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A PRIORI ESTIMATES OF THE DEGENERATE MONGE–AMPÈRE EQUATION ON KÄHLER MANIFOLDS OF NON-NEGATIVE BISECTIONAL CURVATURE

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Abstract. The regularity theory of the degenerate complex Monge–Ampère equation is studied. The equation is considered on a closed compact Kähler manifold (M, g) with non-negative orthogonal bisectional curvature of dimension m. Given a solution φ of the degenerate complex Monge–Ampère equation $\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = f \det(g_{i\bar{j}})$, it is shown that the Laplacian of φ can be controlled by a constant depending on (M, g), $\sup f$, and $\inf_M \Delta f^{1/(m-1)}$.

1. Introduction

We will be looking at the regularity theory of the degenerate complex Monge–Ampère equation. Let us consider the equation on a compact Kähler manifold (M, g) without boundary of dimension m. The problem of solving the complex Monge–Ampère equation on M was first motivated by the Calabi conjecture. The conjecture was reduced to solving a non-degenerate Monge–Ampère equation, and the question was solved by Yau [18]. This result had significant geometric implications, in particular, leading to the theory of Calabi–Yau manifolds, which now plays a central role in string theory and complex geometry. One of the major steps in the proof of the Calabi conjecture was establishing an a priori estimate on the Laplacian of the solution; this was done independently by Aubin [1] and Yau [18].

Although the Calabi conjecture deals with a non-degenerate Monge–Ampère equation, Yau's paper [18] also treated the degenerate Monge–Ampère equation, with an application to holomorphic sections of line bundles over M. More recently, the results of Yau were generalized by Kolodziej [14]. The degenerate Monge–Ampère equation in complex geometry has become an active area of research, with connections to the Minimal Model Program [5,9], or geodesics joining two Kähler potentials in the space of Kähler metrics [7,8,16]. For a survey of some of these topics, see [15].

In this paper, we shall consider the following complex Monge–Ampère equation:

$$\det(\varphi_{i\bar{j}} + g_{i\bar{j}}) = f \det g_{i\bar{j}}, \tag{1.1}$$

where $f : M \to \mathbb{R}$ and $f \geq 0$.

The objective is the following: given a solution φ to (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive semi-definite, we seek an estimate $|\Delta\varphi| \leq C$ depending only on (M, g), $\sup f$, and a constant A such that

$$\inf_M \Delta f^{\frac{1}{m-1}} \geq -A. \tag{1.2}$$

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This problem is motivated by a similar result obtained for the real Monge–Ampère equation by Guan [11]. Previously, a result on Kähler manifolds was obtained by Blocki [3] while assuming that $f^{\frac{1}{m-1}} \in C^{1,1}$. Later, Blocki improved his result in [4] to requiring the assumption that $f^{\frac{1}{m-1}}\Delta(\log f)$ is bounded below, which is equivalent to

$$f\Delta f - |\nabla f|^2 \geq -Af^{2-\frac{1}{m-1}}.$$

In comparison, our desired condition (1.2) is equivalent to

$$f\Delta f - \frac{m-2}{m-1}|\nabla f|^2 \geq -Af^{2-\frac{1}{m-1}}.$$

We solve the problem in the case when M has non-negative orthogonal bisectional curvature. Similar results were obtained by Hou [13] by obtaining a Laplacian estimate for complex Hessian equations depending on $\inf_M \Delta f^{\frac{1}{m}}$ in the case of non-negative orthogonal bisectional curvature. The main difficulty in improving the exponent from $1/m$ to $1/(m-1)$ is that we can no longer discard terms by using the concavity of $(\det B)^{1/m}$, and the resulting third-order terms must be handled carefully.

We use the following definition.

Definition 1. *A Kähler manifold (M, g) is said to have non-negative orthogonal bisectional curvature if at each point $p \in M$, for any orthonormal basis $\{e_1, \ldots, e_m\}$ of the tangent space at p, we have $R(e_i, \overline{e_i}, e_j, \overline{e_j}) = R_{i\bar{i}j\bar{j}} \geq 0$.*

Manifolds satisfying this curvature condition are well-understood; indeed, compact Kähler manifolds of non-negative orthogonal bisectional curvature were classified by Gu and Zhang [10]. Our main theorem is the following.

Theorem 2. *Let (M, g) be a compact Kähler manifold with non-negative orthogonal bisectional curvature and empty boundary. Let $f > 0$ be a function on M such that $\inf_M \Delta f^{\frac{1}{m-1}} \geq -A$ for some constant A. For all solutions $\varphi \in C^4(M)$ of*

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = f \det g_{i\bar{j}},$$

such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive definite, we have

$$(\sup_M \varphi - \inf_M \varphi) + ||\nabla\varphi||_\infty + ||\Delta\varphi||_\infty \leq C,$$

where C depends on (M, g), A, and $\sup f$.

Although this theorem assumes $f > 0$, such an estimate is applicable to the degenerate case when $f \geq 0$ via a limiting process. This shall be illustrated in Section 3, which contains a proof of the following application.

