

COMBINATORIAL METHODS FOR THE TWISTED COHOMOLOGY OF ARTIN GROUPS

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ABSTRACT. In this paper we introduce certain “combinatorial sheaves” on posets, which we call *weighted sheaves*, and we relate their cohomology, computed by a *weighted complex*, with the twisted cohomology of Artin groups. It turns out that to each Artin group one can associate a weighted sheaf, where the poset is given by the simplicial complex of all finite parabolic subgroups, and the cohomology of the Artin group with coefficients in a module of Laurent polynomials (interesting for geometrical reasons) is computed by the associated weighted complex. We connect this theory with the so called “Discrete Morse Theory”: a *weighted matching* on the weighted complex gives rise to a Morse complex computing the cohomology. We give a natural filtration of the weighted complex, which is compatible with the weighted matching, so obtaining a converging spectral sequence. We use such machinery to compute the twisted cohomology for all exceptional type affine Artin groups.

1. Introduction

Let (\mathbf{W}, S) be a (finitely generated) Coxeter system and let $\mathbf{G}_{\mathbf{W}}$ be its associated Artin group. For example, one can take as \mathbf{W} the symmetric group Σ_n with set of Coxeter generators the transpositions $(i, i + 1)$, and as $\mathbf{G}_{\mathbf{W}}$ the braid group Br_n . There is a naturally associated *configuration space* \mathbf{Y} where \mathbf{W} acts freely, which is conjectured to be a $K(\pi, 1)$ -space; such conjecture is verified for some (few) cases, including the case when \mathbf{W} is a finite group. In all cases, the topology of the *orbit space* $\mathbf{Y}_{\mathbf{W}} := \mathbf{Y}/\mathbf{W}$ is interesting to study. Its cellular description is given in [23] (see also [6–8, 21]), where a CW-complex $\mathbf{X}_{\mathbf{W}}$ which is a deformation retract is explicitly described.

We are interested in a natural representation of the Artin group over the ring of Laurent polynomials in one variable q , taking each standard generator to the q -multiplication. For finite-type Artin groups and Laurent polynomials over \mathbb{Q} , the cohomology with coefficients in such representation is fully computed (see [10, 11, 16], [9]; see [5] for Laurent polynomials over \mathbb{Z}). For affine-type Artin groups some cases are worked out (see [6–8]).

In this paper, we apply a new combinatorial procedure (introduced by the first author and partially described also in [20, 21, 24]) to calculate the twisted cohomology of the orbit space.

The main combinatorial ingredient is the introduction of a very special kind of sheaves over posets, which we call *weighted sheaves over posets*, which are interesting objects in themselves. The most important case for us is when such poset is the face poset of a simplicial complex K . We define a weighted sheaf as a triple (K, R, w)

where R is a ring (usually a Principal Ideal Domain, PID for brevity) and w is a map between the face poset of K and the ring R , which respects the inclusion relation in K and the divisibility relation in R : $\sigma \subset \tau \Rightarrow w(\sigma) | w(\tau)$. To these data one associates an algebraic complex L_* , which we call here the *weighted complex*, which is a torsion complex generated in dimension n as $L_n := \bigoplus_{|\sigma|=n} \frac{R}{(w(\sigma))} \bar{e}_\sigma$. The boundary is induced by the boundary in K and by the inclusion maps.

A weighted sheaf (K, R, w) is naturally associated to an Artin group. Here K is the simplicial complex, defined over S , of all subsets $J \subset S$ such that the parabolic subgroup \mathbf{W}_J is finite; R is the ring of Laurent polynomials, and $w(J)$ is the Poincaré polynomial of \mathbf{W}_J . From the homology of the weighted complex, one easily derives that of the orbit space, since the two are related in a long exact sequence, where (for affine type Artin groups) there are isomorphisms except in the top dimension.

One can localize the weighted complex at the irreducible factors of the Poincaré polynomials (which turn out to be cyclotomic polynomials); there is a natural filtration of the localized complexes, and therefore an associated spectral sequence whose E^1 term is computed as relative (untwisted) homology of simplicial subcomplexes of K (Theorem 1).

The second combinatorial ingredient is *Discrete Morse Theory*, which was introduced in [14, 15] (see also [19]). We use a small variation of the standard theory, by calling *weighted matching* on a weighted complex a matching \mathcal{M} in the standard sense on the poset of the faces in K , together with the additional condition that two matched elements must have the same weight. Such matching gives rise to an associated *Morse complex* $C_*^{\mathcal{M}}$: as a module, $C_*^{\mathcal{M}}$ is the torsion complex generated by the critical cells; the boundary is given by “combinatorial integration” of the flow over alternating paths.

We show that such notions fit very well into the theory of weighted sheaves. First, we prove that the Morse complex computes the (co-)homology of the weighted complex, so generalizing the known results for free modules (Theorem 2).

Second, the above filtration of the weighted complex induces a filtration onto the Morse complex and an isomorphism of spectral sequences (Theorems 3 and 4).

To apply the previous theory and perform computations, we take a suitable weighted matching on the poset of subgraphs of the Coxeter diagram of the group. We describe some natural ways to take such matchings, and we give in details the example of the affine case \tilde{F}_4 .

We succeed to make complete calculations (by hand) for all the exceptional affine-type Artin groups. We include at the end the tables containing all the homology groups and their generators.

2. Weighted sheaves over posets

Let (P, \prec) be a poset.

Definition 2.1. A sheaf of rings over P (also called *diagram of rings*) is a collection

$$\{R_x, x \in P\}$$

of commutative rings together with a collection of ring homomorphisms

$$\{\rho_{x,y} : R_y \rightarrow R_x, x \preceq y\}$$

satisfying

$$\begin{aligned} \rho_{x,x} &= id_{R_x}; \\ x \preceq y \preceq z &\Rightarrow \rho_{x,z} = \rho_{x,y} \rho_{y,z} \end{aligned}$$

(see also [1, 26]).

It is convenient to consider P as a category with objects all elements of P , and a unique morphism between x and y if $x \preceq y$ (or no morphism otherwise). So, one can see a sheaf of rings as a contravariant functor from P as a category to the category of rings.

We take now a *unique factorization domain* R (later we assume that R is a PID).

We consider in R the *transitive relation* given by divisibility in R : we write $x|y$ when $\exists z \in R$ such that $y = xz$. Analog to P and \preceq , the relation of divisibility gives R the structure of a small category (of course, such relation induces a structure of poset on the set R/R_* , where R_* is the multiplicative subgroup of units in R).

In particular, any morphism

$$w : (P, \prec) \rightarrow (R, |) : x \longrightarrow w(x) := w_x \in R$$

defines a sheaf over P by the collections

$$\{R/(w_x), x \in P\}$$

and

$$\{i_{x,y} : R/(w_y) \rightarrow R/(w_x)\},$$

where $i_{x,y}$ is the map induced by the identity of R , well defined since by definition $(w_y) \subset (w_x)$.

