \quad D_{23} = h, \quad D_{24} = {}_{c_3^{-1}}(c_4).
\end{gathered}$$

We observe that $\beta_{21} = 1$ in $\pi_1(X)$, implies that c_1 is trivial element in $\pi_1(X)$. Using $\beta_{10} = \beta_{20} = 1$, we obtain c_2 and c_3 are trivial in $\pi_1(X)$. Furthermore, the relations $\beta_5 = \beta_{22} = \beta_{23} = \beta_{24} = 1$ imply that the elements c_4 and c_5 are null-homotopic and thus X is simply connected.

Using the blow up formula for the Seiberg–Witten function [15], we have $SW_{K3\#2\overline{\mathbb{CP}}^2} = SW_{K3} \cdot \prod_{j=1}^{2}(e^{E_i} + e^{-E_i}) = (e^{E_1} + e^{-E_1})(e^{E_2} + e^{-E_2})$, where E_i is an exceptional class coming from the ith blow up. Consequently, it follows from this formula, the set of basic classes of $K3\#2\overline{\mathbb{CP}}^2$ are given by $\pm E_1 \pm E_2$, and the Seiberg–Witten invariants on these classes are ± 1. Moreover, after performing two rational blowdowns along a copy of the configuration C_2, the resulting manifold is diffeomorphic to $K3$ by Lemma 15. Thus, the only basic class is the zero class, which descends from the top classes $\pm(E_1 + E_2)$ in $K3\#2\overline{\mathbb{CP}}^2$. Next, using the Corollary 8.6 in [14], we observe that X has Seiberg–Witten simple type. Furthermore, by applying Theorems 11 and 12, we completely determine the Seiberg–Witten invariants of X using the basic classes and invariants of $K3$: up to sign the symplectic manifold X has only one basic class which descends from the canonical class of $K3$. By Theorem 12 (or by Taubes theorem [24]), the value of the Seiberg–Witten function on these classes, $\pm K_X$, are ± 1.

Alternatively, we can determine the Seiberg–Witten invariants of X directly by computing the algebraic intersection number of the classes $\pm E_1 \pm E_2$, with the classes of -4 spheres of six C_2 configurations. Observe that these -4 spheres are the components of the singular fibers of $K3\#2\overline{\mathbb{CP}}^2$. Furthemore, by considering three regions on the genus two surface, where the rational blowdowns are performed (see Figures 7 and 8), and the location of the two points where we did blow up the genus two pencil (see Figures 3 and 4), we compute the intersection numbers as follows: let S denote the homology class of -4 sphere of C_2. We have $S \cdot E_1 = S \cdot E_2 = 1$. Consequently, $S \cdot \pm(E_1 + E_2) = \pm 2$ and $S \cdot \pm(E_1 - E_2) = 0$. Since among the four classes $\pm E_1 \pm E_2$ only $E_1 + E_2$ and $-(E_1 + E_2)$ have intersection ± 2 with -4 spheres of C_2, it follows from Theorem 11 that these are only two classes that descend to X.

Next, we apply the connected sum theorem for the Seiberg–Witten invariant and show that SW function is trivial for $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$. Since the Seiberg–Witten invariants are diffeomorphism invariants, we conclude that X is not diffeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}}^2$.

The minimality of X follows from the fact that X has no two basic classes K and K' such that $(K-K')^2 = -4$. Notice that $(K_X - (-K_X))^2 = 4({K_X}^2) = 16$ in our case. □

Theorem 20. *There exists an infinite family of irreducible symplectic and an infinite family of irreducible non-symplectic pairwise non-diffeomorphic 4-manifolds all homeomorphic to X.*