Using the a priori estimates, we solve the Dirichlet problem for the degenerate complex Monge–Ampère equation on a domain Ω in $\mathbb{C}^m$. This type of question was investigated by Bedford and Taylor [2], and Caffarelli, Kohn, Niremberg and Spruck [6]. The problem is of the following form:

$$\det u_{i\bar{j}}(z) = f(z) \ \text{ in } \Omega, \qquad (1.3)$$

$$u = 0 \ \text{ on } \partial\Omega.$$

Before stating our result, we first establish some terminology. We say a real-valued function u is pluri-subharmonic if $(u_{i\bar{j}})$ is positive semi-definite. We say a real-valued function u is strictly pluri-subharmonic if $(u_{i\bar{j}})$ is positive definite. Following the terminology of Blocki [3], if $|\Delta u|$ is bounded, we say u is almost $C^{1,1}$. A domain $\Omega \subset \mathbb{C}^m$ with smooth boundary $\partial\Omega$ is called strongly pseudo-convex if there exists a smooth real-valued function r defined on a neighbourhood of $\overline{\Omega}$ such that $r < 0$ in Ω, $r = 0$ on $\partial\Omega$, $r > 0$ outside of $\overline{\Omega}$, $dr \neq 0$, and $(r_{i\bar{j}}(z))$ is positive-definite at each point in its domain. With these definitions in place, we can now state the following result.

Theorem 3. *Let Ω be a bounded strongly pseudo-convex domain in $\mathbb{C}^m$. Let $f : \Omega \to \mathbb{R}$ be a function such that $f \geq 0$, $|\nabla f^{1/m}| \leq A_1$, and $\Delta f^{\frac{1}{m-1}} \geq -A_2$. Then there exists a unique pluri-subharmonic, almost $C^{1,1}$ solution u such that*

$$\det u_{i\bar{j}}(z) = f(z) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Furthermore, $||u||_{C^1(\bar{\Omega})} + ||\Delta u||_\infty \leq C$, where C depends only on A_1, A_2, $\sup(f)$, and Ω.

2. Preliminaries

In this section, we establish some notation and recall previous results relating to the proof of Theorem 2. First, we remind the reader that the constant C denotes a positive quantity that is under control, and may change line by line. We will use the convention $\varphi_{i\bar{j}} = \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}$; as opposed to Yau [18], subscripts do not indicate covariant derivatives. Also, for a function $h : M \to \mathbb{C}$, we will use any of the following notation interchangeably: $\frac{\partial h}{\partial z^i}$, $\partial_i h$, and h_i. We shall denote

$$g'_{i\bar{j}} := g_{i\bar{j}} + \varphi_{i\bar{j}}. \tag{2.1}$$

It is well-known, as seen, for example, in the exposition of Siu [17], that for Theorem 2 we have an a priori estimate

$$(\sup_M \varphi - \inf_M \varphi) \leq C.$$

The objective of this paper is to estimate $|\Delta\varphi| \leq C$ directly from the uniform bound. Assuming such a bound on the Laplacian, we can obtain a bound on the gradient. Indeed, if we look at $\Delta\varphi(z) := G(z)$, then $|G(z)| \leq C$. Then by the Schauder estimates

$$\sup_M |\nabla\varphi| \leq C_0(||G||_\infty + ||\varphi||_2) \leq C. \tag{2.2}$$

Furthermore, we can easily obtain a lower bound on $\Delta\varphi$. Since $g'_{i\bar{j}} = g_{i\bar{j}} + \varphi_{i\bar{j}}$ is positive definite, we have $0 < Tr(g_{i\bar{j}} + \varphi_{i\bar{j}})$. At any point $p \in M$, we may choose coordinates such that $g_{i\bar{j}} = \delta_{i\bar{j}}$. Thus

$$0 < m + \Delta\varphi, \tag{2.3}$$

and it only remains to bound $\Delta\varphi$ from above.

3. Second-order estimate

As shown in the previous section, Theorem 2 will following from the following estimate.

Proposition 4. *Let (M,g) be a closed, compact Kähler manifold with non-negative orthogonal bisectional curvature. Let $f>0$ be a positive function on M such that $\inf_M \Delta f^{\frac{1}{m-1}} \geq -A$ for some constant A. For all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}}+g_{i\bar{j}})$ is positive-definite, we have*

$$|\Delta\varphi| \leq C,$$

where C depends on (M,g), $(\sup\varphi - \inf\varphi)$, A, and $\sup f$.