The following is the main definition of this paper.

Definition 2.2. We call the triple (P, R, w) a *weighted sheaf* over P and the coefficients w_x the *weights* of the sheaf.

Example 1. When the poset is given by a simplicial complex (or a simplicial scheme) K with partial ordering

$$(2.1) \quad \sigma \prec \tau \Leftrightarrow \sigma \subset \tau$$

then a weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w_\sigma \in R$, with

$$\sigma \prec \tau \Rightarrow w_\sigma | w_\tau.$$

Remark 2.3. Recall that the *order complex* $\Delta(P)$ of a poset P is the simplicial complex with vertex set P and q -simplices the $q + 1$ -chains of P .

A weighted sheaf (P, R, w) over a poset P induces a weighted sheaf $(\Delta(P), R, w')$ over $\Delta(P)$. It is sufficient to define the weight w' of a chain c of P as

$$w'(c) := w(\max(c)).$$

Remark 2.4. The previous definition 2.2 (and part of the following theory) could be extended to the more general setting where one associates to any $x \in P$ an ideal I_x in some fixed ring A , with the condition $x \prec y \Rightarrow I_y \subset I_x$. Our setting corresponds to the case, where I_x is a *principal ideal*, for all $x \in P$.

In the following, we consider mainly the case when the poset is a simplicial scheme K (see Example 1), defined over some finite set S .

Let $C_*^0(K)$ be the algebraic complex for the *reduced* simplicial homology of K with coefficients in R , shifted in dimension by one. In dimension k one has

$$(2.2) \quad C_k^0(K) = \bigoplus_{\substack{\sigma \in K, \\ |\sigma|=k}} R \cdot e_\sigma^0,$$

where e_σ^0 is a generator associated to a given orientation of σ . In dimension 0 by definition, we have $C_0^0(K) = R e_\emptyset^0$.

The boundary is given by

$$(2.3) \quad \partial^0(e_\sigma^0) = \sum_{|\tau|=k-1} [\tau : \sigma] e_\tau^0,$$

where $[\tau : \sigma]$ denotes the incidence number holding ± 1 if $\tau \prec \sigma$ and vanishing otherwise.

The second main definition concerns a modified complex for the homology of weighted sheaves over a simplicial scheme K .

Definition 2.5. The *weighted complex* associated to the weighted sheaf (K, R, w) is the algebraic complex

$$L_* := L_*(K)$$

defined by

$$L_k := \bigoplus_{|\sigma|=k} \frac{R}{(w_\sigma)} \bar{e}_\sigma, \quad k \geq 0,$$

and boundary

$$\partial : L_k \rightarrow L_{k-1}$$

induced by ∂^0 :

$$\partial(a_\sigma \bar{e}_\sigma) = \sum_{\tau \prec \sigma} [\tau : \sigma] i_{\tau, \sigma}(a_\sigma) \bar{e}_\tau.$$

The condition $\partial^2 = 0$ is straightforward from the definitions.

We will be interested to the homology $H_*(L_*)$, which we will refer to as the homology of the weighted sheaf (K, R, w) .

Remark 2.6. If (P, R, w) is a weighted sheaf over a poset P and $(K = \Delta(P), R, w')$ is the induced weighted sheaf as in Remark 2.3 then the associated weighted complex

$$L_*(K) = \bigoplus_{x_1 \prec \dots \prec x_k} \frac{R}{(w(x_k))} \bar{e}(x_1 \prec \dots \prec x_k)$$

has boundary

$$\partial(a \bar{e}(x_1 \prec \dots \prec x_k)) = \sum_{m=1}^k (-1)^m a' \bar{e}(x_1 \prec \dots \prec \widehat{x_m} \prec \dots \prec x_k),$$

where $a' := a$ if $m < k$, while $a' = i_{x_{k-1}, x_k}(a)$ if $m = k$.

2.0.1. *Decomposition and filtration of $L_*(K)$.* We assume hereafter that R is a PID.

Let $L := L_*(K)$ be the algebraic complex associated to (K, R, w) , where K is as in Example 1.

Each single factor $\frac{R}{(w_\sigma)}$ of L decomposes into cyclic modules according to the irreducible factorization of w_σ :

$$w_\sigma = \prod_i \varphi_i^{m_i} \Rightarrow \frac{R}{(w_\sigma)} \cong \bigoplus_i \frac{R}{(\varphi_i^{m_i})}.$$

Given any irreducible $\varphi \in R$, we define the φ -primary component of the weighted sheaf through the φ -adic valuation as follows. For $\sigma \in K$, let

$$v_\varphi(\sigma) := \text{max power of } \varphi \text{ dividing } w_\sigma$$

and let $w_{(\varphi)} : K \rightarrow R$ be defined by

$$w_{(\varphi)}(\sigma) := \varphi^{v_\varphi(\sigma)}.$$

Then $(K, R, w_{(\varphi)})$ defines a weighted sheaf, which we call the φ -primary component of the given sheaf, and denote also by $(K, R, w)_{(\varphi)}$. The associated complex is $L_{(\varphi)}$, defined in degree k by:

$$(2.4) \quad (L_{(\varphi)})_k := \bigoplus_{|\sigma|=k} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \bar{e}_\sigma.$$

There is a natural increasing filtration into subcomplexes of $L_{(\varphi)}$ by the powers of φ :

$$(2.5) \quad F^s(L_{(\varphi)}) := \bigoplus_{v_\varphi(\sigma) \leq s} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \bar{e}_\sigma.$$

The latter is associated to an increasing filtration of the simplicial complex K into subcomplexes:

$$K_{(\varphi),s} := \{\sigma \in K \mid v_\varphi(\sigma) \leq s\}.$$

Then $F^s(L_{(\varphi)})$ is the weighted complex associated to the weighted sheaf $(K_{(\varphi),s}, R, w_{(\varphi)}|_{K_{(\varphi),s}})$.

We get the following theorem.

Theorem 1. *Let (K, R, w) be a weighted sheaf, with associated weighted complex L_* . For any irreducible $\varphi \in R$, there exists a spectral sequence*

$$E_{p,q}^0 \Rightarrow H_*(L_{(\varphi)})$$

that abuts to the homology of the φ -primary component of the associated algebraic complex L_* .

Moreover the E^1 -term:

$$(2.6) \quad E_{p,q}^1 = H_{p+q}(F^p/F^{p-1}) \cong H_{p+q}(K_{(\varphi),p}, K_{(\varphi),p-1}; R/(\varphi^p))$$

is isomorphic to the relative homology with trivial coefficients of the simplicial complexes pair $(K_{(\varphi),p}, K_{(\varphi),p-1})$.