Proof. We will use both the branched cover and the fiber sum descriptions of $K3\#2\overline{\mathbb{CP}}^2$ given as in Lemma 6 and Proposition 7 to show that X contains at least two disjoint tori that are disjoint from the singular fibers of genus two Lefschetz fibration on $K3\#2\overline{\mathbb{CP}}^2$ over S^2. An alternative proof, using the homology classes, is given in Example 9. These tori descend from Gompf nucleus of $E(2) = K3$ (see Example 9), and survive in X after the rational blowdowns along C_2. Let us explain this more precisely using a geometric argument. First, note that according to the proof of Lemma 6 any genus two fiber, which arises from a two-fold cover of the sphere $h - e_1$, meets the torus $a \times b$ descending from an elliptic fiber of $K3$ at two points. Let $f_1, \ldots, f_6$ denote the complicated singular fibers of the genus two fibration on $K3\#2\overline{\mathbb{CP}}^2$. We perturb these singular fibers into Lefschetz type upon which we perform the rational blowdowns along C_2. Consider the tubular neighborhoods of these singular fibers in the manifold $K3\#2\overline{\mathbb{CP}}^2$. One should think of each neighborhood where we perturb one complicated singular fiber into five Lefschetz-type singular fibers. The normal disks of these neighborhoods on the torus $a \times b$ are denoted in Figure 9 as $\delta_1, \ldots, \delta_6$. The dots in the disks indicate there are five singulars fibers over each disk δ_i. We choose an open set U on the torus $a \times b$ which contains the disks $\delta_1, \ldots, \delta_6$, and away from the loops a and b (see Figure 9). Thus, we can assume that the rational blowdown surgeries have no effect on the outside the tubular neighborhood of U. Next, we will choose a rim circle μ away from the tubular neighborhood of U. The rim tori that are not affected by six rational blowdowns are $\mu \times a$ and $\mu \times b$ (Figure 9). Note that each of these tori has a dual sphere of self-intersection -2, arising from the dual circles b and a (see Example 9). These tori are Lagrangian, but we can perturb the symplectic form so that one of them, say $T = \mu \times a$ becomes symplectic. Moreover, $\pi_1(X \backslash T) = 1$, which follows from the Van Kampen's theorem using the facts that $\pi_1(X) = 1$ and the rim torus has a dual sphere (see Proposition 1.2 in [19] or Gompf [16], p. 564). Hence, we have a symplectic torus T in X of self-intersection 0 such that $\pi_1(X \backslash T) = 1$. By performing a knot surgery on T, inside X, we acquire an irreducible 4-manifold X_K that is homeomorphic to X. By varying our choice of the knot K, we can realize infinitely many pairwise non-diffeomorphic 4-manifolds, either symplectic or non-symplectic. □

Remark 21. Let $X(m)$ denote the symplectic 4-manifold obtained from $K3\#2\overline{\mathbb{CP}}^2$ by applying m copies of the lantern substitutions, performed in the order given as in Lemma 19, where $1 \leq m \leq 5$. Using Gompf's result 14 and 15, we observe that $X(1)$ and $X(2)$ are diffeomorphic to $K3\#\overline{\mathbb{CP}}^2$ and $K3$, respectively. Similarly as in Theorem 17, we show that $X(m)$ is an exotic copy of $3\mathbb{CP}^2\#(21-m)\overline{\mathbb{CP}}^2$ for $3 \leq m \leq 5$. Moreover, the knot surgery on torus yields the infinite family of symplectic and non-symplectic irreducible 4-manifolds all homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#(21-m)\overline{\mathbb{CP}}^2$. We leave the details to the reader as an exercise.

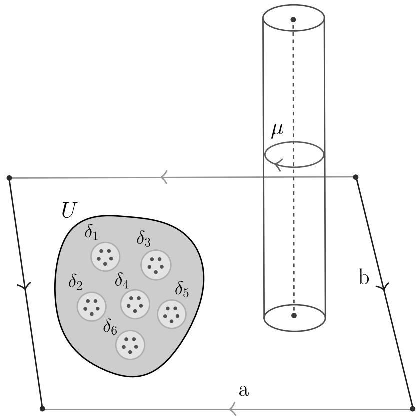

FIGURE 9. The rim tori in X.

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