Proof. We will estimate the maximum value of the following test function:

$$H = (m+\Delta\varphi)e^{-\alpha(\varphi)}, \tag{3.1}$$

where $\alpha : [2,\lambda] \to \mathbb{R}$ is a function that will be specified later. In view of the L^∞ estimate, we may shift φ by a constant and assume that $\varphi(p) \in [2,\lambda]$ for all $p \in M$. We start by computing the first two derivatives of H.

$$\begin{aligned} H_\gamma &= (\Delta\varphi)_\gamma e^{-\alpha(\varphi)} - \alpha'(m+\Delta\varphi)\varphi_\gamma e^{-\alpha(\varphi)}, \\ H_{\gamma\bar{\gamma}} &= \left((\Delta\varphi)_{\gamma\bar{\gamma}} - \alpha'(m+\Delta\varphi)\varphi_{\gamma\bar{\gamma}} - \alpha''(m+\Delta\varphi)\varphi_\gamma\varphi_{\bar{\gamma}}\right)e^{-\alpha(\varphi)} \\ &\quad + \left(-\alpha'\left((\Delta\varphi)_\gamma\varphi_{\bar{\gamma}} + \overline{(\Delta\varphi)_\gamma}\,\varphi_\gamma\right) + (\alpha')^2(m+\Delta\varphi)\varphi_\gamma\varphi_{\bar{\gamma}}\right)e^{-\alpha(\varphi)}. \end{aligned} \tag{3.2}$$

Let $p \in M$ be the point where H achieves its maximum value. Since the manifold is Kähler, we may choose coordinates such that at p we have $g_{i\bar{j}} = \delta_{ij}$, $\frac{\partial}{\partial z^k} g_{i\bar{j}} = 0$ and $\varphi_{i\bar{j}} = \delta_{ij}\varphi_{i\bar{j}}$. At p, the gradient of H is equal to zero, and hence,

$$(\Delta\varphi)_\gamma = \alpha'\varphi_\gamma(m+\Delta\varphi). \tag{3.3}$$

We recall the notation (2.1). Since g' defines a Kähler metric on M, we denote $\Delta' = g'^{i\bar{j}}\partial_i\bar{\partial}_j$ to be the Laplacian of (M,g'). By the maximum principle, $\Delta' H(p) \leq 0$, hence if we use this fact while substituting the gradient equation (3.3) into (3.2), we obtain

$$\begin{aligned} \Delta'\Delta\varphi &\leq \alpha'(m+\Delta\varphi)\Delta'\varphi + (\alpha')^2(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} \\ &\quad + \alpha''(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}}. \end{aligned} \tag{3.4}$$

Next, we raise both sides of (1.1) to the power of $1/(m-1)$ and take derivatives. This yields

$$\begin{aligned} &(m-1)(\det g_{i\bar{j}})^{\frac{1}{m-1}}\partial_\gamma f^{\frac{1}{m-1}} + f^{\frac{1}{m-1}}(\det g_{i\bar{j}})^{\frac{1}{m-1}} g^{i\bar{j}}\partial_\gamma g_{i\bar{j}} \\ &\quad = (\det g'_{i\bar{j}})^{\frac{1}{m-1}} g'^{i\bar{j}}(\partial_\gamma g_{i\bar{j}} + \varphi_{i\bar{j}\gamma}). \end{aligned}$$

We then take another derivative of the previous equation, noting that our choice of coordinates will simplify the expression.

$$\begin{aligned} (m-1)\partial_\gamma\bar{\partial}_\gamma f^{\frac{1}{m-1}} + f^{\frac{1}{m-1}}\delta^{i\bar{j}}\partial_\gamma\bar{\partial}_\gamma g_{i\bar{j}} &= \frac{1}{m-1} f^{\frac{1}{m-1}} g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}\gamma}\varphi_{j\bar{j}\bar{\gamma}} \\ + f^{\frac{1}{m-1}}\left(g'^{i\bar{i}}(\partial_\gamma\bar{\partial}_\gamma g_{i\bar{i}} + \varphi_{i\bar{i}\gamma\bar{\gamma}}) + \varphi_{i\bar{j}\gamma}\bar{\partial}_\gamma g'^{i\bar{j}}\right). & \end{aligned}$$

Expanding out $\partial_\gamma g'^{i\bar{j}}$ and using the definition of the curvature tensor, we obtain

$$(3.5)\qquad \begin{aligned}&(m-1)f^{\frac{-1}{m-1}}\partial_\gamma\bar{\partial}_\gamma f^{\frac{1}{m-1}} - \delta^{i\bar{j}}R_{i\bar{j}\gamma\bar{\gamma}}\\ &= \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}\gamma}\varphi_{j\bar{j}\bar{\gamma}} - g'^{i\bar{i}}R_{i\bar{i}\gamma\bar{\gamma}} + g'^{i\bar{i}}\varphi_{i\bar{i}\gamma\bar{\gamma}} - g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{j}\gamma}\varphi_{j\bar{i}\bar{\gamma}}.\end{aligned}$$

Also, at the point in consideration we have

$$\begin{aligned}\Delta'\Delta\varphi &= g'^{k\bar{l}}\partial_k\bar{\partial}_l(g^{i\bar{j}}\varphi_{i\bar{j}})\\ &= g'^{k\bar{l}}\varphi_{i\bar{j}}\partial_k\bar{\partial}_l g^{i\bar{j}} + g'^{k\bar{l}}g^{i\bar{j}}\varphi_{i\bar{j}k\bar{l}}\\ &= -g'^{k\bar{l}}g^{i\bar{t}}g^{n\bar{j}}\varphi_{i\bar{j}}\partial_k\bar{\partial}_l g_{n\bar{t}} + g'^{k\bar{l}}g^{i\bar{j}}\varphi_{i\bar{j}k\bar{l}}\\ &= \sum_{k,i} g'^{k\bar{k}}\varphi_{i\bar{i}}R_{i\bar{i}k\bar{k}} + \sum_{k,i} g'^{i\bar{i}}\varphi_{i\bar{i}k\bar{k}}.\end{aligned}$$