Proof. The only non-standard fact is the second isomorphism in (2.6), with a relative homology group with trivial coefficients. This follows straightforward since by definition the boundary ∂ of $L(\varphi)$ restricted to $F^p L(\varphi)/F^{p-1} L(\varphi)$ reduces to the standard boundary $\partial^0 \otimes id$ of $C_*^0 \otimes \frac{R}{(\varphi^p)}$. \square

3. Discrete Morse theory for weighted sheaves

We show here how the discrete Morse theory for complexes generalizes to weighted sheaves (we refer in this section to [14, 15, 19]).

Given x, y in a poset P , the notation $x \triangleleft y$ means that y covers x , that is $x \prec y$ and there is no other $z \in P$ such that $x \prec z \prec y$.

Recall that a *matching* in a poset P is defined as a set $\mathcal{M} \subset P \times P$ such that $(x, y) \in \mathcal{M} \Rightarrow x \triangleleft y$ and each $x \in P$ belongs to at most one pair of \mathcal{M} .

An *alternating path* is a sequence

$$(3.1) \quad y_0 \triangleright x_1 \triangleleft y_1 \triangleright x_2 \triangleleft y_2 \triangleright \cdots \triangleright x_m \triangleleft y_m (\triangleright x_{m+1})$$

such that each pair $x_i \triangleleft y_i$ belongs to \mathcal{M} and no pair $x_i \triangleleft y_{i-1}$ belongs to \mathcal{M} .

A *cycle* is a *closed* alternating path with $y_0 = y_m$.

An *acyclic matching* over P is a matching with no cycles.

(The original definition of *discrete vector field* over the cells of a *CW-complex*, see [14], translates into the above notion of matching in the associated face-poset, see for example [19]). An acyclic matching can be associated to a *discrete Morse function* [14].

Definition 3.1. A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching \mathcal{M} on P such that

$$x \triangleleft y \in \mathcal{M} \Rightarrow w(x) = w(y)$$

(We will refer briefly just to an acyclic matching on the weighted sheaf, if it is clear from the context).

Remark 3.2. In the more general situation of Remark 2.4, one could define a weighted acyclic matching on a weighted sheaf (P, R, w) as an acyclic matching \mathcal{M} such that $x \triangleleft y \in \mathcal{M}$ implies that the ideal I_x equals the ideal I_y .

Let now $\mathcal{S} := (K, R, w)$ be a weighted sheaf as in the previous section, where K is a simplicial complex with natural partial ordering (2.1). Given an irreducible φ , let $\mathcal{S}_{(\varphi)} := (K, R, w_{(\varphi)})$ be the φ -primary component defined in the previous section, with associated weighted complex $L_{(\varphi)}$ (see (2.4)).

Let \mathcal{M} be a weighted acyclic matching for $\mathcal{S}_{(\varphi)}$. The standard theory generalizes as follows.

Definition 3.3. (1) A *critical simplex* of K is a simplex σ which does not belong to any pair of \mathcal{M} .

(2) The *Morse complex* of \mathcal{S} , with respect to \mathcal{M} , is defined as the (torsion) complex

$$C_*^{\mathcal{M}} := \bigoplus_{\sigma \text{ critical}} \frac{R}{(w_\sigma)} \bar{e}_\sigma$$

with boundary

$$(3.2) \quad \partial^{\mathcal{M}}(a\bar{e}_\sigma) := \sum_{\substack{\tau \text{ critical} \\ |\tau| = |\sigma| - 1}} \left(\sum_p \iota_{p,\tau,\sigma}(a) \right) \bar{e}_\tau.$$

Here the sum is over all alternating paths starting with σ and ending with τ . Such path

$$p = \sigma \triangleright \tau_1 \triangleleft \sigma_1 \triangleright \tau_2 \triangleleft \sigma_2 \triangleright \cdots \triangleright \tau_m \triangleleft \sigma_m \triangleright \tau$$

contributes by

$$(3.3) \quad \begin{aligned} \iota_{p,\tau,\sigma}(a) := & (-1)^m ([\tau : \sigma_m] i_{\tau,\sigma_m}) \circ ([\tau_m : \sigma_m]^{-1} i_{\tau_m,\sigma_m}^{-1}) \circ \cdots \\ & \cdots \circ ([\tau_2 : \sigma_1] i_{\tau_2,\sigma_1}) \circ ([\tau_1 : \sigma_1]^{-1} i_{\tau_1,\sigma_1}^{-1}) \circ ([\tau_1 : \sigma] i_{\tau_1,\sigma})(a), \end{aligned}$$

where the inverse of each i_{τ_j,σ_j} is well defined: by definition of weighted matching, $w_{\tau_j} = w_{\sigma_j}$, so i_{τ_j,σ_j} is actually the identity map.

So we can rewrite (3.3) in a more compact way as

$$(3.4) \quad \iota_{p,\tau,\sigma}(a) := (-1)^m [\tau : \sigma_m] i_{\tau,\sigma_m} \circ \left[\prod_{j=m-1}^0 \frac{[\tau_{j+1} : \sigma_j]}{[\tau_{j+1} : \sigma_{j+1}]} i_{\tau_{j+1},\sigma_{j+1}}^{-1} \circ i_{\tau_{j+1},\sigma_j} \right] (a),$$

where we set $\sigma_0 = \sigma$.

Next, we generalize some standard results in “algebraic Morse theory” to our context.

Let A be a PID and let D be an A -module. For $d \in D$, we define the torsion of d as the principal ideal

$$\text{tors}(d) := \{a \in A : a \cdot d = 0\}.$$

We need the following property of PID’s.

Proposition 3.4. *Let A be a PID and let D be an A -module with generators f_1, \dots, f_k , and a direct sum decomposition into cyclic submodules*

$$D = \bigoplus_{i=1}^k A.f_i,$$

where $\text{tors}(f_i) = (d_i)$, so $A.f_i \cong \frac{A}{(d_i)}$, $i = 1, \dots, k$.

Let

$$(3.5) \quad \begin{cases} g_1 = f_1, \\ g_2 = \pm f_2 + \sum_{j \geq 3} c_{j2} f_j, \\ g_i = f_i + c_{1i} f_1, \quad i = 3, \dots, k \end{cases}$$

be another set of generators of D , with same torsion

$$\text{tors}(g_i) = \text{tors}(f_i), \quad i = 1, \dots, k.$$

Then we have a decomposition

$$D = \bigoplus_{i=1}^k A.g_i.$$

Proof. We need to proof that

$$\sum_{i=1}^k a_i g_i = 0, \quad a_i \in A, \quad i = 1, \dots, k \Rightarrow a_i \in \text{tors}(g_i), \quad i = 1, \dots, k.$$