After summing the γ in (3.5) and substituting the previous identity, one obtains the following at the point p:

$$\begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &= \Delta'\Delta\varphi + \sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}\\ &\quad - \sum_k g'^{i\bar{i}}(1+\varphi_{k\bar{k}})R_{i\bar{i}k\bar{k}} + \sum_{i,k} R_{i\bar{i}k\bar{k}}.\end{aligned}$$

We substitute (3.4), define $S := \sum_{i,k} R_{i\bar{i}k\bar{k}}$, and obtain

$$(3.6)\qquad \begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &\le \alpha' m(m+\Delta\varphi) - \alpha'(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big)\\ &\quad + (\alpha')^2(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} + \alpha''(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}}\\ &\quad + \sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}\\ &\quad - \inf_{i,k} R_{i\bar{i}k\bar{k}}(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big) + S.\end{aligned}$$

If M has non-negative orthogonal bisectional curvature, then $\inf_{i,k} R_{i\bar{i}k\bar{k}}$ is non-negative and the term involving it can be dropped. We are left with

$$(3.7)\qquad \begin{aligned}(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &\le \alpha' m(m+\Delta\varphi) + S - \alpha'(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big)\\ &\quad + (\alpha'' + (\alpha')^2)(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}}\\ &\quad + \sum_k \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}.\end{aligned}$$

The troublesome terms are those involving third-order derivatives, and we shall follow the argument of Guan [11] to control the following quantity for a fixed k:

$$(3.8)\qquad \frac{1}{m-1}g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}.$$

First, we drop mixed terms $|\varphi_{\bar{i}jk}|^2$ for $i \neq j$ and obtain

$$\frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}k} \varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{\bar{i}jk} \varphi_{i\bar{j}\bar{k}} \leq \frac{1}{m-1} \left| g'^{i\bar{i}} \varphi_{i\bar{i}k} \right|^2 - (g'^{i\bar{i}})^2 |\varphi_{i\bar{i}k}|^2.$$

We recall that $\varphi_{i\bar{i}}(z)$ is a locally defined real-valued function. Also, $\varphi_{i\bar{i}k} = \partial_{z^k} \varphi_{i\bar{i}}$ where $\partial_{z^k} = \frac{1}{2}(\partial_{x^k} - i\partial_{y^k})$. Thus

$$|\varphi_{i\bar{i}k}|^2 = \frac{1}{4}(\varphi_{i\bar{i}x}^2 + \varphi_{i\bar{i}y}^2),$$

where we write f_x for $\partial_{x^k} f$, and there is no confusion since k is fixed. Thus we get

$$\begin{aligned} \frac{4}{m-1} & g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{i\bar{i}k} \varphi_{j\bar{j}\bar{k}} - 4 g'^{i\bar{i}} g'^{j\bar{j}} \varphi_{\bar{i}jk} \varphi_{i\bar{j}\bar{k}} \\ & \leq \frac{1}{m-1} \left(\sum_i g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 - \sum_i (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 \\ & \quad + \frac{1}{m-1} \left(\sum_i g'^{i\bar{i}} \varphi_{i\bar{i}y} \right)^2 - \sum_i (g'^{i\bar{i}} \varphi_{i\bar{i}y})^2. \end{aligned} \tag{3.9}$$

We shall show how to control the terms containing real derivatives in the x direction. Let $I = \{1 \leq i \leq m : \varphi_{i\bar{i}x}(p) > 0\}$ and $J = \{1 \leq i \leq m : \varphi_{i\bar{i}x}(p) < 0\}$. We consider the two following cases. Case 1: I and J are both non-empty, or case 2: either I or J is empty. In case 1, we have $|I| \leq m-1$ and $|J| \leq m-1$. Using $(\sum_i^n a_i)^2 \leq n \sum a_i^2$ for $a_i \geq 0$, we can compute the following:

$$\begin{aligned} & \frac{1}{m-1} \left(\sum_i g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 - \sum_i (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 \\ & = \frac{1}{m-1} \left(\left(\sum_I g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 + \left(\sum_J g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 + 2 \left(\sum_I g'^{i\bar{i}} \varphi_{i\bar{i}x} \right) \left(\sum_J g'^{i\bar{i}} \varphi_{i\bar{i}x} \right) \right) \\ & \quad - \sum_I (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 - \sum_J (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 \\ & \leq \frac{1}{m-1} \left(\sum_I g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 - \sum_I (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 + \frac{1}{m-1} \left(\sum_J g'^{i\bar{i}} \varphi_{i\bar{i}x} \right)^2 - \sum_J (g'^{i\bar{i}} \varphi_{i\bar{i}x})^2 \\ & \leq 0. \end{aligned}$$

Case 2 is a little bit more delicate. Without loss of generality, we assume that $J = \emptyset$. Therefore, $\varphi_{i\bar{i}x}(p) > 0$ for all i. Using (3.3), we obtain the following at p:

$$\varphi_{i\bar{i}x} \leq \sum_{j=1}^m \varphi_{j\bar{j}x} = 2Re \left(\frac{\partial}{\partial z^k} \Delta\varphi \right) \leq 2|(\Delta\varphi)_k| \leq 2\alpha' |\nabla\varphi| (m + \Delta\varphi). \tag{3.10}$$

We now compute

$$\begin{aligned}
&\frac{1}{m-1}\left(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&= \frac{1}{m-1}\left(\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} + g'^{m\bar{m}}\varphi_{m\bar{m}x}\right)^2 - \sum_{i=1}^{m}(g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&= \frac{2}{m-1} g'^{m\bar{m}}\varphi_{m\bar{m}x}\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} + \frac{1}{m-1}\left(\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_{i=1}^{m-1}(g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 \\
&\quad + \frac{1}{m-1}(g'^{m\bar{m}}\varphi_{m\bar{m}x})^2 - (g'^{m\bar{m}}\varphi_{m\bar{m}x})^2.
\end{aligned}$$

Without loss of generality, we can assume $\varphi_{m\bar{m}}(p) \geq \varphi_{i\bar{i}}(p)$ for all i. Therefore, using (3.10) we have

$$\begin{aligned}
\frac{1}{m-1}\left(\sum_i g'^{i\bar{i}}\varphi_{i\bar{i}x}\right)^2 - \sum_i (g'^{i\bar{i}}\varphi_{i\bar{i}x})^2 &\leq \frac{2}{m-1} g'^{m\bar{m}}\varphi_{m\bar{m}x}\sum_{i=1}^{m-1} g'^{i\bar{i}}\varphi_{i\bar{i}x} \\
&\leq \frac{8}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)^2 g'^{m\bar{m}}\sum_{i=1}^{m-1} g'^{i\bar{i}} \\
&\leq \frac{8m}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)\sum_{i=1}^{m} g'^{i\bar{i}}.
\end{aligned}$$

The terms involving y derivatives in (3.9) can be controlled in the same way as the x derivatives. Thus combining both cases and (3.9), we obtain

$$\begin{aligned}
&\frac{1}{m-1} g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - g'^{i\bar{i}} g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}} \\
&\quad \leq \frac{4m}{m-1}(\alpha')^2|\nabla\varphi|^2(m+\Delta\varphi)\sum_{i=1}^{m} g'^{i\bar{i}}.
\end{aligned} \tag{3.11}$$

We substitute (3.11) into (3.7) and obtain

(3.12)

$$\begin{aligned}
(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} &\leq \alpha' m(m+\Delta\varphi) + S - \alpha'(m+\Delta\varphi)(\sum_i g'^{i\bar{i}}) \\
&\quad + \left(\alpha'' + (\alpha')^2\left(1+\frac{4m^2}{m-1}\right)\right)(m+\Delta\varphi)|\nabla\varphi|^2\left(\sum_i g'^{i\bar{i}}\right).
\end{aligned}$$

Denote $C_0 := 1 + 4m^2/(m-1)$. Following an idea of Blocki in his gradient estimate [4], we pick $\alpha(x) = (C_0)^{-1}\log x$. We know that $\alpha(\varphi)$ is well-defined, since φ was renormalized such that $2 \leq \varphi \leq \lambda$. This choice of α yields $\alpha'' + C_0(\alpha')^2 = 0$ and hence we are left with

$$(m-1)f^{\frac{-1}{m-1}}\Delta f^{\frac{1}{m-1}} \leq \frac{1}{2C_0}m(m+\Delta\varphi) + S - \frac{1}{C_0\lambda}(m+\Delta\varphi)\left(\sum_i g'^{i\bar{i}}\right). \tag{3.13}$$

We next notice that for $B_i > 0$, the following inequality holds:

$$\left(\sum_{i=1}^{m} \frac{1}{B_i}\right)^{m-1} \geq \frac{\sum_{i=1}^{m} B_i}{\prod_{i=1}^{m} B_i}.$$

Since $g^{i\bar{i}} > 0$, we thus have at the point p,

$$\sum_i g'^{i\bar{i}} \geq \left(\frac{m+\Delta\varphi}{f}\right)^{\frac{1}{m-1}}. \tag{3.14}$$

Substituting (3.14) into (3.13) and using the definition of A, we get

$$A(m-1) \geq \frac{1}{C_0\lambda}(m+\Delta\varphi)^{1+\frac{1}{m+1}} - S\sup_M f^{\frac{1}{m+1}} - \left(\frac{m}{2C_0}\sup_M f^{\frac{1}{m+1}}\right)(m+\Delta\varphi).$$