By substituting relations (3.5) we get

$$\left(a_1 + \sum_{i \geq 3} a_i c_{1i} \right) f_1 \pm a_2 f_2 + \sum_{i \geq 3} (a_i + a_2 c_{i2}) f_i = 0$$

which gives $a_2 \in \text{tors}(f_2) = \text{tors}(g_2)$. Therefore, we get

$$(3.6) \quad a_1 g_1 + \sum_{i=3}^k a_i g_i = 0,$$

and consequently

$$\left(a_1 + \sum_{i=3}^k a_i c_{1i} \right) f_1 + \sum_{i=3}^k a_i f_i = 0,$$

which gives $a_i \in \text{tors}(f_i) = \text{tors}(g_i)$ for $i = 3, \dots, k$. So (3.6) reduces to

$$a_1 g_1 = a_1 f_1 = 0,$$

which gives $a_1 \in \text{tors}(f_1) = \text{tors}(g_1)$. □

By using proposition 3.4 we get:

Theorem 2. *Let $\mathcal{S} := (K, R, w)$ be a weighted sheaf over a simplicial complex K . Let \mathcal{M} be an acyclic matching for \mathcal{S} . Then one has an isomorphism*

$$H_*(L_*(K), \partial) \cong H_*(C_*^{\mathcal{M}}, \partial^{\mathcal{M}})$$

between the homology of the associated weighted complex $L_(K)$ and the homology of the Morse complex $C_*^{\mathcal{M}}$ of \mathcal{S} (definition 3.3).*

Proof. The proof is a Gauss-type triangularization of the set of generators of L_* (see [19] for free modules).

Let us give an ordering to the pairs in \mathcal{M} : $p_1 = (\tau_1, \sigma_1), p_2 = (\tau_2, \sigma_2), \dots, p_m = (\tau_m, \sigma_m)$. Let

$$\mathcal{M}^{(k)} := \{p_i \in \mathcal{M} : i \leq k\}$$

$k = 0, \dots, m$, where we set $\mathcal{M}^{(0)} = \emptyset$ (and we have $\mathcal{M}^{(m)} = \mathcal{M}$). Clearly $\mathcal{M}^{(k)}$ is also an acyclic matching for \mathcal{S} . Let Cr_k be the set of critical simplices for $\mathcal{M}^{(k)}$.

Let $\bar{e}_\sigma^{(0)} = \bar{e}_\sigma$, for all $\sigma \in K$. By induction on $k \geq 0$, assume we have constructed generators

$$\bar{e}_\sigma^{(k)}, \quad \sigma \in K, \quad k \geq 0,$$

for L_* such that:

- (1) there is a direct sum decomposition into subcomplexes

$$L_* = A_k \oplus_{\sigma \in Cr_k} \frac{R}{(w_\sigma)} \bar{e}_\sigma^{(k)},$$

where A_k decomposes as

$$A_k = \bigoplus_{i=1}^k \Gamma_i$$

into elementary acyclic subcomplexes

$$\Gamma_i := \left[0 \rightarrow \frac{R}{(w_{\sigma_i})} \bar{e}_{\sigma_i}^{(k)} \rightarrow \frac{R}{(w_{\tau_i})} \bar{e}_{\tau_i}^{(k)} \rightarrow 0 \right]$$

(by definition: $w_{\sigma_i} = w_{\tau_i}$; it follows that A_k is acyclic);

- (2) the boundary ∂ induces on Cr_k a boundary $\partial^{(k)}$, which is given by formula (3.2) with respect to $\mathcal{M}^{(k)}$.

When $k = 0$ we have $A_0 = 0$ and $\partial^{(0)} = \partial$.

Now we consider $\mathcal{M}^{(k+1)} = \mathcal{M}^{(k)} \cup \{(\tau_{k+1}, \sigma_{k+1})\}$. By the acyclicity of $\mathcal{M}^{(k+1)}$, there were no alternating paths in $\mathcal{M}^{(k)}$ between σ_{k+1} and τ_{k+1} except the trivial one $\sigma_{k+1} \triangleright \tau_{k+1}$, so that the coefficient of $\bar{e}_{\tau_{k+1}}^{(k)}$ in $\partial^{(k)} \bar{e}_{\sigma_{k+1}}^{(k)}$ is just the incidence number $[\tau_{k+1} : \sigma_{k+1}]$. Now we substitute

$$(3.7) \quad \bar{e}_{\tau_{k+1}}^{(k+1)} := \partial^{(k)} \bar{e}_{\sigma_{k+1}}^{(k)} = [\tau_{k+1} : \sigma_{k+1}] \bar{e}_{\tau_{k+1}}^{(k)} + \sum_{\substack{\tau \in Cr_k \\ |\tau|=|\sigma|-1}} c_{\tau, \sigma_{k+1}} \bar{e}_{\tau}^{(k)},$$

where the coefficients are here as in (3.2).

Note that $\bar{e}_{\tau_{k+1}}^{(k+1)}$ has the same torsion as $\bar{e}_{\tau_{k+1}}^{(k)}$ (and as $\bar{e}_{\sigma_{k+1}}^{(k)}$) by definition of weighted matching and by the fact that going down into the poset gives an element whose weight is a divisor of the weight above.

Now, for all σ such that $\sigma \in Cr_k$ and $\dim(\sigma) = \dim(\sigma_{k+1})$ we make the substitution

$$(3.8) \quad \begin{cases} \bar{e}_{\sigma}^{(k+1)} = \bar{e}_{\sigma}^{(k)}, & \text{if } \sigma = \sigma_{k+1} \\ \bar{e}_{\sigma}^{(k+1)} = \bar{e}_{\sigma}^{(k)} - [\tau_{k+1} : \sigma_{k+1}] c(\bar{e}_{\tau_{k+1}}^{(k)}, \partial^{(k)} \bar{e}_{\sigma}^{(k)}) \bar{e}_{\sigma_{k+1}}^{(k)}, & \text{if } \sigma \neq \sigma_{k+1}, \end{cases}$$

where we set $c(\bar{e}_{\tau_{k+1}}^{(k)}, \partial^{(k)} \bar{e}_{\sigma}^{(k)})$ as the coefficient of $\bar{e}_{\tau_{k+1}}^{(k)}$ in $\partial^{(k)} \bar{e}_{\sigma}^{(k)}$ (as given in (3.2)). As before, the torsion of $\bar{e}_{\sigma}^{(k+1)}$ equals that of $\bar{e}_{\sigma}^{(k)}$.