Thus there are constants C_1,C_2 under control such that

$$(m+\Delta\varphi(p))^{1+1/(m-1)} \leq C_1(m+\Delta\varphi(p)) + C_2.$$

It follows that there exists a constant C_3 under control such that

$$m+\Delta\varphi(p) \leq C_3.$$

Now that we have control of $(m+\Delta\varphi)$ at p, we have control of $(m+\Delta\varphi)$ at all $z \in M$. Indeed,

$$(m+\Delta\varphi(z))e^{-\alpha(\varphi(z))} \leq (m+\Delta\varphi(p))e^{-\alpha(\varphi(p))} \leq C_3 e^{-\alpha(\varphi(p))}.$$

Since $\alpha(x) = C_0^{-1}\log x$, we have

$$(m+\Delta\varphi(z)) \leq C_3\left(\frac{\lambda}{2}\right)^{1/C_0}.$$

□

By dropping the assumption on the bisectional curvature of M, the curvature terms break down the previous argument. It is unknown whether Proposition 4 holds without this condition on the curvature. Before ending this section, we give a partial result working towards the removal of this assumption. To attempt to control these curvature terms, we strengthen our hypothesis to match the direct gradient estimate given by Blocki [4] or Guan [12]: we assume $f^{1/m}$ is Lipschitz continuous. In the case $m = 2$, this additional assumption makes dealing with the terms (3.8) particularly easy, and we can thus obtain the following estimate.

Proposition 5. *Let (M, g) be a closed, compact Kähler manifold of dimension $m = 2$. Let $f > 0$ be a positive function on M such that $\inf_M \Delta f \geq -A$ for some constant A, and $f^{\frac{1}{2}}$ is Lipschitz. For all $\varphi \in C^4(M)$ satisfying (1.1) such that $(\varphi_{i\bar{j}} + g_{i\bar{j}})$ is positive-definite, we have*

$$|\Delta\varphi| \leq C, \tag{3.15}$$

where C depends on (M, g), $(\sup\varphi - \inf\varphi)$, A, the Lipschitz constant of $f^{\frac{1}{2}}$, and $\sup f$.

Proof. We run the same argument as the proof of Proposition 4 up until equation (3.6). In this case, we can simply let $\alpha(x) = \alpha_0 x$, where $0 < \alpha_0$ is a constant. Equation (3.6) becomes

$$f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq \alpha_0 m(m+\Delta\varphi) + S - (\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}})(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big) + (\alpha_0)^2(m+\Delta\varphi)g'^{i\bar{i}}\varphi_i\varphi_{\bar{i}} + \sum_k \frac{1}{m-1} g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}.$$

We see that if we choose $\alpha_0 > \inf_{i,k} R_{i\bar{i}k\bar{k}}$, the coefficient on the third term is negative. To eliminate the α_0^2 term, we substitute the gradient equation (3.3):

$$f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq \alpha_0 m(m+\Delta\varphi) + S - (\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}})(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big) + (m+\Delta\varphi)^{-1}g'^{i\bar{i}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{i}} + \sum_k \frac{1}{m-1} g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}} - \sum_k g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{\bar{i}jk}\varphi_{i\bar{j}\bar{k}}.$$

Using Cauchy–Bunyakowsky–Schwarz, we obtain

$$\begin{aligned}(m+\Delta\varphi)^{-1}g'^{i\bar{i}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{i}} &= (m+\Delta\varphi)^{-1}\sum_i g'^{i\bar{i}}(\sum_k \varphi_{k\bar{k}i})(\sum_k \varphi_{k\bar{k}\bar{i}}) \\ &\leq (m+\Delta\varphi)^{-1}\sum_i g'^{i\bar{i}}\left(\sum_k \frac{|\varphi_{k\bar{k}i}|^2}{(1+\varphi_{k\bar{k}})}\right)\left(\sum_k (1+\varphi_{k\bar{k}})\right) \\ &= \sum_{i,k} g'^{i\bar{i}}g'^{k\bar{k}}|\varphi_{k\bar{k}i}|^2 \\ &\leq \sum_{i,j,k} g'^{i\bar{i}}g'^{j\bar{j}}|\varphi_{i\bar{j}k}|^2.\end{aligned}$$

We are left with

$$f^{\frac{-1}{m-1}}\Delta(f^{\frac{1}{m-1}}) \leq \alpha_0 m(m+\Delta\varphi) + S - (\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}})(m+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big) + \sum_k \frac{1}{m-1} g'^{i\bar{i}}g'^{j\bar{j}}\varphi_{i\bar{i}k}\varphi_{j\bar{j}\bar{k}}.$$

It is at this point that we use the hypotheses that $m = 2$ and $f^{1/2}$ is Lipschitz continuous. From taking the derivative of both sides of $(\det g'_{i\bar{j}})^{1/m} = (f \det g_{i\bar{j}})^{1/m}$, we see that $g'^{i\bar{j}}\varphi_{i\bar{j}k} = 2f^{-1/2}\partial_k f^{1/2}$. Therefore,

$$f^{-1}\Delta f \leq 2\alpha_0(2+\Delta\varphi) + S - (\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}})(2+\Delta\varphi)\Big(\sum_i g'^{i\bar{i}}\Big) + 4\frac{|\nabla f^{1/2}|^2}{f}.$$

Since we choose α_0 such that $\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}} > 0$, we use (3.14) and get

$$-A \leq S \sup_M f + 4||\nabla f^{1/2}||^2_\infty + 2\alpha_0 (\sup_M f)(2+\Delta\varphi) - (\alpha_0 + \inf_{i,k} R_{i\bar{i}k\bar{k}})(2+\Delta\varphi)^2.$$

As shown in the previous argument, it follows that $(m+\Delta\varphi) \leq C$. □

4. Dirichlet problem in $\mathbb{C}^m$

As an application of the a priori estimates shown previously, we shall solve a Dirichlet problem in $\mathbb{C}^m$, following the footsteps of Caffarelli, Kohn, Nirenberg and Spruck [6]. In order to prove Theorem 3, we will make use of estimates previously established in the literature and combine them with our result.

Let Ω be a strongly pseudo-convex domain and u be a strictly pluri-subharmonic solution to (1.3), where $f > 0$. We let ψ be a strictly pluri-subharmonic function on Ω such that $\psi = 0$ on $\partial\Omega$ and

$$\det(\psi_{i\bar{j}}) > \sup_{\bar{\Omega}} f \geq \det(u_{i\bar{j}}).$$

By a maximum principle such as the one given in [6], we have have $\psi \leq u$ in $\overline{\Omega}$. To get a upper bound, we solve the Laplace equation for a harmonic function h: $\Delta h = 0$ in Ω and $h = 0$ on $\partial\Omega$. Then since $\Delta u \geq 0$ in Ω, we have $u \leq h$. Using $\psi \leq u \leq h$ in Ω and $\psi = u = h$ on $\partial\Omega$, we can obtain $|\nabla u(z)| \leq \max\{|\nabla\psi(z)|, |\nabla h(z)|\}$ for all $z \in \partial\Omega$. Since ψ and h depend only on Ω, we have a gradient estimate on the boundary.

To push the interior gradient estimate to the boundary as done in [6], we need to introduce the additional hypothesis that $f^{1/m}$ is Lipschitz. It is an open question to determine whether this assumption can be improved to requiring that $f^{1/(m-1)}$ is Lipschitz. Such a result would lead to a more natural statement for Theorem 3, which would be analogous to the result given in [11] for the real Monge–Ampère equation. In our case, we assume the hypothesis that $|\nabla f^{1/m}| \leq A_1$, and thus have

$$||u||_{C^1(\overline{\Omega})} \leq C.$$

We can obtain second-order estimates of u from our current result. Define $\varphi := u - |z|^2$. We have $\det(\delta_{i\bar{j}} + \varphi_{i\bar{j}}) = f \det \delta_{i\bar{j}}$. Consider the test function $H = (m + \Delta\varphi)e^{-\alpha(\varphi)}$ from Proposition 4. If H attains its maximum on $\partial\Omega$, then $|m+\Delta\varphi| \leq C||m+\Delta\varphi||_{L^\infty(\partial\Omega)}$. If H attains its maximum at $p \in \Omega$, then we obtain $|m+\Delta\varphi| \leq C$ if we follow the proof of Proposition 4 with $g_{i\bar{j}} = \delta_{i\bar{j}}$. Therefore, we have

$$||\Delta u||_{L^\infty(\overline{\Omega})} \leq C(1 + ||\Delta u||_{L^\infty(\partial\Omega)}).$$

Control of second-order derivatives on $\partial\Omega$ follows from the argument in [6], and this argument relies on a C^1 estimate and $f^{1/m}$ to be Lipschitz. Therefore, we have the following result.

Proposition 6. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^m$. Let $f : \Omega \to \mathbb{R}$ be a function such that $f > 0$, $|\nabla f^{1/m}| \leq A_1$, and $\Delta f^{\frac{1}{m-1}} \geq -A_2$. Suppose there exists a strictly pluri-subharmonic solution $u \in C^\infty(\bar{\Omega})$ such that*

$$\det u_{i\bar{j}}(z) = f(z) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Then there exists a constant C which depends only on Ω, $\sup(f)$, A_1 and A_2 such that

$$||u||_{C^1(\overline{\Omega})} + ||\Delta u||_{L^\infty(\overline{\Omega})} \leq C.$$

Using Proposition 6, we shall now prove Theorem 3. The strategy will be to solve the non-degenerate Dirichlet problem for $f > 0$, and then use a limiting process. Let $g_\varepsilon = f^{\frac{1}{m-1}} + \varepsilon$, with $\varepsilon > 0$. We extend f such that it is defined on all of $\mathbb{C}^m$. Let $\gamma_\rho = \gamma(|z|/\rho)$, where $\gamma : \mathbb{C}^m \to \mathbb{R}$ is a C^∞ function of compact support such that $0 \leq \gamma \leq 1$, $\int_{\mathbb{C}^m} \gamma = 1$. We define $h_{\varepsilon,\rho} : \Omega \to \mathbb{R}$ in the following way:

$$h_{\varepsilon,\rho}(x) = (g_\varepsilon * \gamma_\rho(x))^{m-1}.$$

Since $\overline{\Omega}$ is compact, we know that $g_\varepsilon * \gamma_\rho \to g_\varepsilon$ uniformly on $\overline{\Omega}$. For $\rho, \varepsilon > 0$ small enough, it can be shown that

$$|\nabla h_{\varepsilon,\rho}^{1/m}| \leq 2A_1, \quad \Delta h_{\varepsilon,\rho}^{1/(m-1)} \geq -A_2.$$