For the remaining elements, we set

$$\bar{e}_{\sigma}^{(k+1)} = \bar{e}_{\sigma}^{(k)}.$$

By applying proposition (3.4) with

$$f_1 = \bar{e}_{\sigma_{k+1}}^{(k)}, \quad f_2 := \bar{e}_{\tau_{k+1}}^{(k)},$$

and enumerating f_i , $i \geq 3$, subsequently all other $\bar{e}_{\sigma}^{(k)} \in Cr_k \setminus (\sigma_{k+1} \cup \tau_{k+1})$, and setting, respectively

$$g_1 = f_1 = \bar{e}_{\sigma_{k+1}}^{(k+1)}, \quad g_2 = \partial^{(k)} \bar{e}_{\sigma_{k+1}}^{(k+1)} = \bar{e}_{\tau_{k+1}}^{(k+1)}$$

and all others g_i as in (3.8), we get

$$L_* = A_{k+1} \oplus_{\sigma \in Cr_{k+1}} \frac{R}{(w_{\sigma})} \bar{e}_{\sigma}^{(k+1)},$$

where

$$A_{k+1} = A_k \oplus \Gamma_{k+1},$$

with

$$\Gamma_{k+1} := \left[0 \rightarrow \frac{R}{(w_{\sigma_{k+1}})} \bar{e}_{\sigma_{k+1}}^{(k+1)} \rightarrow \frac{R}{(w_{\tau_{k+1}})} \bar{e}_{\tau_{k+1}}^{(k+1)} \rightarrow 0 \right].$$

By (3.8) the boundary changes only for the alternating paths containing the edge $(\bar{e}_{\sigma_{k+1}}, \bar{e}_{\tau_{k+1}})$, so we get formula (3.2). □

Let us consider the filtration of the Morse complex into increasing weights:

$$F^p C_*^{\mathcal{M}} := \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma) \leq p}} \frac{R}{(\varphi^{v_\varphi(\sigma)})} \bar{e}_\sigma$$

and let

$$\mathcal{F}^p C_*^{\mathcal{M}} := F^p C_*^{\mathcal{M}} / F^{p-1} C_*^{\mathcal{M}}$$

be the quotient complex. As an R -module we have

$$\mathcal{F}^p C_*^{\mathcal{M}} \cong \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma) = p}} \frac{R}{(\varphi^p)} \bar{e}_\sigma$$

Then as a corollary of theorem 2 we can reformulate theorem 1 in terms of the Morse complex as follows.

Theorem 3. *Let \mathcal{S} and φ be as above and let \mathcal{M} be an acyclic matching for $\mathcal{S}_{(\varphi)}$. Then the spectral sequence of theorem 1 is computed by*

$$E_{p,q}^1 \cong H_{p+q}(\mathcal{F}^p C_*^{\mathcal{M}}),$$

where $C_*^{\mathcal{M}}$ is the Morse complex of $\mathcal{S}_{(\varphi)}$.

Proof. We note that, as in the proof of theorem 2, $L_{(\varphi)}$ is the direct sum $A_m \oplus C_*$ of an acyclic complex and of the Morse complex. This splitting is compatible with the filtrations, therefore the theorem follows straightforward. □

4. Artin groups

We consider here finitely generated *Coxeter systems* (\mathbf{W}, S) , (i.e., S is finite). Recall that

$$(4.1) \quad \mathbf{W} = \langle s \in S \mid (ss')^{m(s,s')} = 1 \rangle,$$

where $m(s, s') \in \mathbb{N} \cup \{\infty\}$, $m(s, s') = m(s', s)$, $m(s, s) = 1$ ([2, 17, 25]). We recall some general facts.

The group \mathbf{W} can be realized as a group generated by (in general, non-orthogonal) reflections in \mathbb{R}^n , $n = |S|$. Let \mathcal{A} be the *reflection arrangement*, i.e.,

$$\mathcal{A} = \{H \subset \mathbb{R}^n \mid H \text{ is fixed by some reflection in } \mathbf{W}\}.$$

We consider the stratification into *facets* of \mathbb{R}^n induced by \mathcal{A} , where the co-dimension-0 facets, (i.e., the connected components of the complement to the arrangement) are called *chambers*. Fix a base-chamber C_0 . The Coxeter generator set S corresponds to the set of reflections with respect to the *walls* of C_0 .

Let $U := \mathbf{W} \cdot \bar{C}_0$ be the orbit of the closure of the base chamber. U is called the *Tits cone* of the Coxeter system.

Each chamber contained inside U is of the shape $w.C_0$ for a unique $w \in \mathbf{W}$. Of course, \mathcal{A} is not locally finite if \mathbf{W} is infinite (e.g., 0 is contained in all the hyperplanes). The orbits of the facets of C_0 give a “stratification” of U into relatively open cells, also called facets (in general, U is neither open nor closed in \mathbb{R}^n).

Recall also:

- (1) U is a convex cone in \mathbb{R}^n with vertex 0 .
- (2) $U = \mathbb{R}^n$ iff \mathbf{W} is finite
- (3) The stabilizer of a facet F in U is the subgroup \mathbf{W}_F generated by all the reflections with respect to hyperplanes (in \mathcal{A}) containing F . So, in general \mathbf{W}_F is not finite.
- (4) $U^0 := \text{int}(U)$ is open in \mathbb{R}^n and a (relatively open) facet $F \subset \overline{C}_0$ is contained in U^0 iff the stabilizer \mathbf{W}_F is finite.

By property 4 the arrangement is locally finite in the interior part U^0 .

Let $H_{\mathbb{C}} := H + iH \subset \mathbb{C}^n$ be the *complexification* of the hyperplane H , and set

$$\mathbf{Y} := [U^0 + i\mathbb{R}^n] \setminus \cup_{H \in \mathcal{A}} H_{\mathbb{C}},$$

which corresponds to complexifying only the interior part of the Tits cone. The group \mathbf{W} acts (as before) diagonally over \mathbf{Y} , and one shows easily (by property 3 above) that the action is free. Therefore, one has an *orbit space*

$$\mathbf{Y}_{\mathbf{W}} := \mathbf{Y}/\mathbf{W},$$

which is still a manifold.

Definition 4.1. Define

$$\mathbf{G}_{\mathbf{W}} := \pi_1(\mathbf{Y}_{\mathbf{W}})$$

as the *Artin group* of type \mathbf{W} .

(see [3, 13] for the case when \mathbf{W} is finite). It is known that the orbit space $\mathbf{Y}_{\mathbf{W}}$ deformation retracts onto a *CW-complex* $\mathbf{X}_{\mathbf{W}}$, which is explicitly described in combinatorial terms (see [23] and also [12, 22], and the recent paper [21], where also a survey of the theory is given). From the explicit description of $\mathbf{X}_{\mathbf{W}}$ one derives a presentation of $\mathbf{G}_{\mathbf{W}}$

$$(4.2) \quad \mathbf{G}_{\mathbf{W}} = \langle g_s, s \in S \mid g_s g_{s'} g_s \cdots = g_{s'} g_s g_{s'} \cdots \rangle$$

(same number $m(s, s')$ of factors on each side, where we have to consider only pairs such that $m(s, s')$ is finite) as well as an algebraic complex, which computes the homology (and cohomology) of any local system over the orbit space $\mathbf{Y}_{\mathbf{W}}$. When such space is of type $K(\pi, 1)$ this cohomology equals that of the group $\mathbf{G}_{\mathbf{W}}$.

We recall formulas in case of the following representation (one can see that such representation corresponds, for the case of the braid group, to the determinant of the Burau representation; in general, there is a precise relation with the (co)-homology of the Milnor fiber of a suitable fibration; see, for example, [4, 9]).

Let $R := A[q, q^{-1}]$ be the ring of Laurent polynomials over a ring A . One can represent $\mathbf{G}_{\mathbf{W}}$ by

$$(4.3) \quad g_s \mapsto [\text{multiplication by } -q], \quad \forall s \in S$$

($\in \text{Aut}(R)$).

First we denote by $K := K(\mathbf{W})$ the simplicial scheme, defined over S , of the subsets $J \subset S$, which generate a finite parabolic subgroup \mathbf{W}_J (we include the empty set which by definition generates the trivial subgroup). The algebraic complex, which computes the trivial simplicial homology of K with coefficients in \mathbb{Z} is given in (2.2) and (2.3).

Notice that C_*^0 is graduated according to the cardinality of the subsets, so there is a degree-1 shift isomorphism with the standard complex for the simplicial homology of K , i.e., $H_m(C_*^0) \cong H_{m-1}(K)$ for $m \geq 1$.

The complex which computes the homology of $\mathbf{Y}_{\mathbf{W}}$ is given by

$$(4.4) \quad C_k := \bigoplus_{\substack{J \subset S \\ |J|=k \\ \mathbf{W}_J \text{ finite}}} R \cdot e_J$$

with boundary

$$(4.5) \quad \partial(e_J) = \sum_{\substack{I \subset J \\ |I|=|J|-1}} [I : J] \frac{\mathbf{W}_J(q)}{\mathbf{W}_I(q)} e_I$$

where

$$\mathbf{W}_J(q) := \sum_{w \in \mathbf{W}_J} q^{\ell(w)}$$

is the Poincaré series of the group \mathbf{W}_J (here, a polynomial since \mathbf{W}_J is finite by definition). The denominator $\mathbf{W}_I(q)$ divides the numerator $\mathbf{W}_J(q)$ so the quotient is still a polynomial.

We can formally rewrite the boundary map in (4.5) as:

$$(4.6) \quad \partial \left(\frac{1}{\mathbf{W}_J(q)} e_J \right) = \sum_{\substack{I \subset J \\ |J|=|I|+1}} [I : J] \frac{1}{\mathbf{W}_I(q)} \cdot e_I$$

That means that the fractions $e_J/\mathbf{W}_J(q)$ behave like the cells of the simplicial scheme K .

Consider the diagonal map:

$$(4.7) \quad \Delta : C_* \rightarrow C_*^0, \quad e_J \mapsto \mathbf{W}_J(q) e_J^0.$$

Clearly Δ is an injective chain-complex homomorphism, so there is an exact sequence of complexes:

$$(4.8) \quad 0 \longrightarrow C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \longrightarrow 0$$

where

$$L_k := \bigoplus_{\substack{J \subset S \\ |J|=k \\ |\mathbf{W}_J| < \infty}} \frac{R}{(\mathbf{W}_J(q))} \cdot \bar{e}_J$$

is the quotient complex.

Passing to the associated long exact sequence we get:

$$(4.9) \quad \xrightarrow{\pi_*} H_{k+1}(L_*) \xrightarrow{\delta} H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \xrightarrow{\delta} H_{k-1}(C_*) \xrightarrow{\Delta_*}$$

Remark 4.2. With the preceding notations, we can consider the map $w : K \rightarrow R$ given by

$$J \longrightarrow w_J := \mathbf{W}_J(q) \in R.$$

Since $w_I \mid w_J$ if $I \subset J$ we can see the w_J 's as weights associated to the simplices of the simplicial complex K . It is clear that

$$(K, R, w)$$

is a weighted sheaf over K , and that L_* is the associated weighted complex.

5. Matchings for the affine groups

From the fundamental Remark 4.2, and from sequences (4.8) and (4.9), we can compute the local homology of the Artin group from the knowledge of the associated weighted complex L_* (we assume A is a field so $R = A[q^{\pm 1}]$ is a PID).

In this section, we show how to find good matchings for the affine-type Artin groups, so that we can fruitfully use the theory developed in Section 3.

We make use of theorems 1 and 3. In particular, we need a weighted matching which allows us to reduce computations to small Morse complexes.

Recall that a standard way to produce acyclic matchings is given by the following easy lemma (see [18, Lemma 4.2], [19, Theorem 11.10])

Lemma 5.1. *Let $f : \mathcal{P} \rightarrow \mathcal{Q}$ be a poset map and for any $q \in \mathcal{Q}$ let \mathcal{M}_q be an acyclic matching on $f^{-1}(q)$. Then*

$$\mathcal{M} := \cup_{q \in \mathcal{Q}} \mathcal{M}_q$$

is an acyclic matching on \mathcal{P} .

In case of an affine Artin group $\mathbf{G}_{\mathbf{W}}$, associated to a Coxeter system (\mathbf{W}, S) , notice that the associated simplicial scheme $K = K(\mathbf{W})$, defined in part 4, is given by the scheme of all proper subsets of S .

Let us give any total ordering $<$ to $S := \{s_1 < s_2 < \dots < s_n\}$. We have

Lemma 5.2. *The map $f : K \rightarrow S$ defined by*

$$(5.1) \quad K \ni J \longrightarrow f(J) := \min\{s \in S : s \notin J\}$$

is well defined and it is a poset map.

Proof. The map f is well defined because each $J \in K$ misses some element from S .

If $I \subset J$ then clearly $f(I) \leq f(J)$, so f is a poset map. □

Remark 5.3. Let

$$K_i := \{J \subset K : s_i \notin J\}.$$

Clearly K_i is the full simplex of all the subsets of $S \setminus \{s_i\}$ (including the empty set). So one has an isomorphism

$$f^{-1}(s_i) \cong St_{K_i}(\{s_1, \dots, s_{i-1}\}), \quad i \geq 0.$$

Note that all the $f^{-1}(s_i)$ correspond to the case where one vertex is removed from the Coxeter diagram, so reducing to a finite-type Artin group. Starting from exceptional affine cases, one will have exceptional finite-type Artin groups, which were worked out in [9] by different methods.

6. Computational results

The computations of the E_1 -page of the spectral sequence introduced in Section 2 use a weighted matching on L_* of the type described in the previous section, together with theorems 1 and 3. As a matter of fact, in all cases that we consider we succeed to get matchings such that the Morse complex is *minimal* at each level p , meaning that the boundary of the Morse complex of $F^p L_{(\varphi)}/F^{p-1} L_{(\varphi)}$ vanishes.

One can work out the differential d_1 in the spectral sequence introduced before as follows.

Theorem 4. *Let \mathcal{S} , φ , be as before. The differential*

$$d_{p,q}^1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1$$

is induced by the boundary of the Morse complex

$$H_{p+q}(\mathcal{F}^p C_*^{\mathcal{M}}) \rightarrow H_{p+q-1}(\mathcal{F}^{p-1} C_*^{\mathcal{M}}).$$

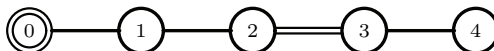
The boundary of a critical class $[\bar{e}_J] \in H_{p+q}(F^p/F^{p-1})$ is computed as in formula (3.2), through all alternating paths p starting from \bar{e}_J .