Now, we consider the non-degenerate Monge–Ampère Dirichlet problem

$$\det(u_{\varepsilon,\rho})_{i\bar{j}} = h_{\varepsilon,\rho} \quad \text{in } \Omega,$$
$$u_{\varepsilon,\rho} = 0 \quad \text{on } \partial\Omega.$$

Caffarelli, Kohn, Nirenberg and Spruck [6], since $h_{\varepsilon,\rho}$ is smooth, we know that there exists a smooth strictly pluri-subharmonic solution $u_{\varepsilon,\rho}$. By Proposition 6, we have

$$||u_{\varepsilon,\rho}||_{C^1(\overline{\Omega})} + ||\Delta u_{\varepsilon,\rho}||_\infty \leq C,$$

for some constant C independent of ε and ρ. We let $\rho \to 0$ and obtain a strictly pluri-subharmonic solution u_ε of

$$\det(u_\varepsilon)_{i\bar{j}} = (f^{\frac{1}{m-1}} + \varepsilon)^{m-1} \quad \text{in } \Omega,$$
$$u_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

such that $||u_\varepsilon||_{C^1(\overline{\Omega})} + ||\Delta u_\varepsilon||_\infty \leq C$. Finally, we let $\varepsilon \to 0$ and obtain a pluri-subharmonic solution u of (3) such that $||u||_{C^1(\overline{\Omega})} + ||\Delta u||_\infty \leq C$. Uniqueness is well-known and can be found in [6].

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References

[1] T. Aubin, *Equations du type Monge–Ampere sur les varietes kahleriennes compactes*, C. R. Acad. Sci. Paris **283** (1976), 119–121.

[2] E. Bedford and B. Taylor, *The Dirichlet problem for a complex Monge–Ampere equation*, Invent. Math. **37** (1976), 1–44.

[3] Z. Blocki, *Regularity of the degenerate Monge–Ampere equation on compact Kahler manifolds*, Math. Z. **244** (2003), 153–161.

[4] Z. Blocki, *A gradient estimate in the Calabi–Yau theorem*, Math. Ann. **334** (2009), 317–327.

[5] S. Boucksom, P. Esyssidieux, V. Guedj and A. Zeriahi, *Monge–Ampere equations in big cohomology classes*, Acta Math. **205** (2010), 199–262.

[6] L. Caffarelli, J. Kohn, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. ii. Complex Monge–Ampere, and uniformly elliptic equations*, Commun. Pure Appl. Math. **XXXVIII** (1985), 209–252.
[7] X. Chen, *The space of Kahler metrics*, J. Differ. Geom. **56** (2000), 189–234.
[8] S. Donaldson, *Symmetric spaces, Kahler geometry, and Hamiltonian dynamics*, Northern California Symplectic Geometry Seminar, **196** (1999), 13–33.
[9] P. Esyssidieux, V. Guedj and A. Zeriahi, *Singular Kahler–Einstein metrics*, J. Amer. Math. Soc. **22** (2009), 607–639.
[10] H. Gu and Z. Zhang, *An extension of Mok's theorem on the generalized Frankel conjecture*, Sci. China Math. **53**(5) (2010), 1253–1264.
[11] P. Guan, c^2 *a priori estimate for degenerate Monge–Ampere equations*, Duke Math. J. **86** (1997), 323–346.
[12] P. Guan, *A gradient estimate for complex Monge–Ampere equation*, Preprint, 2008.
[13] Z. Hou, *Complex Hessian equation on Kahler manifold*, Int. Math. Res. Not. **16** (2009), 3098–3111.
[14] S. Kolodziej. *The Monge–Ampere equation on compact Kahler manifolds*, Indiana Univ. Math. **52**(3) (2003), 667–686.
[15] D. Phong, J. Song and J. Sturm, *Complex Monge–Ampere equations*, Surv. Differ. Geom. **17** (2012), 327–411.
[16] D. Phong and J. Sturm, *The Monge–Ampere operator and geodesics in the space of Kahler potentials*, Invent. Math. **166**(1) (2006), 125–149.
[17] Y. Siu, *Lectures on Hermitian–Einstein metrics for stable bundles and Kahler–Einstein metrics*, Birkhauser Verlag, Basel, 1987.
[18] S. Yau, *On the Ricci curvature of a compact Kahler manifold and the complex Monge–Ampere equation*, Commun. Pure Appl. Math. **XXXI** (1978), 339–411.

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