Proof. In fact, the decomposition of $L_{(\varphi)}$ in $A_m \oplus C_*^{\mathcal{M}} f$ obtained in the proof of theorem 2, where A_m is an acyclic subcomplex while $C_*^{\mathcal{M}}$ is the Morse complex, clearly induces an isomorphism of spectral sequences

$$E_{p,q}^*(L_{(\varphi)}) \cong E_{p,q}^*(C_*^{\mathcal{M}}).$$

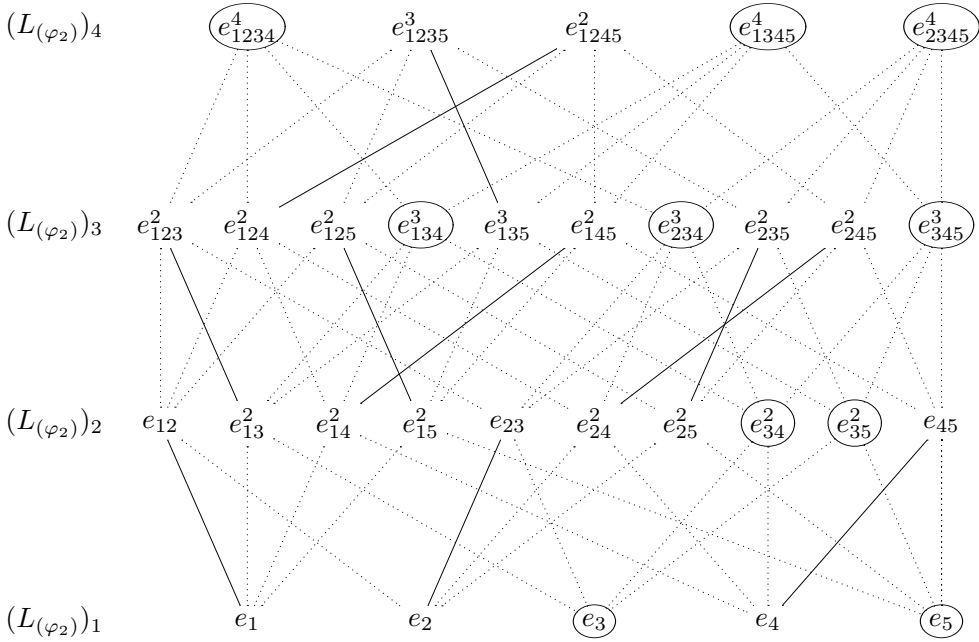
Then we use that the Morse complex is minimal at each level p (see above) to identify homology classes with classes of the shape $[\bar{e}_J]$. □

To explain the procedure, we give in detail the example of \tilde{F}_4 , in the case of φ_2 -primary component. The Coxeter diagram is given here by:



The weighted sheaf $(K, R, w_{(\varphi_2)})$ corresponding to the φ_2 -primary component is associated to the weighted complex $L_{(\varphi_2)} := L_{(\varphi_2)}(\tilde{F}_4)$.

We have the following weighted acyclic matching over $L_{(\varphi_2)}$. We give the Hasse diagram, where the pairs of the matching are joined by bold lines.



In view of theorem 3, we have that the Morse complex $\mathcal{C}_*^{\mathcal{M}}$ associated to the given matching is generated by the critical elements, which are circled in the diagram.

Note that the boundaries in $(F^i/F^{i-1})_*$ over the critical cells are vanishing. This allows us to use the same notations for the classes generating the homology groups. We obtain the first page of the spectral sequence. The only non-zero entries are (we drop the index φ_2 for brevity):

$$\begin{aligned}
 E_{1,0}^1 &:= H_1(\mathcal{F}^1 C_*^{\mathcal{M}}) \cong H_1(F^1/F^0) = \frac{R}{(\varphi_2)} e_3 \oplus \frac{R}{(\varphi_2)} e_5, \\
 E_{2,0}^1 &:= H_2(\mathcal{F}^2 C_*^{\mathcal{M}}) = H_2(F^2/F^1) = \frac{R}{(\varphi_2^2)} e_{34} \oplus \frac{R}{(\varphi_2^2)} e_{35}, \\
 E_{3,0}^1 &:= H_3(\mathcal{F}^3 C_*^{\mathcal{M}}) = H_3(F^3/F^2) = \frac{R}{(\varphi_2^3)} e_{134} \oplus \frac{R}{(\varphi_2^3)} e_{234} \oplus \frac{R}{(\varphi_2^3)} e_{345}, \\
 E_{4,0}^1 &:= H_4(\mathcal{F}^4 C_*^{\mathcal{M}}) = H_4(F^4/F^3) = \frac{R}{(\varphi_2^4)} e_{1234} \oplus \frac{R}{(\varphi_2^4)} e_{1345} \oplus \frac{R}{(\varphi_2^4)} e_{2345}.
 \end{aligned}$$

Furthermore, we have only three non-zero differentials, which are calculated using formula (3.2):

1. $d_{2,0}^1 : E_{2,0}^1 \rightarrow E_{1,0}^1$.

The alternating paths starting from e_{34} and ending in e_3 or in e_5 are unique:

$$e_{34} \triangleright e_3 \quad e_{34} \triangleright e_4 \trianglelefteq e_{45} \triangleright e_5$$

TABLE 1. z as defined at the end

	\tilde{A}_1	\tilde{B}_2	$\tilde{I}_2(6)$	\tilde{F}_4
H_0	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$
H_1	Rz	$\frac{R}{(\varphi_2\varphi_4)} \frac{\partial(e_{12})}{(\varphi_2\varphi_4)}$ $\oplus \frac{R}{(\varphi_2)} \frac{\partial(e_{13})}{(\varphi_2)}$	$\frac{R}{(\varphi_2\varphi_3)} \frac{\partial(e_{23})}{(\varphi_2\varphi_3)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_{34})}{(\varphi_2)}$
H_2	0	Rz	Rz	$\frac{R}{(\varphi_2\varphi_3)} \frac{\partial(e_{234})}{\varphi_2\varphi_3}$
H_3	0	0	0	$\frac{R}{(\varphi_2\varphi_3\varphi_4\varphi_8)} \frac{\partial(e_{2345})}{(\varphi_2\varphi_3\varphi_4\varphi_8)}$ $\oplus \frac{R}{(\varphi_2)} \frac{\partial(e_{2345})}{(\varphi_2)}$
H_4	0	0	0	Rz

So $d_{2,0}^1(e_{34}) = e_3 - e_5$. The same if we consider alternating paths between e_{35} and e_3 or e_5 :

$$e_{35} \triangleright e_3 \quad e_{35} \triangleright e_5.$$

In this case, we obtain $d_{2,0}^1(e_{35}) = e_3 - e_5$.

- $d_{3,0}^1 : E_{3,0}^1 \rightarrow E_{2,0}^1$.

The alternating paths between critical cells are:

$$\begin{aligned} e_{134} \triangleright e_{34} \quad e_{134} \triangleright e_{14} \trianglelefteq e_{145} \triangleright e_{15} \trianglelefteq e_{125} \triangleright e_{25} \trianglelefteq e_{235} \triangleright e_{35} \\ e_{234} \triangleright e_{34} \quad e_{234} \triangleright e_{24} \trianglelefteq e_{245} \triangleright e_{25} \trianglelefteq e_{235} \triangleright e_{35} \\ e_{345} \triangleright e_{34} \quad e_{345} \triangleright e_{35} \end{aligned}$$

They determine the differential $d_{3,0}^1$:

$$d_{3,0}^1(e_{134}) = d_{3,0}^1(e_{234}) = d_{3,0}^1(e_{345}) = -e_{34} + e_{35}.$$

- $d_{4,0}^1 : E_{4,0}^1 \rightarrow E_{3,0}^1$.

Finally, the alternating paths between critical cells are:

$$e_{1234} \triangleright e_{134} \quad e_{1234} \triangleright e_{234} \quad e_{1345} \triangleright e_{134} \quad e_{1345} \triangleright e_{345} \quad e_{2345} \triangleright e_{234} \quad e_{2345} \triangleright e_{345}$$

And the differential $d_{4,0}^1$ is given by:

$$\begin{aligned} d_{4,0}^1(e_{1234}) &= -e_{234} + e_{134}, \\ d_{4,0}^1(e_{1345}) &= -e_{345} + e_{134}, \\ d_{4,0}^1(e_{2345}) &= -e_{345} + e_{234}. \end{aligned}$$

TABLE 2. $w_6(q) = \varphi_2\varphi_3\varphi_6\varphi_9\varphi_{12}$, $w_7(q) = \varphi_2\varphi_3\varphi_4\varphi_6\varphi_8\varphi_{10}\varphi_{14}\varphi_{18}$ and $w_8(q) = \varphi_2\varphi_3\varphi_4\varphi_5\varphi_8\varphi_9\varphi_{14}$. z as before

	\tilde{E}_6	\tilde{E}_7	\tilde{E}_8
H_0	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$	$\frac{R}{(\varphi_2)} \frac{\partial(e_1)}{(\varphi_2)}$
H_1	0	0	0
H_2	0	0	0
H_3	$\frac{R}{(\varphi_3)} \frac{\partial(e_{245} - e_{435} - e_{125} - e_{135})}{(\varphi_3)}$	$\frac{R}{(\varphi_3)} \frac{\partial(e_{2681} - e_{2581} + e_{3581} - e_{3681})}{(\varphi_3)}$	0
H_4	$\frac{R}{(\varphi_5\varphi_8)} \frac{\partial(\varphi_8(e_{13} - e_{12} + e_{24}) + \varphi_5e_{27})}{(\varphi_5\varphi_8)}$	0	0
H_5	$\frac{R}{(w_6(q))} \frac{\partial\left(\varphi_2e_1 + \frac{w_6(q)}{(\varphi_2)}e_6\right)}{(w_6(q))}$	0	0
	$\oplus \frac{R}{(\varphi_3\varphi_6\varphi_9\varphi_{12})} \frac{\partial(\varphi_6\varphi_9\varphi_{12}e_2 + \varphi_3e_7)}{(\varphi_3\varphi_6\varphi_9\varphi_{12})}$		
	$\oplus \frac{R}{(\varphi_3)} \frac{\partial(e_3)}{(\varphi_3)}$		
H_6	Rz	$\frac{R}{(w_7(q))} \frac{\partial\left(\varphi_4\varphi_8\varphi_{10}e_1 + \frac{w_7(q)}{(\varphi_8)}e_5 + \frac{w_7(q)}{(\varphi_{10})}(e_2 - e_3)\right)}{(w_7(q))}$	$\frac{R}{(\varphi_5\varphi_8)} \frac{\partial(\varphi_8(e_{45} - e_{35} + e_{25} - e_{15}) + \varphi_5e_{16})}{(\varphi_5\varphi_8)}$
		$\oplus \frac{R}{(\varphi_2)} \frac{\partial(e_2)}{(\varphi_2)} \oplus \frac{R}{(\varphi_2)} \frac{\partial(e_5)}{(\varphi_2)}$	
H_7	Rz	0	$\frac{R}{(w_8(q))} \frac{\partial\left(\varphi_9\varphi_{14}e_1 + \frac{w_8(q)}{(\varphi_9)}e_6 + \frac{w_8(q)}{(\varphi_{14})}e_9\right)}{(w_8(q))}$
H_8	0	0	Rz

We obtain the homology groups for the complex $(L_{(\varphi_2)})_*$:

$$\begin{aligned} H_4(L_{(\varphi_2)}) &= \frac{R}{(\varphi_2^4)}(e_{1234} - e_{1235} + e_{1245} - e_{1345} + e_{2345}) \\ &\quad \oplus \frac{R}{(\varphi_2)}(\varphi_2^3 e_{1234}) \oplus \frac{R}{(\varphi_2)}(\varphi_2^3 e_{2345}), \\ H_3(L_{(\varphi_2)}) &= \frac{R}{(\varphi_2)}(\varphi_2^2 e_{234}), \\ H_2(L_{(\varphi_2)}) &= \frac{R}{(\varphi_2)}(\varphi_2 e_{34}), \\ H_1(L_{(\varphi_2)}) &= \frac{R}{(\varphi_2)}(e_3). \end{aligned}$$

Similar computation is done for the φ_d -torsion, $d \geq 3$.

We collect in Tables 1 and 2 the final results of our computations. We use the computation of the homology of the weighted complex L_* together with the exact sequences (4.8) and (4.9).

As remarked in Section 5 the simplicial complex of subsets generating finite parabolic subgroups is given by all the proper subsets of S , so it is the boundary of a simplex, having reduced homology only in the top dimension. By this remark and the exact sequence (4.9) it is easy to pass from the homology of the weighted complex to that of the Artin group.

The entries of the tables give the module H_* of the corresponding group, together with representative generators.

For an affine Dynkin graph Γ with $n + 1$ vertices, we define $\mathbf{W}_n(q)$ to be the lcm between the Poincaré polynomials ranging over the subgraphs with exactly n vertices. We denote by (see formula 4.5) $z = \sum_{i \in I} [\Gamma \setminus \{i\} : \Gamma] \frac{\mathbf{W}_n(q)}{\mathbf{W}_{\Gamma \setminus \{i\}}(q)} e_{\Gamma \setminus \{i\}}$ the weighted alternating sum.

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